The Formula of Auslander and Buchsbaum

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Homological conjectures for finite dimensional algebras

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Notation. During this talk R is always assumed to be a commutative – to avoid technicalities – Noetherian ring.

1. Regular Sequences

Definition 1.1. Let M be an R-module.

(1) We call $x \in R$ M-regular, if the multiplication with x on M is an injective map, otherwise x is called a zero divisor of M. The set of zero divisors of M is denoted by $\mathrm{ZD}_R(M)$.

(2) A sequence $\underline{x} := (x_1, \ldots, x_n)$ of elements of R is called an M-regular sequence or for short an M-sequence provided the following conditions are fulfilled:

(RS1) The element x_1 is M-regular and x_i is $M/(x_1, \ldots, x_{i-1})M$-regular for $i = 2, \ldots, n$.

(RS2) $M/(x_1, \ldots, x_n)M \neq 0$.

In this situation we call M a $\underline{x}$-regular module.

(3) An R-sequence, i.e. a regular sequence for the R-module R, is for short called a regular sequence.

(4) A sequence $\underline{x} \subset R$ which satisfies the condition (RS1) is called a weak M-sequence.

Lemma 1.2. Let R be a Noetherian ring and $0 \neq M$ an R-module. We have

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p} = \mathrm{ZD}_R(M).$$

Notation. If R is local with maximal ideal $\mathfrak{m}$ and residue class field k we will refer to R as $(R, \mathfrak{m})$ or if we need the residue class field of R as $(R, \mathfrak{m}, k)$.

An easy application of Nakayama's Lemma yields that for a local ring $(R, \mathfrak{m})$ the following definition is equivalent to Definition 1.1. For the local case we will use both equivalent definitions.

Definition 1.3. Let $(R, \mathfrak{m})$ be a local ring and M an R-module. A sequence $\underline{x} := (x_1, \ldots, x_n) \subset R$ is called an M-regular sequence if it satisfies:

(LRS1) $\underline{x} \subset \mathfrak{m}$.

(LRS2) $\underline{x}$ is a weak M-sequence in the sense of Definition 1.1.

Example 1.4. Let S denote an arbitrary ring and $X_1, \ldots, X_n$ pairwise commuting indeterminates over S. We set $R := S[X_1, \ldots, X_n]$. Then $\underline{X} := (X_1, \ldots, X_n)$ is a regular sequence.

Proposition 1.5. Let

$$N_* : \cdots\cdots\; N_m \longrightarrow \cdots\cdots\; N_0 \xrightarrow{\varphi_0} N_{-1} \longrightarrow 0$$

be an exact sequence of R-modules. If $\underline{x}$ is a weak N_i-sequence for all i we have that $N_*/\underline{x}N_*$ is an exact sequence.

Proof. This result is obtained by some straightforward calculations. $\square$

2. Grade and Depth

Proposition 2.1. Let M and N be R-modules. We set $I := \mathrm{Ann}_R(N)$.

(1) If I contains an M-regular element, we have $\mathrm{Hom}_R(N, M) = 0$.
(2) If R is Noetherian, M, N are finite R-modules and $\mathrm{Hom}_R(N, M) = 0$ is fulfilled, we find an M-regular element in I.

Proof. Compare [BRHZ][Proposition 1.2.3]. □

Lemma 2.2. Let M and N be two R-modules and $\underline{x} = (x_1, \ldots, x_n)$ a weak M-sequence in $\mathrm{Ann}_R(N)$. Then we have

$$\mathrm{Hom}_R(N, M/\underline{x}M) \simeq \mathrm{Ext}_R^n(N, M).$$

Proof. We will use induction on n:

- If $n = 0$ we have nothing to prove, because of the natural isomorphism
$$\mathrm{Hom}_R(N, M) \simeq \mathrm{Ext}_R^0(N, M).$$
- Now assume $n > 0$. We set $\underline{x}^{'} := (x_1, \ldots, x_{n-1})$. By the induction hypothesis we have $\mathrm{Hom}_R(N, M/\underline{x}^{'}M) \simeq \mathrm{Ext}_R^{n-1}(N, M)$. The element x_n is $M/\underline{x}^{'}M$ regular. An application of Proposition 2.1 yields $\mathrm{Hom}_R(N, M/\underline{x}^{'}M) = 0$, so by the above isomorphism $\mathrm{Ext}_R^{n-1}(N, M) = 0$. The element x_1 is M-regular, so we find an exact sequence of R-modules
$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0.$$
We apply the long exact homology sequence to deduce that the following sequence is also exact:
$$\cdots\cdots \mathrm{Ext}_R^{n-1}(N, M) \xrightarrow{x_1} \underbrace{\mathrm{Ext}_R^{n-1}(N, M)}_{=0 \text{ (by ab. consid.)}} \longrightarrow \mathrm{Ext}_R^{n-1}(N, M/x_1M)$$
$$\longrightarrow \mathrm{Ext}_R^n(N, M) \xrightarrow{x_1} \mathrm{Ext}_R^n(N, M) \longrightarrow \mathrm{Ext}_R^n(N, M/x_1M) \cdots\cdots .$$
In addition we have $x_1 \in \mathrm{Ann}_R(N)$, this yields: $x_1 \cdot \mathrm{Ext}_R^n(N, M) = 0$, so we have by the exactness of the last sequence an isomorphism
$$\mathrm{Ext}_R^n(N, M) \simeq \mathrm{Ext}_R^{n-1}(N, M/x_1M).$$
A second application of the induction hypothesis yields
$$\mathrm{Ext}_R^{n-1}(N, M/x_1M) \simeq \mathrm{Hom}_R(N, \frac{M/x_1M}{(x_2, \ldots, x_n)(M/x_1M)}),$$
and hence the desired result.

□

Remark 2.3. Let R be Noetherian and $\mathfrak{a} \subset R$ an ideal and M an R-module. Consider $\underline{x} = (x_1, \ldots, x_n)$ a weak M-sequence in $\mathfrak{a}$. Assume that there is an i with $\langle x_1, \ldots, x_{i-1}\rangle = \langle x_1, \ldots, x_{i-1}, x_i\rangle$. Then $x_i(M/\langle x_1, \ldots, x_{i-1}\rangle M) = 0$, a contradiction. Therefore we have – using the fact that R is Noetherian – maximal weak M-, respectively maximal M-sequences in $\mathfrak{a}$.

Theorem 2.4 (Rees)**.** Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finite R-module with $\mathfrak{a}M \neq M$. Then all maximal M-sequence have as common length

$$l = \min\{i \in \mathbb{N} \mid \mathrm{Ext}_R^i(R/\mathfrak{a}, M) \neq 0\}.$$

Proof.

- We choose $\underline{x} = (x_1, \ldots, x_n)$ to be a weak M-sequence in $\mathfrak{a}$. Because of the existence of $\underline{x}$ we find for $1 \leq i \leq n$ an $M/(x_1, \ldots, x_{i-1})$-regular element in $\mathfrak{a} = \mathrm{Ann}_R(R/\mathfrak{a})$. So by Proposition 2.1 we have
$$\mathrm{Hom}_R(R/\mathfrak{a}, M/(x_1, \ldots, x_{i-1})M) = 0.$$
Applying Lemma 2.2 we get
$$0 = \mathrm{Ext}_R^{i-1}(R/\mathfrak{a}, M) \simeq \mathrm{Hom}_R(R/\mathfrak{a}, M/(x_1, \ldots, x_{i-1})M).$$
This yields $n \leq l$.

- Now assume $\underline{x} = (x_1, \ldots, x_n)$ to be a maximal M-sequence in $\mathfrak{a}$.

  Claim. $\operatorname{Ext}_R^n(R/\mathfrak{a}, M) \neq 0$ holds.
  If we have shown the above claim, we have obviously $l \leq n$ and we are done.

  Proof of the claim. Assume $\operatorname{Ext}_R^n(R/\mathfrak{a}, M) = 0$. Because of our assumption $\mathfrak{a}M \neq M$ we have both $M \neq 0$ and $\mathfrak{a} \neq R$. By Lemma 2.2 we have
  $$\operatorname{Ext}_R^n(R/\mathfrak{a}, M) \simeq \operatorname{Hom}_R(R/\mathfrak{a}, M/\underline{x}M) = 0.$$
  An application of Proposition 2.1 yields the existence of an $M/\underline{x}M$-regular element $\tilde{x}$ in $\mathfrak{a}$, so $(\underline{x}, \tilde{x})$ is, using $\mathfrak{a}M \neq M$, an M-sequence in $\mathfrak{a}$. This contradicts the fact, that $\underline{x}$ is maximal. In the case of $\mathfrak{a}M = M$ we can in general not show, that $M/(\underline{x}, \tilde{x})M \neq 0$ holds, so it could indeed happen, that $(\underline{x}, \tilde{x})$ is a just a weak M-sequence but not an M-sequence. □

□

Remark 2.5. For a local ring $(R, \mathfrak{m})$ and a finite R-module $M \neq 0$ we have always that $\mathfrak{m}M \neq M$ is fulfilled (we use Nakayama's Lemma). So we can apply Theorem 2.4 to M-sequences in $\mathfrak{m}$.

Before stating the next definition we need a lemma:

Lemma 2.6. *Let R be a Noetherian ring, M a finite R-module and I an ideal of R. The following facts are equivalent:*

(1) $IM = M$.
(2) $\operatorname{Ext}_R^i(R/I, M) = 0 \ \forall i \in \mathbb{N}$.

Proof. (1) $\Longrightarrow$ (2) : Assume $IM = M$.

Claim. Let R be a Noetherian ring, M a finite R-module and I an ideal of R with $IM = M$. Then we have $\operatorname{Sup}_R(R/I) \cap \operatorname{Sup}_R(M) = \varnothing$.

Proof of the claim. Easy application of Nakayama's lemma. □

Lemma 2.7. *Let R be a Noetherian ring, N a finite and M an arbitrary R-module. Then we have for all $i \in \mathbb{N}$:*
$$\operatorname{Sup}_R(\operatorname{Ext}_R^i(N, M)) \subset \operatorname{Sup}_R(N) \cap \operatorname{Sup}_R(M).$$

Proof. Straightforward. □

An application of Lemma 2.7 yields
$$\operatorname{Sup}_R(\operatorname{Ext}_R^i(R/I, M)) \subset \operatorname{Sup}_R(R/I) \cap \operatorname{Sup}_R(M) = \varnothing.$$
Because R is Noetherian and M is finite over R the R-module $\operatorname{Ext}_R^i(R/I, M)$ is also finite over R, so we can conclude $\operatorname{Ext}_R^i(R/I, M) = 0$.
(2) $\Longrightarrow$ (1) : Let us assume, that $IM \neq M$ holds. We have by Theorem 2.4 that $\min\{i \in \mathbb{N} \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}$ is finite and we are done. □

Definition 2.8. Let R be a Noetherian ring, M a finite R-module and I an ideal of R. We set $\operatorname{grade}(I, M)$ to be

- the common length of maximal M-sequences in I, if we have $IM \neq M$.
- ∞, if $IM = M$ is fulfilled.

Remark 2.9. The last definition is "consistent" with Theorem 2.4, i.e. by applying Theorem 2.4 and Lemma 2.6 we obtain (in both cases)
$$\operatorname{grade}(I, M) = \inf\{i \in \mathbb{N} \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

Definition 2.10. Let $(R, \mathfrak{m})$ be a local ring and M a finite R-module. We denote $\operatorname{grade}(\mathfrak{m}, M)$ by $\operatorname{depth}(M)$ and call it the depth of M (with respect to R).

Remark 2.11. For the local case we have the following version of Theorem 2.4:

Theorem 2.12. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite R-module. Then we have*
$$\operatorname{depth}(M) = \inf\{i \in \mathbb{N} \mid \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

Remark 2.13. For a local ring $(R, \mathfrak{m})$ and a finite R-module $M \neq 0$ we have always $\operatorname{depth}(M) < \infty$, i.e. $\mathfrak{m}M \neq M$ holds.

Proof. This follows from easy applications of Nakayama's Lemma and Lemma 2.6. □

Example 2.14. Let $(R, \mathfrak{m})$ be a local ring and M an R-module. Equivalent are:

(1) $\operatorname{depth}(M) = 0$.
(2) $\mathfrak{m} \in \operatorname{Ass}_R(M)$.

Proof.

$$\operatorname{depth}(M) = 0 \underset{R \text{ is local}}{\Leftrightarrow} \mathfrak{m} = \operatorname{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p} \underset{\substack{\text{prime avoidance} \\ \text{and } |\operatorname{Ass}_R(M)| < \infty}}{\Leftrightarrow} \mathfrak{m} \in \operatorname{Ass}_R(M).$$

□

Now we consider the connection between grade and short exact sequences, which is easily obtained from Theorem 2.4, respectively Remark 2.9 and the long exact homology sequence.

Proposition 2.15. Let R be a Noetherian ring and I an ideal of R. If

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence of R-modules, the following inequalities are satisfied:

(1) $\operatorname{grade}(I, M) \geq \min\{\operatorname{grade}(I, U), \operatorname{grade}(I, N)\}$.
(2) $\operatorname{grade}(I, U) \geq \min\{\operatorname{grade}(I, M), \operatorname{grade}(I, N) + 1\}$.
(3) $\operatorname{grade}(I, N) \geq \min\{\operatorname{grade}(I, U) - 1, \operatorname{grade}(I, M)\}$.

Proposition 2.16. Let M a finite R-module. Assume $\underline{x} = (x_1, \ldots, x_n)$ to be an M-sequence in I. Then

$$\operatorname{grade}(I/(\underline{x}), M/\underline{x}M) = \operatorname{grade}(I, M/\underline{x}M) = \operatorname{grade}(I, M) - n.$$

Proof.
- The first equality is obtained by some straightforward calculations.
- For the second equality, we have to distinguish two cases:
  Case 1: $IM = M$. So we have $\operatorname{grade}(I, M) = \infty$. Obviously we have $\overline{IM} = \overline{M}$, hence $\operatorname{grade}(I, \overline{M}) = \operatorname{grade}(\overline{I}, \overline{M}) = \infty$. Using $\infty - n = \infty$ we are in good shape.
  Case 2: $IM \neq M$: In this case, Theorem 2.4 states that the length of a maximal N-sequence in I is well defined for every R-module N with $IN \neq N$. So we can "blow up" $\underline{x}$ to a maximal M-sequence in I. Using this new sequence we are done.

□

3. The formula of Auslander and Buchsbaum

In this section let $(R, \mathfrak{m}, k)$ be a fixed local Noetherian ring.

Theorem 3.1 (Auslander-Buchsbaum)**.** Let M be a finite R-module of finite projective dimension n. Then we have

$$\operatorname{pdim}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Before proving the Formula of Auslander and Buchsbaum we need some lemmas.

Lemma 3.2. Let $\varphi : F \longrightarrow G$ be an injective morphism of finite R-modules. Assume F to be free and $\mathfrak{m} \in \operatorname{Ass}_R(R)$. Then $\varphi \otimes k$ is injective.

Proof. By $\mathfrak{m} \in \operatorname{Ass}_R(R)$ we have an injection $\iota : k \longrightarrow R$. F is free over R and so in particular a flat R-module, hence $F \otimes \iota : F \otimes k \longrightarrow F \otimes R$ is also injective. Obviously we have a commutative diagram:

$$\begin{array}{ccc} F\otimes k & \xrightarrow{F\otimes \iota} & F\otimes R \\ \Big\downarrow{\scriptstyle \varphi\otimes k} & & \Big\downarrow{\scriptstyle \varphi\otimes R} \\ G\otimes k & \xrightarrow{G\otimes \iota} & G\otimes R. \end{array}$$

So we have

$$(\varphi\otimes k)(G\otimes\iota) = \underbrace{(F\otimes\iota)}_{\substack{\text{injective}\\ \text{by the above}\\ \text{considerations}}} \quad \underbrace{(\varphi\otimes R)}_{\substack{\text{injective}\\ \text{by assumption}}} \quad ,$$

hence $\varphi\otimes k$ is injective. □

Lemma 3.3. Let M be an R-module, we have:

$$\operatorname{pdim}_R(M) = \sup\{n\in\mathbb{N} \mid \operatorname{Tor}_n^R(M,k)\neq 0\}.$$

Lemma 3.4. Let M be a finite R-module. If $x\in\mathfrak{m}$ is both an R-regular and an M-regular element, we have

$$\operatorname{pdim}_R(M) = \operatorname{pdim}_{R/(x)}(M/xM).$$

Proof. Compare [BRHZ][Lemma 1.3.5] □

Proof of the Formula of Auslander and Buchsbaum. We use induction on $\operatorname{depth}(R)$:

- Let us consider the case $\operatorname{depth}(R) = 0$. By Example 2.14 we have $\mathfrak{m}\in\operatorname{Ass}_R(R)$. By assumption there is a minimal free resolution of M:

$$F_*:\quad 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \cdots\cdots F_0 \longrightarrow M \longrightarrow 0.$$

  Let us $n\geq 1$ assume. The morphism $\varphi_n\otimes R$ is injective by Lemma 3.2. The long exact homology sequence for Tor yields that we have $\operatorname{Ker}(\varphi\otimes k) = \operatorname{Tor}_n^R(M,k)\neq 0$ by Lemma 3.3, a contradiction. Therefore M is projective, hence free over the local ring R. But this means $\operatorname{depth}(M) = \operatorname{depth}(R^n) = 0$ for a suitable $n\in\mathbb{N}$. So the desired equation is in the case $\operatorname{depth}(R)=0$ satisfied.
- Let us now $\operatorname{depth}(R)\geq 1$ assume. First we consider the case: $\operatorname{depth}(M) = 0$. Now choose a projective presentation of M:

$$0 \longrightarrow M_1 \longrightarrow P \longrightarrow M \longrightarrow 0.$$

  The R-module P can be chosen to be finitely generated over R (because M is a finite module over a Noetherian ring). So P is a free module over R. Therefore $\operatorname{depth}(P)\geq 1$ because $\operatorname{depth}(R)\geq 1$ holds. By Proposition 2.15 we have

$$0 = \operatorname{depth}(M) \geq \min\{\operatorname{depth}(M_1)-1, \underbrace{\operatorname{depth}(P)}_{\geq 1}\}.$$

  It is easy to get $\operatorname{depth}(M_1) = 1$, by using that M_1 is a submodule of a free module, if $M_1\neq 0$ is fulfilled. The case $M_1 = 0$ yields that M is projective, hence $\operatorname{depth}(M)\geq 1$, a contradiction. By the finiteness of $\operatorname{pdim}_R(M)$ and the fact that M_1 is a first syzygy of M we find $\operatorname{pdim}_R(M_1) = \operatorname{pdim}_R(M) - 1$. So we can w.l.o.g. $\operatorname{depth}(M)\geq 1$ assume. Hence we are now in the case that both $\operatorname{depth}(R)>0$ and $\operatorname{depth}(M)>0$ holds. Therefore we have $\mathfrak{m}\notin\operatorname{Ass}_R(M)$ and $\mathfrak{m}\notin\operatorname{Ass}_R(R)$ by using Example 2.14 a second time.

  Claim. There is an element $x\in\mathfrak{m}$ such that x is both R- and M-regular.

  Proof of the claim. Using Lemma 1.2 we find

$$\operatorname{ZD}_R(M) = \bigcup_{\underbrace{\mathfrak{p}\in\operatorname{Ass}_R(M)}_{\subset_{R\text{ local}}\ \mathfrak{m}}} \mathfrak{p}\,; \quad \operatorname{ZD}_R(R) = \bigcup_{\underbrace{\mathfrak{p}\in\operatorname{Ass}_R(R)}_{\subset_{R\text{ local}}\ \mathfrak{m}}} \mathfrak{p}\,.$$

If such an element does not exist we have

$$\mathfrak{m} = \Big(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)} \mathfrak{p}\Big) \bigcup \Big(\bigcup_{\mathfrak{q}\in \mathrm{Ass}_R(M)} \mathfrak{q}\Big).$$

Both sets $\mathrm{Ass}_R(M)$ and $\mathrm{Ass}_R(R)$ respectively are finite, so by prime avoidance we conclude either $\mathfrak{m} \in \mathrm{Ass}_R(M)$ or $\mathfrak{m} \in \mathrm{Ass}_R(R)$, a contradiction. □

Now choose $x \in \mathfrak{m}$ to be both R- and M-regular. By Proposition 2.16 we find $\mathrm{depth}(R/(x)) = \mathrm{depth}(R) - 1$ and $\mathrm{depth}_{R/(x)}(M/xM) = \mathrm{depth}_R(M) - 1$. An application of Lemma 3.4 yields $\mathrm{pdim}_{R/(x)}(M/xM) = \mathrm{pdim}_R(M)$. We conclude

$$\begin{aligned} \mathrm{depth}(R) - 1 = \mathrm{depth}(R/(x)) &\underset{\text{Induction}}{=} \mathrm{pdim}_{R/(x)}(M/xM) + \mathrm{depth}_{R/(x)}(M/xM) \\ &= \mathrm{pdim}_R(M) + \mathrm{depth}_R(M) - 1, \end{aligned}$$

hence

$$\mathrm{depth}(R) = \mathrm{depth}(M) + \mathrm{pdim}_R(M)$$

and we are done.

□

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