GRADED AND KOSZUL CATEGORIES

ROBERTO MARTÍNEZ-VILLA AND ØYVIND SOLBERG

Abstract. Koszul algebras have arisen in many contexts; algebraic geometry, combinatorics, Lie algebras, non-commutative geometry and topology. The aim of this paper and several sequel papers is to show that for any finite dimensional algebra there is always a naturally associated Koszul theory. To obtain this, the notions of Koszul algebras, linear modules and Koszul duality are extended to additive (graded) categories over a field. The main focus of this paper is to provide these generalizations and the necessary preliminaries.

Introduction

Koszul theory is usually applied to graded algebras Λ which are Koszul. The theory has been extended to non-graded semiperfect Noetherian algebras Λ through the notion of weakly Koszul algebras [MVZ]. The aim of this paper is to show that Koszul theory can be applied to any finite dimensional algebra by associating a Koszul object and therefore a Koszul theory for any finite dimensional algebra. This theory is found by considering the category of all additive contravariant functors from finitely generated Λ-modules to vector spaces. The simple objects in this category are known to be weakly Koszul by a result of Igusa-Todorov ([IT]). Similarly as for algebras, we then pass to a naturally associated graded category, where the simple objects are linear. In this way Koszul theory can be applied to study any finite dimensional algebra. This application serves as a motivation for most of the definitions and the results in the present paper and the subsequent papers based on the current paper. The generalization of the Koszul theory we introduce goes through extending the notions of Koszul algebras, linear modules and Koszul duality, to additive K-categories over a field K. For related work, but a different focus, we point out the work of Mazorchuk, Ovsienko and Stroppel in [MOS].

The process of associating a Koszul object to any finite dimensional algebra goes through utilizing the analogy with algebras. As mentioned above, Koszul theory has been extended to non-graded finite dimensional algebras Λ through the notion of weakly Koszul algebras [MVZ]: Λ is *weakly Koszul* if all simple Λ-modules S have a minimal projective resolution $\cdots \to P_2 \to P_1 \to P_0 \to S \to 0$ satisfying $\mathfrak{r}^{i+1}P_j \cap \Omega_\Lambda^{j+1}(S) = \mathfrak{r}^i\Omega_\Lambda^{j+1}(S)$ for all $j \geq 0$ and $i \geq 0$, where $\mathfrak{r}$ is the Jacobson radical

Date: February 2, 2009.

2000 *Mathematics Subject Classification.* 18A25, 18G10, 18G20.

Key words and phrases. Graded categories, Koszul theory, quadratic categories.

The first author thanks the Universidad Nacional Autonoma de Mexico program PAPITT for funding the research project. In addition he thanks his coauthor and Department of Mathematical Sciences (NTNU) for their kind hospitality and support through Research Council of Norway Storforsk grant no. 167130.

The second author thanks Universidad Nacional Autonoma de Mexico and his coauthor for their kind hospitality and support, and the Department of Mathematical Sciences (NTNU) and Research Council of Norway Storforsk grant no. 167130 also for the support.

of Λ. If Λ is weakly Koszul, then the associated graded ring $\mathcal{A}_{\mathrm{gr}}(\Lambda) = \amalg_{i\geq 0}\mathfrak{r}^i/\mathfrak{r}^{i+1}$ is Koszul. Extending this to $\mathrm{Mod}(\mathrm{mod}\,\Lambda)$, the category of all additive contravariant functors from the category of finitely generated Λ-modules, $\mathrm{mod}\,\Lambda$, to vector spaces over the field and using results of Igusa-Todorov [IT], we show that $\mathrm{Mod}(\mathrm{mod}\,\Lambda)$ is weakly Koszul. Again as for algebras, we show that the naturally associated graded category is Koszul. To demystify this object we describe it for finite representation type. For a finite dimensional algebra Λ of finite representation type, the category $\mathrm{Mod}(\mathrm{mod}\,\Lambda)$ is equivalent to the category of modules over the Auslander algebra Γ of Λ. Which by the results of Igusa-Todorov is weakly Koszul. Then the associated Koszul object we describe for this finite dimensional algebra is equivalent to the module category of graded modules over the associated graded ring $\mathcal{A}_{\mathrm{gr}}(\Gamma)$, which is Koszul.

Next we describe the organization of the paper. In Section 1 we recall definitions of graded categories and functors, and in addition discuss fundamental concepts and results as Yoneda's Lemma, ideals, tensor products, Nakayama's Lemma. While Section 2 deals with projective and simple objects, duality and homological dimensions. Section 3 is devoted to defining and proving basic results about Koszul categories, where we end with a brief discussion on our main application appearing in a subsequent paper. An analogue of weakly Koszul algebras for categories is introduced in Section 4. The graded categories we consider are generated in degrees 0 and 1, and with the further assumption we impose they are quotients of free tensor categories over a bimodule. These categories and the Koszul dual of such are discussed in the the last section.

Finally in this introduction we mention the standing assumptions throughout the paper. An additive (graded) K-category $\mathcal{C}$ is said to be *Krull-Schmidt* if any object in $\mathcal{C}$ is a finite direct sum of objects with a (graded) local endomorphism ring. Throughout we are assuming that in any category we consider, all idempotents split. Under this assumption using [AF, Theorem 27.6] it follows that an additive (graded) K-category is Krull-Schmidt if and only if $\mathrm{End}_{\mathcal{C}}(C)$ (or in the graded case $\mathrm{End}_{\mathcal{C}}(C)_0$) is semiperfect for all C in $\mathcal{C}$. Throughout we assume that all categories we consider are skeletally small.

1. Graded categories and functors

This section is devoted to recalling definitions of graded categories and functors between graded categories. Throughout the paper K denotes a fixed field. In further detail, after graded categories and functors between them are introduced, we discuss Yoneda's Lemma, ideals, tensor products of functors and Nakayama's Lemma. All modules throughout the paper are left modules unless otherwise explicitly said otherwise.

1.1. **Graded categories.** First we introduce graded categories. Let $\mathcal{C}$ be a K-category. The category $\mathcal{C}$ is called *graded* if for each pair of objects C and D in $\mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(C,D) = \amalg_{i\in\mathbb{Z}}\,\mathrm{Hom}_{\mathcal{C}}(C,D)_i$$

as a $\mathbb{Z}$-graded vector space over K, such that if f is in $\mathrm{Hom}_{\mathcal{C}}(C,C')_i$ and g is in $\mathrm{Hom}_{\mathcal{C}}(C',C'')_j$, then gf is in $\mathrm{Hom}_{\mathcal{C}}(C,C'')_{i+j}$. In particular the identity maps are concentrated in one degree and this degree is 0.

Just having a graded K-category is normally too general, so further conditions are often needed. The following two restrictions are central in what follows.

Definition 1.1. (a) A graded K-category $\mathcal{C}$ is *locally finite* if $\operatorname{Hom}_{\mathcal{C}}(C, D)_j$ is finite dimensional over K for all objects C and D in $\mathcal{C}$ and all integers j.

(b) A graded K-category $\mathcal{C}$ is called *positively graded* if

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) = \amalg_{i\geq 0} \operatorname{Hom}_{\mathcal{C}}(X, Y)_i$$

for all objects X and Y in $\mathcal{C}$.

Below we give some examples of graded categories where we return to some of them later. To this end we fix the following notation. For a $\mathbb{Z}$-graded module over a $\mathbb{Z}$-graded ring we define the i-th shift functor $[i]$ as follows: $M[i]_j = M_{i+j}$ and for a homomorphism $f\colon M \to N$ of graded modules $f[i] = f\colon M[i] \to N[i]$.

Example 1.2. The category of graded modules over a graded ring is the first example we review. Let $\Lambda = \amalg_{i\geq 0}\Lambda_i$ be a positively graded algebra over K. Denote by $\operatorname{Gr}(\Lambda)$ the category having as objects all $\mathbb{Z}$-graded Λ-modules and

$$\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M, N) = \amalg_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}(\Lambda)_0}(M, N[i]),$$

where $\operatorname{Gr}(\Lambda)_0$ denotes the category of graded Λ-modules with degree zero homomorphisms. Then $\operatorname{Gr}(\Lambda)$ is a graded category with

$$\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M, N)_i = \operatorname{Hom}_{\operatorname{Gr}(\Lambda)_0}(M, N[i])$$

for all M and N in $\operatorname{Gr}(\Lambda)$. Note here that even though Λ is positively graded, the category $\operatorname{Gr}(\Lambda)$ is never positively graded (as long as $\Lambda \neq (0)$).

Example 1.3. Here we define the associated graded category of an additive category $\mathcal{C}$ with respect to the radical of $\mathcal{C}$. This construction can in fact be done with respect to any ideal in the category $\mathcal{C}$. See subsection 1.4 for a short discussion about ideals in a category.

Let $\mathcal{C}$ be an additive K-category, and denote by $\operatorname{rad}_{\mathcal{C}}$ the radical of $\mathcal{C}$. Recall that the radical, $\operatorname{rad}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$, as a subfunctor of $\operatorname{Hom}_{\mathcal{C}}(-,-)$ is given by

$$\operatorname{rad}_{\mathcal{C}}(C, D) = \{f \in \operatorname{Hom}_{\mathcal{C}}(C, D) \mid gf \in \operatorname{rad}(\operatorname{End}_{\mathcal{C}}(C)) \text{ for all } g \in \operatorname{Hom}_{\mathcal{C}}(D, C)\}.$$

Observe that we also have

$$\operatorname{rad}_{\mathcal{C}}(C, D) = \{f \in \operatorname{Hom}_{\mathcal{C}}(C, D) \mid fh \in \operatorname{rad}(\operatorname{End}_{\mathcal{C}}(D)) \text{ for all } h \in \operatorname{Hom}_{\mathcal{C}}(D, C)\}.$$

(see [M]). Therefore $\operatorname{rad}_{\mathcal{C}} = \operatorname{rad}_{\mathcal{C}^{\mathrm{op}}}$. Furthermore, a morphism f in $\operatorname{Hom}_{\mathcal{C}}(C, D)$ is in $\operatorname{rad}^2_{\mathcal{C}}(C, D)$ if and only if f is a finite sum of maps of the form $C \xrightarrow{f'_i} X \xrightarrow{f''_i} D$ with f'_i in $\operatorname{rad}_{\mathcal{C}}(C, X)$ and f''_i in $\operatorname{rad}_{\mathcal{C}}(X, D)$ for $i = 1, 2, \ldots, n$ and $f = \sum_{i=1}^n f''_i f'_i$. Inductively define $\operatorname{rad}^n_{\mathcal{C}} = \operatorname{rad}_{\mathcal{C}} \cdot \operatorname{rad}^{n-1}_{\mathcal{C}}$.

The associated graded category, $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$, of $\mathcal{C}$ (with respect to the radical) has the same objects as $\mathcal{C}$ while the morphisms are given by

$$\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(A, B) = \amalg_{i\geq 0} \operatorname{rad}^i_{\mathcal{C}}(A, B)/\operatorname{rad}^{i+1}_{\mathcal{C}}(A, B).$$

Then $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is a positively graded category with

$$\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(A, B)_i = \operatorname{rad}^i_{\mathcal{C}}(A, B)/\operatorname{rad}^{i+1}_{\mathcal{C}}(A, B)$$

for all objects A and B in $\mathcal{C}$.

The prime example and application of this construction for us, is to consider a finite dimensional algebra Λ and let $\mathcal{C} = \operatorname{Mod}(\operatorname{mod}\Lambda)$, the category of all additive functors from $(\operatorname{mod}\Lambda)^{\mathrm{op}}$ to $\operatorname{Mod} K$. In Section 3 and in subsequent papers we return to this example. More generally, for a K-category $\mathcal{D}$, we later consider the category, and subcategories of, $\operatorname{Mod}(\mathcal{D})$, which denotes the category of all additive functors from $\mathcal{D}^{\mathrm{op}}$ to $\operatorname{Mod} K$.

Example 1.4. The final example deals with the Ext-category associated to a subcategory of an abelian category. The Koszul dual of a Koszul category, that we define in Section 3, is obtained in this way.

Let $\mathcal{C}$ be an abelian category. For a full subcategory $\mathcal{C}'$ consider the Ext-category $E(\mathcal{C}')$ of $\mathcal{C}'$, which has the same objects as $\mathcal{C}'$ and the homomorphisms are given by

$$\operatorname{Hom}_{E(\mathcal{C}')}(A,B) = \amalg_{i\geq 0} \operatorname{Ext}^i_{\mathcal{C}}(A,B)$$

for all objects A and B in $E(\mathcal{C}')$. Then $E(\mathcal{C}')$ is a positively graded category with

$$\operatorname{Hom}_{E(\mathcal{C}')}(A,B)_i = \operatorname{Ext}^i_{\mathcal{C}}(A,B)$$

for all objects A and B in $E(\mathcal{C}')$.

Here again one of the most important example and application for us is the category $\mathcal{C} = \operatorname{Mod}(\operatorname{mod}\Lambda)$ for a finite dimensional algebra Λ and $\mathcal{C}'$ the full subcategory consisting of all simple functors.

Remark. For both applications to finite dimensional algebras in Example 1.3 and Example 1.4 the graded parts of the categories in question are always semisimple.

1.2. **Functor categories of graded categories.** Next we discuss functors between graded categories. Let $\mathcal{C}$ and $\mathcal{D}$ be two graded K-categories. A covariant functor $F\colon \mathcal{C}\to\mathcal{D}$ of graded categories is a functor between the (ungraded) categories $\mathcal{C}$ and $\mathcal{D}$ such that F induces a degree zero homomorphism of the $\mathbb{Z}$-graded vector spaces $\operatorname{Hom}_{\mathcal{C}}(C,D)$ and $\operatorname{Hom}_{\mathcal{D}}(F(C),F(D))$; that is,

$$F\colon \operatorname{Hom}_{\mathcal{C}}(C,D) = \amalg_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(C,D)_i \to \amalg_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(F(C),F(D))_i = \operatorname{Hom}_{\mathcal{D}}(F(C),F(D))$$

is a degree zero homomorphism. A contravariant functor between graded K-categories is defined similarly.

Example 1.5. Let $\mathcal{C}$ be a graded K-category. For an object C in $\mathcal{C}$ the representable functors $\operatorname{Hom}_{\mathcal{C}}(-,C)\colon \mathcal{C}^{\mathrm{op}}\to \operatorname{Gr}(K)$ and $\operatorname{Hom}_{\mathcal{C}}(C,-)\colon \mathcal{C}\to\operatorname{Gr}(K)$ are covariant functors from the graded K-categories $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{C}$ into the category of graded K-vector spaces, respectively.

Let $\mathcal{C}$ be an additive graded K-category. Denote by $\operatorname{Gr}(\mathcal{C})_0$ the category having as objects the additive graded functors $F\colon \mathcal{C}^{\mathrm{op}}\to\operatorname{Gr}(K)$ and morphisms being the natural transformations $\eta\colon F\to G$ of F and G, where $\eta_C\colon F(C)\to G(C)$ is a degree zero homomorphism for each object C in $\mathcal{C}$. This is an abelian category.

Given F in $\operatorname{Gr}(\mathcal{C})_0$ we define a shift operation $[j]$ for any integer j on the functor F by letting

$$(F[j])(C) = F(C)[j]$$

and

$$(F[j])(f) = F(f)[j]$$

for any objects C and D in $\mathcal{C}$ and any map $f\colon C \to D$. In other words $F[j] = [j] \circ F$, where the last $[j]$ denotes the j-th shift operator in $\mathrm{Gr}(K)$.

We define the category $\mathrm{Gr}(\mathcal{C})$ as the category with the same objects as $\mathrm{Gr}(\mathcal{C})_0$ and morphisms given by

$$\mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, G) = \amalg_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})_0}(F, G[i])$$

for all objects F and G in $\mathrm{Gr}(\mathcal{C})$. In this way $\mathrm{Gr}(\mathcal{C})$ becomes a graded K-category.

1.3. **Yoneda's Lemma.** The Yoneda's Lemma is fundamental in the theory of functors. Here we give a graded version, and the proof is given for completeness.

Lemma 1.6. *Let $\mathcal{C}$ be a graded K-category. The morphism*

$$\alpha\colon \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, C), F) \simeq F(C)$$

given by $\alpha(\eta) = \eta_C(1_C)$ for any $\eta\colon \mathrm{Hom}_C(-, C) \to F$, is a degree zero isomorphism for any C in $\mathcal{C}$ and for any F in $\mathrm{Gr}(\mathcal{C})$.

Proof. Consider a homogeneous element ψ of degree j in

$$\mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, C), F) = \amalg_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})_0}(\mathrm{Hom}_{\mathcal{C}}(-, C), F[i]).$$

This means that $\psi = \{\psi_X\}_{X\in\mathcal{C}}$ and each ψ_X is a natural transformation with $\psi_X\colon \mathrm{Hom}_{\mathcal{C}}(X, C) \to F[j](X)$ a degree zero homomorphism for all X in $\mathcal{C}$. In particular, $\alpha(\psi) = \psi_C(1_C)$ is in $F[j](C)_0 = F(C)_j$, and therefore α is a degree zero homomorphism of K-vector spaces. Hence to show that α is an isomorphism, it is enough to consider homogeneous elements.

Next we show that α is injective. Suppose that $\alpha(\psi) = 0$ for a homogeneous element ψ of degree j in $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, C), F)$. We want to prove that $\psi_X = 0$ for all X in $\mathcal{C}$. Let $h = \{h_i\}_{i\in\mathbb{Z}}$ be in $\mathrm{Hom}_{\mathcal{C}}(X, C) = \amalg_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, C)_i$. Then we have the following commutative diagram for all i in $\mathbb{Z}$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(C, C) & \xrightarrow{\psi_C} & F[j](C) \\
\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(h_i, C)} & & \downarrow{\scriptstyle F[j](h_i)} \\
\mathrm{Hom}_{\mathcal{C}}(X, C)[i] & \xrightarrow{\psi_X} & F[j](X)[i]
\end{array}$$

so that $F[j](h_i)\psi_C(1_C) = \psi_X(h_i)$. Since $\psi_C(1_C) = 0$, we have that $\psi_X(h_i) = 0$ for all i and all X in $\mathcal{C}$. Hence $\psi = 0$, and α is injective.

Finally we prove that α is surjective. Let μ be in $F(C)_j$. Define $\psi = \{\psi_X\}_{X\in\mathcal{C}}\colon \mathrm{Hom}_{\mathcal{C}}(-, C) \to F[j]$ for all X in $\mathcal{C}$ and $h = \{h_i\}_{i\in\mathbb{Z}}$ in $\mathrm{Hom}_{\mathcal{C}}(X, C)$ by letting

$$\psi_X(h) = (F(h_i)(\mu))_{i\in\mathbb{Z}} = F(h)(\mu)$$

viewing $F(h)(\mu)$ as an element in $F[j](X)$. To see that this all makes sense observe the following. For h_i in $\mathrm{Hom}_{\mathcal{C}}(X, C)_i$ the image of h_i under F is a map $F(h_i)\colon F(C) \to F(X)$ of degree i. Hence $F(h_i)(\mu)$ is in $F(X)_{i+j} = F[j](X)_i$, and $F(h)(\mu)$ is in $F[j](X)$. Clearly we have that $\alpha(\psi) = \mu$. So if ψ is a natural transformation the proof is complete. To this end let $g\colon Y \to X$ be in $\mathcal{C}$ and consider the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(X,C) & \xrightarrow{\psi_X} & F[j](X) \\ \big\downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(g,C)} & & \big\downarrow{\scriptstyle F[j](g)} \\ \operatorname{Hom}_{\mathcal{C}}(Y,C) & \xrightarrow{\psi_Y} & F[j](Y) \end{array}$$

For $h\colon X \to C$ in $\mathcal{C}$ we obtain that

$$F[j](g)\psi_X(h) = F(g)(F(h)(\mu)) = F(hg)(\mu) = \psi_Y(hg) = \psi_Y(\operatorname{Hom}_{\mathcal{C}}(g,C)(h)).$$

Hence ψ is a natural transformation, and α is a degree zero isomorphism. □

1.4. **Ideals.** Secretly we have considered ideals in a category already as we have discussed the radical of a category. Here we review the definition of a (graded) ideal in a category and some elementary constructions and results involving ideals.

Recall that an ideal in a category is a sub-bifunctor of the Hom-functor. In case $\mathcal{C}$ is a $\mathbb{Z}$-graded K-category, a graded ideal in $\mathcal{C}$ is a graded sub-bifunctor of the Hom-functor.

For two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in a category we define in a natural way inclusion, intersection and product. In particular, the product of two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ is given by

$$\mathcal{I}_1\mathcal{I}_2(X,Y) = \{f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \mid f = \sum_{i=1}^{n} h_i g_i, g_i \in \mathcal{I}_2(X,A_i), h_i \in \mathcal{I}_1(A_i,Y)\},$$

and if $\mathcal{I}_1$ and $\mathcal{I}_2$ are graded ideals, then their product $\mathcal{I}_1\mathcal{I}_2$ is a graded ideal with

$$\mathcal{I}_1\mathcal{I}_2(X,Y)_t = \{f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)_t \mid f = \sum_{i=1}^{n} h_i g_i, g_i \in \mathcal{I}_2(X,A_i)_{n_i}, h_i \in \mathcal{I}_1(A_i,Y)_{t-n_i}\}.$$

For an n-fold product of an ideal $\mathcal{I}$ with itself we write $\mathcal{I}^n$.

Let $\mathcal{I}$ be an ideal in a category $\mathcal{C}$. For a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} K$ define $\mathcal{I}F\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} K$ as the subfunctor of F given by

$$\mathcal{I}F(M) = \sum_{\substack{f\in\mathcal{I}(M,X)\\ X\in\mathcal{C}}} \operatorname{Im} F(f)$$

for all objects M in $\mathcal{C}$. If $\mathcal{I}$ is a graded ideal and $F\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Gr}(K)$ is a graded functor, $\mathcal{I}F$ is a graded subfunctor of F, where

$$\mathcal{I}F(M)_n = \sum_{\substack{f\in\mathcal{I}(M,X)_i\\ X\in\mathcal{C}\\ i+j=n}} F(f)(F(X)_j)$$

for all objects M in $\mathcal{C}$. Easy properties of the product of an ideal and a functor are the following.

Lemma 1.7. *Let $\mathcal{I}$ be an ideal and $\eta\colon F \to G$ a morphism of two functors F and G in* $\operatorname{Mod}(\mathcal{C})$.

(a) $\eta(\mathcal{I}F) \subseteq \mathcal{I}G$.
(b) *If $\eta\colon F \to G$ is an epimorphism, then $\eta|_{\mathcal{I}F}\colon \mathcal{I}F \to \mathcal{I}G$ is an epimorphism.*

Given an ideal in an additive K-category $\mathcal{C}$ there is a naturally associated graded category. We saw an example of this already in Example 1.3. Let $\mathcal{C}$ be an additive K-category with an ideal $\mathcal{I}$. Denote by $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ the associated graded category with respect to $\mathcal{I}$ having the same objects as $\mathcal{C}$ while the morphisms are given by

$$\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(A,B) = \amalg_{j\geq 0}\mathcal{I}^j(A,B)/\mathcal{I}^{j+1}(A,B)$$

for all objects A and B in $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$.

For each object and each morphism in $\operatorname{Mod}(\mathcal{C})$ there are naturally associated an object and a morphism in $\operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$. Denote this function on objects and morphisms by G, and $G\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is given by letting

$$G(F) = \amalg_{j\geq 0}\mathcal{I}^j F/\mathcal{I}^{j+1}F$$

for all F in $\operatorname{Mod}(\mathcal{C})$, where $G(F)_j = \mathcal{I}^j F/\mathcal{I}^{j+1}F$. On a morphism $\eta\colon F\to H$ in $\operatorname{Mod}(\mathcal{C})$ let

$$G(\eta) = (\overline{\eta}|_{\mathcal{I}^jF/\mathcal{I}^{j+1}F})_{j\geq 0}\colon G(F)\to G(H).$$

An ideal $\mathcal{I}$ gives a filtration of any object F in $\operatorname{Mod}(\mathcal{C})$ via $\{\mathcal{I}^jF\}_{j\geq 0}$. Any natural transformation $\eta\colon F\to F'$ for F and F' in $\operatorname{Mod}(\mathcal{C})$ we have that $\eta(\mathcal{I}^jF)\subseteq \mathcal{I}^jF'$ for all $j\geq 0$ by Lemma 1.7. So any object in $\operatorname{Mod}(\mathcal{C})$ has a filtration and any morphism in $\operatorname{Mod}(\mathcal{C})$ has degree 0, referring to [NvO, I.2]. Then by [NvO, I.4] the function G is a functor $G\colon \operatorname{Mod}(\mathcal{C})\to \operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$.

The notion of an ideal in a graded category leads to the following natural definition of a graded category generated in degrees 0 and 1.

Definition 1.8. Let $\mathcal{C}$ be a positively graded K-category. The graded category $\mathcal{C}$ is said to be *generated in degrees* 0 *and* 1 if the ideal

$$J = \amalg_{i\geq 1}\operatorname{Hom}_{\mathcal{C}}(-,-)_i \subseteq \operatorname{Hom}_{\mathcal{C}}(-,-)$$

satisfies

$$J^r = \amalg_{i\geq r}\operatorname{Hom}_{\mathcal{C}}(-,-)_i$$

for all $r\geq 1$.

1.5. **Tensor product of functors.** Tensor products of functors were first considered by Mitchell in [M] and then later by Auslander and Bautista et. al. in [A, BCS]. Here we review the construction of tensor products of functors from [A].

Let $\mathcal{C}$ be an additive K-category. For two functors F in $\operatorname{Mod}(\mathcal{C})$ and G in $\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ we want to define the tensor product $G\otimes_{\mathcal{C}} F$ of G and F. There is a unique (up to isomorphism) functor $-\otimes_{\mathcal{C}}-\colon \operatorname{Mod}(\mathcal{C}^{\mathrm{op}})\times\operatorname{Mod}(\mathcal{C})\to \operatorname{Mod}K$ satisfying the following properties:

(i) The tensor product is a right exact functor in each variable.
(ii) The tensor product commutes with direct sums in both variables.
(iii) For each object C in $\mathcal{C}$ we have $\operatorname{Hom}_{\mathcal{C}}(C,-)\otimes_{\mathcal{C}} F = F(C)$ and $G\otimes_{\mathcal{C}}\operatorname{Hom}_{\mathcal{C}}(-,C) = G(C)$ for any F in $\operatorname{Mod}(\mathcal{C})$ and G in $\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$.

A morphism $\amalg_j \operatorname{Hom}_{\mathcal{C}}(Y_j,-)\to \amalg_i\operatorname{Hom}_{\mathcal{C}}(X_i,-)$ is given by morphisms $f_{ij}\colon X_i\to Y_j$ such that (f_{ij}) is in $\prod_j \amalg_i \operatorname{Hom}_{\mathcal{C}}(X_i,Y_j)$, and

$$(\operatorname{Hom}_{\mathcal{C}}(f_{ij},-))\otimes 1_F\colon\ \amalg_j\operatorname{Hom}_{\mathcal{C}}(Y_j,-)\otimes_{\mathcal{C}} F\to \amalg_i\operatorname{Hom}_{\mathcal{C}}(X_i,-)\otimes_{\mathcal{C}} F$$

is by definition given by $(F(f_{ij}))\colon\ \amalg_j F(Y_j)\to \amalg_i F(X_i)$.

Suppose that G is in $\operatorname{Mod}\mathcal{C}$, and let

$$\amalg_j\operatorname{Hom}_{\mathcal{C}}(Y_j,-)\xrightarrow{(f_{ij},-)}\amalg_i\operatorname{Hom}_{\mathcal{C}}(X_i,-)\to G\to 0$$

be a projective presentation of G (See Section 2 for a further discussion on projective functors). Then $G \otimes_{\mathcal{C}} F$ is by definition given by the commutative diagram

$$\begin{array}{ccccccc}
\amalg_j \operatorname{Hom}_{\mathcal{C}}(Y_j, -) \otimes_{\mathcal{C}} F & \longrightarrow & \amalg_i \operatorname{Hom}_{\mathcal{C}}(X_i, -) \otimes_{\mathcal{C}} F & \longrightarrow & G \otimes_{\mathcal{C}} F & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\amalg_j F(Y_j) & \xrightarrow{(F(f_{ij}))} & \amalg_i F(X_i) & \longrightarrow & G \otimes_{\mathcal{C}} F & \longrightarrow & 0
\end{array}$$

It is straightforward to see that the definition of $G \otimes_{\mathcal{C}} F$ is independent of the chosen projective presentation of G. Furthermore, it follows that the tensor product $- \otimes_{\mathcal{C}} -$ is a right exact functor in both variables. An equivalent definition is to use a projective presentation of F to define $G \otimes_{\mathcal{C}} F$. We freely identify these.

Let $B\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \operatorname{Mod} K$ be an additive bifunctor, and let F be in $\operatorname{Mod}(\mathcal{C})$. Then define $B \otimes_{\mathcal{C}} F\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} K$ by

$$(B \otimes_{\mathcal{C}} F)(X) = B(X, -) \otimes_{\mathcal{C}} F$$

and for $f\colon X \to Y$ in $\mathcal{C}$

$$(B \otimes_{\mathcal{C}} F)(f)\colon B(Y, -) \otimes_{\mathcal{C}} F \xrightarrow{B(f,-)\otimes_{\mathcal{C}} 1_F} B(X, -) \otimes_{\mathcal{C}} F.$$

Applying this to an ideal in an additive K-category $\mathcal{C}$ we obtain the following.

Lemma 1.9. *Let $\mathcal{I}$ be an ideal in an additive K-category $\mathcal{C}$. Then*

$$\operatorname{Hom}_{\mathcal{C}}(-, -)/\mathcal{I} \otimes_{\mathcal{C}} F \simeq F/\mathcal{I}F$$

for all F in $\operatorname{Mod}(\mathcal{C})$.

Proof. The exact sequence

$$0 \to \mathcal{I} \to \operatorname{Hom}_{\mathcal{C}}(-, -) \to \operatorname{Hom}_{\mathcal{C}}(-, -)/\mathcal{I} \to 0$$

gives rise to the exact sequence

$$\mathcal{I} \otimes_{\mathcal{C}} F \to \operatorname{Hom}_{\mathcal{C}}(-, -) \otimes_{\mathcal{C}} F \to \operatorname{Hom}_{\mathcal{C}}(-, -)/\mathcal{I} \otimes_{\mathcal{C}} F \to 0.$$

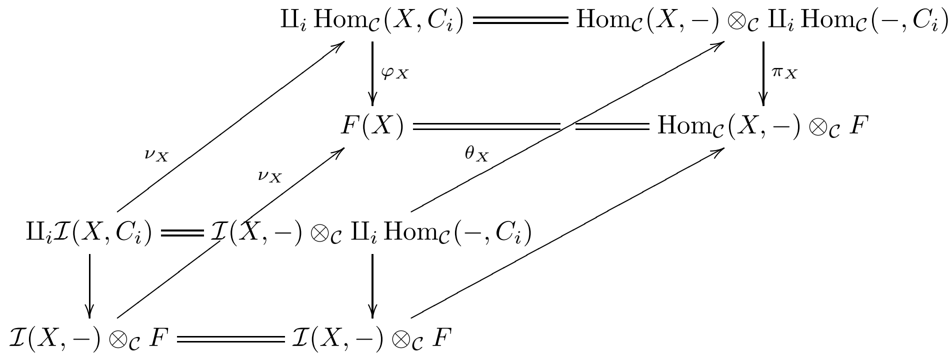
For an object X in $\mathcal{C}$ we obtain

$$\begin{array}{ccccccccc}
 & & \mathcal{I}(X, -) \otimes_{\mathcal{C}} F & \xrightarrow{\theta_X} & \operatorname{Hom}_{\mathcal{C}}(X, -) \otimes_{\mathcal{C}} F & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(X, -)/\mathcal{I}(X, -) \otimes_{\mathcal{C}} F & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow \wr & & \\
0 & \longrightarrow & \operatorname{Im} \theta_X & \longrightarrow & F(X) & \longrightarrow & F(X)/\operatorname{Im} \theta_X & \longrightarrow & 0
\end{array}$$

We want to show that $\operatorname{Im} \theta_X = (\mathcal{I}F)(X)$ for all X in $\mathcal{C}$.

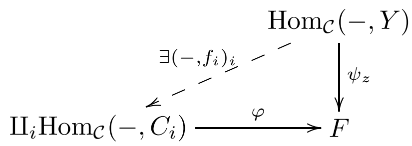
Let $\amalg_j \operatorname{Hom}_{\mathcal{C}}(-, B_j) \to \amalg_i \operatorname{Hom}_{\mathcal{C}}(-, C_i) \xrightarrow{\varphi} F \to 0$ be a projective presentation of F. Recall that by Yoneda's Lemma φ is given as follows. Let $x_i = \varphi(1_{C_i})$ in $F(C_i)$. For $(f_i)_i$ in $\amalg_i(X, C_i)$ we have $\varphi((f_i)_i) = \sum_i F(f_i)(x_i)$.

Consider the following commutative diagram



It is clear from this diagram that $\operatorname{Im}\theta_X = \operatorname{Im}\pi_X\nu_X = \operatorname{Im}\varphi_X\nu_X$. Given $(f_i)_i$ in $\amalg_i\mathcal{I}(X, C_i)$ we have that $\varphi_X\nu_X((f_i)_i) = \varphi_X((f_i)_i) = \sum_i F(f_i)(x_i)$ which is in $(\mathcal{I}F)(X) = \sum_{\substack{h\in\mathcal{I}(X,Y)\\ Y\in\mathcal{C}}} F(h)$. Hence $\operatorname{Im}\theta_X \subseteq (\mathcal{I}F)(X)$.

Conversely, let y be in $(\mathcal{I}F)(X)$. By definition there exists some g in $\mathcal{I}(X, Y)$ and z in $F(Y)$ such that $y = F(g)(z)$. By Yoneda's Lemma z corresponds to a map $\psi_z\colon (-, Y) \to F$. Then the diagram



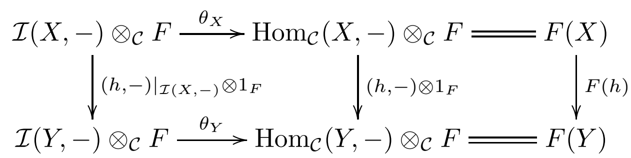
gives a map $(f_i)_i$ in $\amalg_i\mathcal{C}(Y, C_i)$ such that $\psi_z(1_Y) = z = \sum_i F(f_i)(x_i)$. Applying $F(g)$ to this equality we have that

$$y = F(g)(z) = F(g)\sum_i F(f_i)(x_i) = \sum_i F(f_ig)(x_i),$$

which is in $\operatorname{Im}\theta_X$. Hence $\operatorname{Im}\theta_X = (\mathcal{I}F)(X)$. It follows that

$$((-,-)/\mathcal{I}(-,-) \otimes_{\mathcal{C}} F)(X) \simeq F(X)/\operatorname{Im}\theta_X = F(X)/\mathcal{I}F(X) = (F/\mathcal{I}F)(X).$$

Given a map $h\colon Y \to X$ we have a commutative diagram



Hence $\operatorname{Im} F(h)\theta_X \subseteq \operatorname{Im}\theta_Y$. Let z be in $(\mathcal{I}F)(X)$, then $z = F(g)(w)$ for some g in $\mathcal{I}(X, W)$ and w in $F(W)$. For all h in $\operatorname{Hom}_{\mathcal{C}}(Y, X)$ the composition gh is in $\mathcal{I}(Y, W)$. Furthermore $F(gh)(w) = F(h)F(g)(w) = F(h)(z)$, and it is in $(\mathcal{I}F)(Y)$. Therefore $\operatorname{Im}\theta = \mathcal{I}F$ as functors. This completes the proof. □

Let $\mathcal{C}$ be an additive graded K-category. Here we extend the definition of tensor products of functors to tensor products of graded functors and obtain similar results.

For two graded functors F in $\mathrm{Gr}(\mathcal{C})$ and G in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$, then choosing a graded projective presentation

$$\amalg_j \operatorname{Hom}_{\mathcal{C}}(Y_j,-)[n_j] \xrightarrow{(f_{ij})} \amalg_i \operatorname{Hom}_{\mathcal{C}}(X_i,-)[m_i] \to G \to 0$$

of G one similarly defines $G \otimes_{\mathcal{C}} F$ as

$$\amalg_j F(Y_j)[n_j] \xrightarrow{(F(f_{ij}))} \amalg_i F(X_i)[m_i] \to G \otimes_{\mathcal{C}} F \to 0.$$

Since $\amalg_j F(Y_j)[n_j] \xrightarrow{(F(f_{ij}))} \amalg_i F(X_i)[m_i]$ is a degree zero map of graded vector spaces, $G \otimes_{\mathcal{C}} F$ naturally becomes a graded vector space. Again this is independent of the chosen projective presentation of G (or the one of F).

1.6. **Nakayama's Lemma.** As for ring theory in general Nakayama's Lemma is a central result. Here we give a version for graded functors. To this end we need the following definition.

Let $\mathcal{C}$ be a graded K-category. Then a graded functor F in $\mathrm{Gr}(\mathcal{C})$ is said to be *bounded below* if $F(C)_i = (0)$ for all objects C in $\mathcal{C}$ and $i < N$ for some integer N.

Lemma 1.10 (Nakayama's Lemma)**.** *Let $\mathcal{C}$ be an additive graded K-category with radical* $\mathrm{rad}_{\mathcal{C}} = \amalg_{i\geq 1} \operatorname{Hom}_{\mathcal{C}}(-,-)_i$*. Suppose that F in* $\mathrm{Gr}(\mathcal{C})$ *is a bounded below graded functor. Assume that $F/\mathrm{rad}_{\mathcal{C}} F = (0)$, then $F = 0$.*

Proof. Assume that F is bounded below such that $F(C)_i = (0)$ for all $i < N$ and all C in $\mathcal{C}$ and that $F(C)_N \neq (0)$ for some C in $\mathcal{C}$. By definition

$$(\mathrm{rad}_{\mathcal{C}} F)(C) = \sum_{\substack{g\in \mathrm{rad}_{\mathcal{C}}(C,X)\\ X\in\mathcal{C}}} \operatorname{Im} F(g) \subseteq F(C).$$

Since $\mathrm{rad}_{\mathcal{C}}(C,X) = \amalg_{i\geq 1} \operatorname{Hom}_{\mathcal{C}}(C,X)_i$, then for any g in $\mathrm{rad}_{\mathcal{C}}(C,X)$ the morphism $F(g)\colon F(X) \to F(C)$ might be of mixed degree but always of degree greater or equal to 1. Since $F(X)_i = (0)$ for all $i < N$, we have $\operatorname{Im} F(g) \subseteq \amalg_{i>N} F(C)_i$. It follows that $F/\mathrm{rad}_{\mathcal{C}} F \neq 0$. □

The assumption on the radical of the category $\mathcal{C}$ above seems strong. However, the applications we have in mind are coming from Example 1.3 and Example 1.4. In addition, the situation we want to generalize is that of a positively graded algebra $\Lambda = \sum_{i\geq 0} \Lambda_i$ with Λ_0 semisimple.

2. Homological algebra

The main aim in this section is to prove some elementary homological properties we use later for the graded categories discussed in the previous section. When $\mathcal{C}$ is a positively graded Krull-Schmidt category, we characterize the projective and the simple objects in $\mathrm{Gr}(\mathcal{C})$, discuss a duality for subcategories of $\mathrm{Gr}(\mathcal{C})$ and show that the global dimension of $\mathrm{Gr}(\mathcal{C})$ is given by the supremum of the projective dimension of the simple objects.

2.1. **Projective functors and covers.** Using the Yoneda's Lemma it is well-known that the functors $\operatorname{Hom}_{\mathcal{C}}(-,C)$ are projective in $\mathrm{Mod}(\mathcal{C})$ and $\mathrm{Gr}(\mathcal{C})$ for an additive graded K-category $\mathcal{C}$. In addition any projective functor is a direct summand of $\amalg_{i\in\Sigma} \operatorname{Hom}_{\mathcal{C}}(-,C_i)[m_i]$ for some integers m_i and index set Σ. By further assuming that the category $\mathcal{C}$ is Krull-Schmidt, we show next that all projective functors in $\mathrm{Gr}(\mathcal{C})$ are exactly given like this for indecomposable objects C_i.

Lemma 2.1. *Let $\mathcal{C}$ be a graded Krull-Schmidt K-category. The projective functors in $\mathrm{Gr}(\mathcal{C})$ are all of the form*

$$\amalg_{j\in J}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]$$

with C_j indecomposable in $\mathcal{C}$ and m_j an integer for some index set J.

Proof. Let F be a projective functor in $\mathrm{Gr}(\mathcal{C})$. Then there exists an exact sequence $\amalg_{j\in I}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]\xrightarrow{\pi}F$ for some degree zero morphism π and for some indecomposable objects C_j in $\mathcal{C}$, integers m_j and some index set J. Since F is projective in $\mathrm{Gr}(\mathcal{C})$, there is a splitting of the morphism π and this can be chosen also as a degree zero morphism. Standard arguments then shows there is a degree 0 isomorphism

$$\varphi\colon\ \amalg_{j\in I}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]\to F\amalg G$$

for some functor G in $\mathrm{Gr}(\mathcal{C})$ (choose $G=\mathrm{Ker}\,\pi$).

Consider the composition of the maps

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]\xrightarrow{\lambda_j}\amalg_{j\in I}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]\xrightarrow{p_F\varphi}F\xrightarrow{\varphi^{-1}i_F}\\ \amalg_{j\in I}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]\xrightarrow{p_j}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j],\end{aligned}$$

where p_F and i_F are the natural projection and inclusion of F in $F\amalg G$, respectively. Then $p_j\varphi^{-1}i_Fp_F\varphi\lambda_j\colon\ \mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]\to\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]$ is a natural transformation of degree 0 induced by a map $g\colon C_j\to C_j$ in $\mathrm{Hom}_{\mathcal{C}}(C_j,C_j)_0$ by the Yoneda's Lemma. Since $\mathrm{Hom}_{\mathcal{C}}(C_j,C_j)_0$ is a local ring and $1_{F\amalg G}=i_Fp_F+i_Gp_G$, either g or $1-g$ is an isomorphism.

If g is an isomorphism, then $\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]$ is a direct summand of F. And if $1-g$ is an isomorphism, then $\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]$ is a direct summand of G. Hence, either $\mathrm{Hom}_{\mathcal{C}}(-,C_j)[m_j]$ is a direct summand of F or a direct summand of G.

Now one can proceed as in the proof of [AF, Theorem 26.5]. Consider pairs (J,L) of subsets of I with $J\cap L=\emptyset$ and such that $\amalg_{j\in J}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[n_j]$ is a subfunctor of F and $\amalg_{l\in L}\mathrm{Hom}_{\mathcal{C}}(-,C_l)[n_l]$ is subfunctor of G. These pairs (J,L) is naturally ordered by inclusion, and any chain has a upper bound given by the union. So, by Zorn's Lemma we can choose a maximal pair (J,L). Then we have the following commutative diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\amalg_{j\in J\cup L}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[n_j] & = & (\amalg_{j\in J}\mathrm{Hom}_{\mathcal{C}}(-,C_j)[n_j])\amalg(\amalg_{l\in L}\mathrm{Hom}_{\mathcal{C}}(-,C_l)[n_l])\\
\downarrow & & \downarrow\\
\amalg_{i\in I}\mathrm{Hom}_{\mathcal{C}}(-,C_i)[n_i] & \xrightarrow{\sim} & F\amalg G\\
\downarrow & & \downarrow\\
\amalg_{i\in I\setminus(J\cup L)}\mathrm{Hom}_{\mathcal{C}}(-,C_i)[n_i] & \xrightarrow{\sim} & F'\amalg G'\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

where $F' = F/(\amalg_{j\in J} \operatorname{Hom}_{\mathcal{C}}(-, C_j)[n_j])$ and $G' = G/(\amalg_{l\in L} \operatorname{Hom}_{\mathcal{C}}(-, C_l)[n_l])$. Hence, any $\operatorname{Hom}_{\mathcal{C}}(-, C_{j_0})[n_{j_0}]$ with j_0 in $I \setminus (J \cup L)$ is either a direct summand of F' or a direct summand of G'. It follows that either

$$(\amalg_{j\in J} \operatorname{Hom}_{\mathcal{C}}(-, C_j)[n_j]) \amalg \operatorname{Hom}_{\mathcal{C}}(-, C_{j_0})[n_{j_0}]$$

is a subfunctor of F or

$$(\amalg_{l\in L} \operatorname{Hom}_{\mathcal{C}}(-, C_l)[n_l]) \amalg \operatorname{Hom}_{\mathcal{C}}(-, C_{j_0})[n_{j_0}]$$

is a subfunctor of G. This contradicts the choice of the maximal pair (J, L). The claim follows from this. □

Next we discuss projective covers in the category of functors we are considering. Recall that an essential epimorphism $P \to F$ in $\operatorname{Mod}(\mathcal{C})$ is a *projective cover* of F if P is a projective $\mathcal{C}$-module. For the category of modules over a ring, all simple modules have a projective cover if and only if the ring is semiperfect. It was shown by Auslander in [A] that the same condition comes up in having minimal projective presentations of finitely presented functors as we recall next. Denote by $\operatorname{mod}\mathcal{C}$ the full subcategory of $\operatorname{Mod}(\mathcal{C})$ consisting of all finitely presented functors.

Lemma 2.2 ([A, Corollary 4.13]). *Let $\mathcal{C}$ be a positively graded K-category, where (all idempotents split and) $\operatorname{End}_{\mathcal{C}}(C)_0$ is semiperfect for all objects C in $\mathcal{C}$. Then every object in $\operatorname{mod}(\mathcal{C})$ has a minimal projective presentation. In particular, any object in $\operatorname{mod}(\mathcal{C})$ has a projective cover.*

By our remark in the introduction, for a graded Krull-Schmidt K-category $\mathcal{C}$ the category of finitely presented functors $\operatorname{mod}\mathcal{C}$ has minimal projective presentations.

There is another situation where projective covers always exists. To motivate this recall the following situation for graded algebras. Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a positively graded K-algebra with Λ_0 semisimple. Let M be a graded module bounded below, that is, M is generated in some degrees $i_0 < i_1 < i_2 < \cdots$. Then M has a projective cover. An analogue for graded functors is the following, which is also related to the above version of the Nakayama's Lemma.

Lemma 2.3. *Let $\mathcal{C}$ be a positively graded Krull-Schmidt K-category with $\operatorname{rad}_{\mathcal{C}}(-,-) = \amalg_{i\geq 1} \operatorname{Hom}_{\mathcal{C}}(-,-)_i$.*

(a) *Any bounded below functor F in $\operatorname{Gr}(\mathcal{C})$ has a projective cover.*
(b) *Let F in $\operatorname{Gr}(\mathcal{C})$ be bounded below, and let $P \to F$ be a projective cover. Then $P/\operatorname{rad}_{\mathcal{C}} P \simeq F/\operatorname{rad}_{\mathcal{C}} F$.*

Proof. (a) Since any bounded below functor F in $\operatorname{Gr}(\Lambda)$ is a factor of shifts of copies of $\operatorname{Hom}_{\mathcal{C}}(-, C)$ for C indecomposable in $\mathcal{C}$, it follows that $F/\operatorname{rad}_{\mathcal{C}} F$ is a direct sum of shifts of t_i copies of simple functors $(-, C_i)/\operatorname{rad}_{\mathcal{C}}(-, C_i)$ for some integers t_i, say $\amalg_{i\in I} \left((-, C_i)/\operatorname{rad}_{\mathcal{C}}(-, C_i)[n_i]\right)^{t_i}$. Then we obtain an induced morphism $\pi\colon \amalg_{i\in I} \left((-, C_i)[n_i]\right)^{t_i} \to F$. Using Nakayama's Lemma we infer that π is an epimorphism. Let $P = \amalg_{i\in I}(-, C_i)[n_i]^{t_i}$. Then we have that $P/\operatorname{rad}_{\mathcal{C}} P \simeq F/\operatorname{rad}_{\mathcal{C}} F$ and using Nakayama's Lemma again we see that π is an essential epimorphism. Hence $\pi\colon P \to F$ is a projective cover.

(b) The claim follows from the construction in (a). □

2.2. **Simple functors.** In Koszul theory simple modules play a crucial role. So there is no surprise in generalizing to functor categories that the simple functors are of equally great importance. Here we show that for a positively graded Krull-Schmidt K-category, there is one-to-one correspondence between indecomposable objects and simple functors.

To show the above claim we shall need the following considerations. Given a graded module $M = \amalg_{i\in\mathbb{Z}} M_i$ over some positively graded ring, the set $M_{\geq n} = \amalg_{i\geq n} M_i$ is a graded submodule of M. There is a similar construction for graded functors. Let $\mathcal{C}$ be a positively graded K-category, and let F be in $\mathrm{Gr}(\mathcal{C})$. Then we define $F_{\geq n}$ as

$$F_{\geq n}(C) = (F(C))_{\geq n}$$

for all C in $\mathcal{C}$, and for $f\colon C \to C'$ in $\mathcal{C}$

$$F_{\geq n}(f) = F(f)|_{F_{\geq n}(C)}\colon F_{\geq n}(C) \to F_{\geq n}(C').$$

Since $\mathcal{C}$ is positively graded, we infer that $F_{\geq n}$ is a graded subfunctor of F. As a consequence of this we obtain that a simple object S in $\mathrm{Gr}(\mathcal{C})$ is supported only in one degree; that is, $S(C)_i \neq (0)$ for one fixed $i = i_0$ for all objects C in $\mathcal{C}$.

When $\mathcal{C}$ in addition is Krull-Schmidt, we obtain even more as shown next.

Lemma 2.4. *Let $\mathcal{C}$ be a positively graded Krull-Schmidt K-category, and let* $\mathrm{rad}_{\mathcal{C}}(-,-)$ *be the radical of $\mathcal{C}$.*

(a) *Any simple functor in* $\mathrm{Gr}(\mathcal{C})$ *is of the form* $\mathrm{Hom}_{\mathcal{C}}(-,C)/\mathrm{rad}_{\mathcal{C}}(-,C)$ *for some indecomposable object C in $\mathcal{C}$ up to shift.*
(b) *For all finitely generated functors F in* $\mathrm{Gr}(\mathcal{C})$ *the radical of F is given by* $\mathrm{rad}_{\mathcal{C}} F$.
(c) *If F is a finitely generated functor in* $\mathrm{Gr}(\mathcal{C})$ *with* $F = \mathrm{rad}_{\mathcal{C}} F$, *then* $F = 0$.
(d) *All finitely generated functors F in* $\mathrm{Gr}(\mathcal{C})$ *have a projective cover.*

The proof of this result is basically the same as the following result, which we give a proof of.

Lemma 2.5. *Let $\mathcal{C}$ be a Krull-Schmidt K-category, and let* $\mathrm{rad}_{\mathcal{C}}(-,-)$ *be the radical of $\mathcal{C}$.*

(a) *Any simple functor in* $\mathrm{Mod}(\mathcal{C})$ *is of the form* $\mathrm{Hom}_{\mathcal{C}}(-,C)/\mathrm{rad}_{\mathcal{C}}(-,C)$ *for some indecomposable object C in $\mathcal{C}$.*
(b) *For all finitely generated functors F in* $\mathrm{Mod}(\mathcal{C})$ *the radical of F is given by* $\mathrm{rad}_{\mathcal{C}} F$.
(c) *If F is a finitely generated functor in* $\mathrm{Mod}(\mathcal{C})$ *with* $F = \mathrm{rad}_{\mathcal{C}} F$, *then* $F = 0$.
(d) *All finitely generated functors F in* $\mathrm{Mod}(\mathcal{C})$ *have a projective cover.*

Proof. (a) Let S be a simple functor in $\mathrm{Mod}(\mathcal{C})$. Then for some indecomposable object C in $\mathcal{C}$, the vector space $S(C)$ is non-zero. By Yoneda's Lemma there exists a non-zero morphism $\eta\colon \mathrm{Hom}_{\mathcal{C}}(-,C) \to S$, which necessarily is an epimorphism. We claim that this is a projective cover of S.

First we show that for a finitely generated functor F in $\mathrm{Mod}(\mathcal{C})$ and an epimorphism $\eta'\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \to F$, the morphism η' is minimal if and only if η' is an essential epimorphism. Assume that η' is minimal, and let $\rho\colon H \to \mathrm{Hom}_{\mathcal{C}}(-,X)$ be such that $\eta'\rho\colon H \to F$ is an epimorphism. Since $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is projective, there exists a morphism $\sigma\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \to H$ such that $\eta'\rho\sigma = \eta'$. Since η' is minimal, $\rho\sigma$ is an isomorphism, and in particular ρ is an epimorphism. This shows that η' is an essential epimorphism.

Conversely, assume that η' is an essential epimorphism, and let $\gamma\colon \operatorname{Hom}_{\mathcal{C}}(-,X) \to \operatorname{Hom}_{\mathcal{C}}(-,X)$ be such that $\eta'\gamma = \eta'$. It follows that γ is an epimorphism, and since $\operatorname{Hom}_{\mathcal{C}}(-,X)$ is projective, there exists a morphism $\sigma\colon \operatorname{Hom}_{\mathcal{C}}(-,X) \to \operatorname{Hom}_{\mathcal{C}}(-,X)$ such that $\gamma\sigma = 1_{\operatorname{Hom}_{\mathcal{C}}(-,X)}$. Then $\eta'\sigma = \eta'$, and as for γ the morphism σ is an epimorphism. It follows that σ is an isomorphism. Hence also γ is an isomorphism and η' is minimal. This completes the proof of the above claim.

Return to the morphism $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-,C) \to S$ above. If X is indecomposable, then

$$\operatorname{rad}_{\mathcal{C}}(X,C) = \begin{cases} \operatorname{Hom}_{\mathcal{C}}(X,C), & \text{for } X \not\simeq C,\\ \operatorname{rad}\operatorname{End}_{\mathcal{C}}(C), & \text{for } X \simeq C.\end{cases}$$

We infer that $\operatorname{Ker}\eta \subseteq \operatorname{rad}_{\mathcal{C}}(-,C)$ and that $\operatorname{Ker}\eta = \operatorname{rad}_{\mathcal{C}}(-,C)$, since $\operatorname{Ker}\eta$ is a maximal subfunctor. Hence $S \simeq \operatorname{Hom}_{\mathcal{C}}(-,C)/\operatorname{rad}_{\mathcal{C}}(-,C)$. It remains to show that η is minimal (essential epimorphism). Let $\gamma\colon \operatorname{Hom}_{\mathcal{C}}(-,C) \to \operatorname{Hom}_{\mathcal{C}}(-,C)$ be a natural transformation such that $\eta\gamma = \eta$. By Yoneda's Lemma γ is given as $\operatorname{Hom}_{\mathcal{C}}(-,h)$ for some h in $\operatorname{End}_{\mathcal{C}}(C)$. If h is not an isomorphism, then h is in the radical of the local ring $\operatorname{End}_{\mathcal{C}}(C)$, by our assumptions on $\mathcal{C}$. Hence $1-h$ is invertible or equivalently an isomorphism. This implies that $\eta = 0$, which is a contradiction. Therefore h is an isomorphism and $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-,C) \to S$ is a projective cover.

(b) Now let F be finitely generated in $\operatorname{Mod}(\mathcal{C})$ with $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-,B) \to F$ being an epimorphism. Then $\eta(\operatorname{rad}_{\mathcal{C}}(-,B)) = \operatorname{rad}_{\mathcal{C}} F$ and $\operatorname{Hom}_{\mathcal{C}}(-,B)/\operatorname{rad}_{\mathcal{C}}(-,B) \to F/\operatorname{rad}_{\mathcal{C}} F$ is an epimorphism. Therefore $F/\operatorname{rad}_{\mathcal{C}} F$ is semisimple and $\operatorname{rad}(F/\operatorname{rad}_{\mathcal{C}} F) = (0)$. Moreover, $\eta(\operatorname{rad}(-,B)) \subseteq \operatorname{rad} F$. Since $\mathcal{C}$ is Krull-Schmidt, $\operatorname{rad}(-,B) = \operatorname{rad}_{\mathcal{C}}(-,B)$ and therefore $\operatorname{rad}_{\mathcal{C}} F \subseteq \operatorname{rad} F$. It follows that $\operatorname{rad}(F/\operatorname{rad}_{\mathcal{C}} F) = \operatorname{rad} F/\operatorname{rad}_{\mathcal{C}} F$ and consequently $\operatorname{rad} F = \operatorname{rad}_{\mathcal{C}} F$.

(c) Let F be finitely generated in $\operatorname{Mod}(\mathcal{C})$ with $F = \operatorname{rad}_{\mathcal{C}} F$ and $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-,B) \to F$ being an epimorphism. From (b) we have that $\eta(\operatorname{rad}\operatorname{End}_{\mathcal{C}}(B)) = \operatorname{rad}_{\mathcal{C}} F(B)$. Since $\eta_B\colon \operatorname{Hom}_{\mathcal{C}}(B,B) \to F(B)$ is a morphism of $\operatorname{End}_{\mathcal{C}}(B)$-modules, $\operatorname{rad}_{\mathcal{C}} F(B) = (\operatorname{rad}\operatorname{End}_{\mathcal{C}}(B))F(B)$. By Nakayama's Lemma $F(B) = (0)$ and therefore $F = 0$.

(d) Keeping the notation and assumptions from (b), we have

$$F/\operatorname{rad}_{\mathcal{C}} F \simeq \operatorname{Hom}_{\mathcal{C}}(-,C)/\operatorname{rad}_{\mathcal{C}}(-,C)$$

for some C in $\mathcal{C}$. This gives rise to a morphism $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-,C) \to F$ with

$$\overline{\eta}\colon \operatorname{Hom}_{\mathcal{C}}(-,C)/\operatorname{rad}_{\mathcal{C}}(-,C) \to F/\operatorname{rad}_{\mathcal{C}} F$$

an isomorphism. Let $\gamma\colon H \to \operatorname{Hom}_{\mathcal{C}}(-,C)$ such that $\eta\gamma$ is an epimorphism. Then $\operatorname{Hom}_{\mathcal{C}}(-,C)/\operatorname{Im}\gamma = \operatorname{rad}_{\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,C)/\operatorname{Im}\gamma)$, so that by (c) γ is an epimorphism and $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-,C) \to F$ is a projective cover. □

2.3. **Duality.** For finite dimensional algebras, the vector space duality $D = \operatorname{Hom}_K(-,K)$ provides a bridge between left and right finite dimensional modules. For graded K-algebras and graded modules $M = \amalg_{i\in\mathbb{Z}} M_i$ over such, each graded part M_i is a vector space over K. Then one defines the graded dual of M as $\amalg_{i\in\mathbb{Z}} D(M_{-i})$, where the degree i part is $D(M_{-i})$. A similar construction can be carried out for graded functors, and in the following result we describe this construction and some elementary properties and consequences of having such a functor.

Proposition 2.6. *Let $\mathcal{C}$ be a graded K-category. Then there exists a contravariant functor*

$$D\colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$$

defined by

$$D(F)(X) = \amalg_{j\in\mathbb{Z}} \mathrm{Hom}_K(F(X)_j, K)$$

for X in $\mathcal{C}$, where $D(F)(X)_i = \mathrm{Hom}_K(F(X)_{-i}, K)$ for i in $\mathbb{Z}$. Then the following statements hold:

(a) *For a set of functors $\{F_i\}_{i\in I}$ we have that $D(\amalg_{i\in I} F_i) = \prod_{i\in I} D(F_i)$.*

(b) *There exists an injective natural transformation $\eta\colon 1_{\mathrm{Gr}(\mathcal{C})} \to D^2$. If F is locally finite, then η_F is an isomorphism of functors. In particular, if $\mathrm{lfGr}(\mathcal{C})$ denotes the full subcategory of $\mathrm{Gr}(\mathcal{C})$ consisting of locally finite graded contravariant functors, then D induces a duality*

$$D\colon \mathrm{lfGr}(\mathcal{C}) \to \mathrm{lfGr}(\mathcal{C}^{\mathrm{op}}).$$

(c) *If $B\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Gr}(K)$ is a graded bifunctor, G is in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$ and F is in $\mathrm{Gr}(\mathcal{C})$, we have natural isomorphisms*

$$\varphi_{F,G}\colon \mathrm{Hom}_K(G \otimes_{\mathcal{C}} F, K) \simeq \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(G))$$

where $\mathrm{Hom}_K(G \otimes_{\mathcal{C}} F, K)$ is the dual of the graded vector space $G \otimes_{\mathcal{C}} F$, and

$$\psi_{F,B}\colon D(B \otimes_{\mathcal{C}} F) \simeq \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(B))$$

as functors in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$.

(d) *$\mathrm{Gr}(\mathcal{C})$ has enough injectives.*

(e) *Assume in addition that $\mathcal{C}$ is a positively graded Krull-Schmidt category where for each indecomposable object C in $\mathcal{C}$ the factor $\mathrm{End}_{\mathcal{C}}(C)/\operatorname{rad}\mathrm{End}_{\mathcal{C}}(C)$ is finite dimensional over K.*

Then the functors of finite length are in $\mathrm{lfGr}(\mathcal{C})$ (and respectively $\mathrm{lfGr}(\mathcal{C}^{\mathrm{op}})$), and the duality

$$D\colon \mathrm{lfGr}(\mathcal{C}) \to \mathrm{lfGr}(\mathcal{C}^{\mathrm{op}})$$

takes functors of finite length to functors of finite length.

Proof. It is clear that D as defined above gives rise to a functor from $\mathrm{Gr}(\mathcal{C})$ to $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$.

(a) This follows directly from the definitions involved.

(b) Define $\eta_F = \{\eta_F(X)\}_{X\in\mathcal{C}}\colon F \to D^2(F)$ in the following way. For each object X in $\mathcal{C}$, the set $D^2(F)(X)_i = \mathrm{Hom}_K(D(F)(X)_{-i}, K)$ and $D(F)(X)_{-i} = \mathrm{Hom}_K(F(X)_i, K)$. Let f be in $F(X)_i$, then $\eta_F(X)(f)\colon D(F)(X)_{-i} \to K$. Let g be in $D(F)(X)_{-i}$, then $\eta_F(X)(f)(g) = g(f)$. Since $F(X)_i$ is a K-vector space, as usual, $\eta_F(X)$ is injective. Hence, $\eta_F(X)$ is injective for each X in $\mathcal{C}$ and $\eta_F = \{\eta_F(X)\}_{X\in\mathcal{C}}$ is a graded injective natural transformation.

If F is locally finite, then $(\eta_F(X))_i\colon F(X)_i \to (D^2(F)(X))_i$ is an isomorphism for all i in $\mathbb{Z}$ and for all X in $\mathcal{C}$. It follows that $\eta_F(X)\colon F(X) \to D^2(F)(X)$ is an isomorphism and $\eta_F\colon F \to D^2(F)$ is an isomorphism of functors. The last claim follows directly from this.

(c) Let G be in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$. For M in $\mathcal{C}$ we have isomorphisms

$$\begin{aligned} D(G \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-, M)) &\simeq D(G(M)) \\ &\simeq D(G)(M) \\ &\simeq \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, M), D(G)), \end{aligned}$$

and this give rise to a natural isomorphism $\alpha\colon D(G \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-, M)) \to \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, M), D(G))$.

Now let F be in $\mathrm{Gr}(\mathcal{C})$. Assume that

$$\amalg_i \mathrm{Hom}_{\mathcal{C}}(-, M_i)[m_i] \to \amalg_i \mathrm{Hom}_{\mathcal{C}}(-, N_i)[n_i] \to F \to 0$$

is a graded projective presentation of F in $\mathrm{Gr}(\mathcal{C})$. This gives rise to the following commutative diagram with exact rows, where the horizontal morphisms are induced by the natural isomorphism α described above.

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(G)) & \longrightarrow & D(G \otimes_{\mathcal{C}} F) \\ \downarrow & & \downarrow \\ \prod_i \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, N_i)[n_i], D(G)) & \xrightarrow{\sim} & \prod_i D(G \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-, N_i)[n_i]) \\ \downarrow & & \downarrow \\ \prod_i \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, M_i)[m_i], D(G)) & \xrightarrow{\sim} & \prod_i D(G \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-, M_i)[m_i]) \end{array}$$

Therefore $\varphi = \varphi_{F,G}\colon \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(G)) \to D(G \otimes_{\mathcal{C}} F)$ induced by α is an isomorphism, and the first claim follows.

Let $B\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Gr}(K)$ be a graded bifunctor, and let F be as above. Note that $(B \otimes_{\mathcal{C}} F)(X) = B(X, -) \otimes_{\mathcal{C}} F$ and that $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(B))(X) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(B(X, -)))$. For any X in $\mathcal{C}$ define

$$\begin{aligned} \psi_{F,B,X} = \varphi_{F,B(X,-)}\colon D(B \otimes_{\mathcal{C}} F)(X) = D(B(X, -) \otimes_{\mathcal{C}} F) \to \\ \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(B(X, -))) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(F, D(B))(X). \end{aligned}$$

It follows from the above that $\psi_{F,B}$ is a natural isomorphism of functors.

(d) Let G be a functor in $\mathrm{Gr}(\mathcal{C})$. Then $D(G)$ in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$ is a factor of a projective functor $\amalg_{j\in J} \mathrm{Hom}_{\mathcal{C}}(C_j, -)[n_j] \to D(G)$. The inclusions $G \to D^2(G)$ and $D^2(G) \to \prod_{j\in J} D((C_j, -))[-n_j]$ induce an inclusion $G \to \prod_{j\in J} D((C_j, -))[-n_j]$. Hence, if $D(C, -)$ is an injective functor for all objects C in $\mathcal{C}$, the category $\mathrm{Gr}(\mathcal{C})$ has enough injective objects.

Consider an exact sequence of graded functors $0 \to F \to G \to H \to 0$ in $\mathrm{Gr}(\mathcal{C})$. Given the natural isomorphisms

$$\mathrm{Hom}(F, D(\mathrm{Hom}_{\mathcal{C}}(C, -))) \simeq D(\mathrm{Hom}_{\mathcal{C}}(C, -) \otimes_{\mathcal{C}} F) \simeq D(F(C))$$

and the exactness of the sequence $0 \to D(H(C)) \to D(G(C)) \to D(F(C)) \to 0$ for all objects C in $\mathcal{C}$, we infer that $\mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(-, D(\mathrm{Hom}_{\mathcal{C}}(C, -)))$ is an exact functor and $D(\mathrm{Hom}_{\mathcal{C}}(C, -)))$ is an injective object in $\mathrm{Gr}(\mathcal{C})$ for all C in $\mathcal{C}$.

(e) Under the assumptions in (e) the simple functors are given as $S_C = \mathrm{Hom}_{\mathcal{C}}(-, C)/\mathrm{rad}_{\mathcal{C}}(-, C)$ for some indecomposable object C in $\mathcal{C}$ up to shift by Lemma 2.4. Since S_C only has support in C and $S_C(C) = \mathrm{End}_{\mathcal{C}}(C)/\mathrm{rad}\,\mathrm{End}_{\mathcal{C}}(C)$, our assumptions imply that S_C is locally finite for all indecomposable objects C in $\mathcal{C}$. Hence all functors of finite length are locally finite, that is, all functors of finite length are in $\mathrm{lfGr}(\mathcal{C})$.

In (a) we observed that the functor D is a duality from $\mathrm{lfGr}(\mathcal{C})$ to $\mathrm{lfGr}(\mathcal{C}^{\mathrm{op}})$. It follows from this that if S is a simple functor, then $D(S)$ is also a simple functor. Consequently, it follows that if F has finite length, DF also has finite length. □

2.4. **Homological algebra.** Here we discuss some elementary homological facts that we need later in this series of papers. Among other things we show that a bounded below flat functor is projective over a positively graded Krull-Schmidt K-category with the radical given by the positive degrees and that the maximum of the projective dimension of the simple functors in this setting gives the global dimension of $\mathrm{Gr}(\mathcal{C})$.

We start with an immediate homological corollary of the duality considered in the previous subsection.

Corollary 2.7. *Let $\mathcal{C}$ be a graded K-category and consider the functor $D\colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$ defined above. Then*

$$\mathrm{Hom}_K(\mathrm{Tor}_i^{\mathcal{C}}(G,F),K) \simeq \mathrm{Ext}^i_{\mathcal{C}}(F,D(G))$$

for all F in $\mathrm{Gr}(\mathcal{C})$ and G in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$, and all $i \geq 0$.

In the following let $\mathcal{C}$ be a positively graded Krull-Schmidt K-category with $\mathrm{rad}_{\mathcal{C}}(-,-) = \amalg_{i\geq 1}\mathrm{Hom}_{\mathcal{C}}(-,-)_i$. In this setting we discuss bounded below functors. To this end we need to observe the following. As we saw in Lemma 2.4 all simple functors in $\mathrm{Gr}(\mathcal{C})$ are given as $S_C = \mathrm{Hom}_{\mathcal{C}}(-,C)/\mathrm{rad}_{\mathcal{C}}(-,C) = \mathrm{Hom}_{\mathcal{C}}(-,C)_0$ for some indecomposable object C in $\mathcal{C}$. Then we have the following.

Proposition 2.8. *Let $\mathcal{C}$ be a positively graded Krull-Schmidt K-category with $\mathrm{rad}_{\mathcal{C}}(-,-) = \amalg_{i\geq 1}\mathrm{Hom}_{\mathcal{C}}(-,-)_i$. Let F be a bounded below functor in $\mathrm{Gr}(\mathcal{C})$.*

(a) *If $S \otimes_{\mathcal{C}} F = (0)$ for all simple functors S in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$, then F is zero.*
(b) *If $\mathrm{Tor}_1^{\mathcal{C}}(S,F) = (0)$ for all simple functors S in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$, then F is a projective object in $\mathrm{Gr}(\mathcal{C})$. In particular, a bounded below flat functor is a projective functor.*
(c) *Assume that each simple object S in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$ has projective dimension at most n. Then F has projective dimension at most n.*

Proof. (a) Let F be a bounded below functor in $\mathrm{Gr}(\mathcal{C})$. For any simple functor $S = \mathrm{Hom}_{\mathcal{C}}(C,-)/\mathrm{rad}_{\mathcal{C}}(C,-)$ we have that

$$(0) = S \otimes_{\mathcal{C}} F = \mathrm{Hom}_{\mathcal{C}}(C,-)/\mathrm{rad}_{\mathcal{C}}(C,-) \otimes_{\mathcal{C}} F \simeq F(C)/(\mathrm{rad}_{\mathcal{C}} F)(C)$$

for all indecomposable C in $\mathcal{C}$. It follows that $F/\mathrm{rad}_{\mathcal{C}} F = (0)$. By Nakayama's Lemma we infer that $F = (0)$.

(b) Let F be a bounded below functor in $\mathrm{Gr}(\mathcal{C})$, and let

$$0 \to \Omega^1_{\mathcal{C}}(F) \to P \to F \to 0$$

be a projective cover of F. Since $\mathcal{C}$ is positively graded, it follows that $\Omega^1_{\mathcal{C}}(F)$ is also bounded below. The above exact sequence gives rise to the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{C}}(S,F) \to S \otimes_{\mathcal{C}} \Omega^1_{\mathcal{C}}(F) \to S \otimes_{\mathcal{C}} P \to S \otimes_{\mathcal{C}} F \to 0$$

by Lemma 2.3 (b). For $S = \mathrm{Hom}_{\mathcal{C}}(C,-)/\mathrm{rad}_{\mathcal{C}}(C,-)$ with C indecomposable in $\mathcal{C}$, we have that

$$S \otimes_{\mathcal{C}} P \simeq P(C)/\mathrm{rad}_{\mathcal{C}} P(C) \simeq F(C)/\mathrm{rad}_{\mathcal{C}} F(C) \simeq S \otimes_{\mathcal{C}} F.$$

Therefore $(0) = \operatorname{Tor}_1^{\mathcal{C}}(S,F) \simeq S \otimes_{\mathcal{C}} \Omega^1_{\mathcal{C}}(F)$. By (a) we conclude that $\Omega^1_{\mathcal{C}}(F) = (0)$ and that F is projective.

(c) By assumption $\operatorname{Tor}_{n+1}^{\mathcal{C}}(S,-) = (0)$ for all simple functors S in $\operatorname{Gr}(\mathcal{C}^{\mathrm{op}})$. By dimension shift $\operatorname{Tor}_1^{\mathcal{C}}(S,\Omega^n_{\mathcal{C}}(F)) = (0)$ for any bounded below functor F in $\operatorname{Gr}(\mathcal{C})$. Since $\Omega^n_{\mathcal{C}}(F)$ is bounded below, we infer from (b) that $\Omega^n_{\mathcal{C}}(F)$ is projective and that the projective dimension of F is at most n. □

Using this we show that the maximum of the projective dimensions of the simple functors is the global dimension of the category $\operatorname{Gr}(\mathcal{C})$.

Theorem 2.9. *Let $\mathcal{C}$ be a positively graded Krull-Schmidt K-category with $\operatorname{rad}_{\mathcal{C}}(-,-) = \amalg_{i\geq 1} \operatorname{Hom}_{\mathcal{C}}(-,-)_i$.*

- (a) *Assume that all simple functors S in $\operatorname{Gr}(\mathcal{C}^{\mathrm{op}})$ have projective dimension at most n. Then $\operatorname{Gr}(\mathcal{C})$ has global dimension at most n.*
- (b) *The global dimension of $\operatorname{Gr}(\mathcal{C})$ is finite if and only if the global dimension of $\operatorname{Gr}(\mathcal{C}^{\mathrm{op}})$ is finite. When one of the global dimensions is finite, then they are equal.*

Proof. (a) We proved in Proposition 2.8 that any bounded below functor F in $\operatorname{Gr}(\mathcal{C})$ has projective dimension at most n. Let F be any graded functor. Define $G_i = F_{\geq -i}$ for all $i \geq 0$. We obtain a directed system $G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots$, and $\varinjlim G_i = F$.

We have exact sequences of functors

$$0 \to G_i \to G_{i+1} \to F_{-i-1} \to 0,$$

so that the projective objects of a projective resolution of G_{i+1} can be chosen as the direct sum of the projective objects occurring in the projective resolution of G_i and F_{-i-1} and the morphism between the resolutions for G_i and G_{i+1} would just be the natural inclusion. Let P^i and Q^i_j be the projective objects occurring at stage i in the projective resolution of G_0 and of F_j, respectively. If we take the direct limit of this system, we obtain as the projective occurring at stage i to be $P^i \amalg (\amalg_{j<0} Q^i_j)$. Since G_0 and all F_j are bounded below, the projectives occurring at stage $n+1$ are zero, so that the n-th syzygy in the induced resolution of F is $P^n \amalg (\amalg_{j<0} Q^n_j)$. Hence F has projective dimension at most n.

(b) The follows directly from (a) and Proposition 2.8 (c) for $\operatorname{Gr}(\mathcal{C})$. □

3. Koszul categories

This section is devoted to defining Koszul categories and showing some elementary properties of them. In particular it is shown that a Koszul category is generated in degrees 0 and 1, and that there is a naturally associated Koszul dual.

First we define our Koszul categories.

Definition 3.1. Let $\mathcal{C}$ be a positively graded Krull-Schmidt locally finite K-category.

- (a) A functor F in $\operatorname{Gr}(\mathcal{C})$ is said to be *linear* if F has a finitely generated projective graded resolution of the form

$$\cdots \to \operatorname{Hom}_{\mathcal{C}}(-,C_2)[-2] \to \operatorname{Hom}_{\mathcal{C}}(-,C_1)[-1] \to \operatorname{Hom}_{\mathcal{C}}(-,C_0) \to F \to 0$$

with C_i in $\mathcal{C}$ for all $i \geq 0$ and all morphisms have degree zero.

(b) The category $\mathcal{C}$ is said to be *Koszul* if all simple functors S in $\mathrm{Gr}(\mathcal{C})$ are linear.

Our notion of a Koszul category include the classical Koszul algebras defined in [P]. However the generalization in [B] and the further generalization in [CS] of Koszul algebras are not covered.

Let $\mathcal{C}$ be a positively graded K-category. Recall that the category $\mathcal{C}$ is said to be *generated in degrees* 0 *and* 1 if the ideal $J = \amalg_{i\geq 1} \mathrm{Hom}_{\mathcal{C}}(-,-)_i \subseteq \mathrm{Hom}_{\mathcal{C}}(-,-)$ satisfies $J^r = \amalg_{i\geq r} \mathrm{Hom}_{\mathcal{C}}(-,-)_i$ for all $r \geq 1$.

Next we show that Koszul categories share the same property as Koszul algebras being generated in degrees 0 and 1.

Lemma 3.2. *Let $\mathcal{C}$ be a Koszul category. Then the following assertions are true.*

(a) $\mathrm{rad}_{\mathcal{C}}(-,-) = \mathrm{Hom}_{\mathcal{C}}(-,-)_{\geq 1}$.
(b) *The category $\mathcal{C}$ is generated in degrees* 0 *and* 1.

Proof. (a) Since $\mathcal{C}$ is Koszul, the simple functors in $\mathrm{Gr}(\mathcal{C})$ are finitely presented. By Lemma 2.4 the simple functors are given as $S_C = \mathrm{Hom}_{\mathcal{C}}(-,C)/\mathrm{rad}_{\mathcal{C}}(-,C)$ for some indecomposable object C in $\mathcal{C}$ up to shift. The start of the graded projective resolution of S_C is then of the form

$$0 \to \mathrm{rad}_{\mathcal{C}}(-,C) \to \mathrm{Hom}_{\mathcal{C}}(-,C) \to S_C \to 0.$$

The simple functors S_C are only concentrated in one degree, so $\mathrm{Hom}_{\mathcal{C}}(-,C)_{\geq 1}$ is contained in $\mathrm{rad}_{\mathcal{C}}(-,C)$. The next projective in the graded projective resolution of S_C being of the form $\mathrm{Hom}_{\mathcal{C}}(-,C_1)[-1]$, implies that $\mathrm{rad}_{\mathcal{C}}(-,C)$ is contained in $\mathrm{Hom}_{\mathcal{C}}(-,C)_{\geq 1}$. Hence $\mathrm{rad}_{\mathcal{C}}(-,C) = \mathrm{Hom}_{\mathcal{C}}(-,C)_{\geq 1}$ for all indecomposable objects C in $\mathcal{C}$. The claim follows from this.

(b) The epimorphism $\mathrm{Hom}_{\mathcal{C}}(-,C_1)[-1] \to \mathrm{Hom}_{\mathcal{C}}(-,C)_{\geq 1}$ for any indecomposable object C in $\mathcal{C}$ found in (a), implies that $\mathrm{Hom}_{\mathcal{C}}(X,C)_n = \mathrm{Hom}_{\mathcal{C}}(C_1,C)_1 \mathrm{Hom}_{\mathcal{C}}(X,C_1)_{n-1}$ for all $n \geq 2$ and all objects X in $\mathcal{C}$. The claim follows by induction from this. □

Next we show that Koszul categories have many of the same properties as those of Koszul algebras. Let $\mathcal{C}$ be a Koszul K-category, and let $\mathcal{S}(\mathcal{C})$ be the full additive subcategory generated by the simple objects in $\mathrm{Gr}(\mathcal{C})$, which might be viewed as a subcategory of $\mathrm{Mod}(\mathcal{C})$. Recall that the Ext-category (see Example 1.4 in Section 1) $E(\mathcal{S}(\mathcal{C}))$ has the same objects as $\mathcal{S}(\mathcal{C})$ and the morphisms are given by

$$\mathrm{Hom}_{E(\mathcal{S}(\mathcal{C}))}(A,B) = \oplus_{i\geq 0} \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(A,B).$$

Since $E(\mathcal{S}(\mathcal{C}))$ consists of objects in $\mathrm{Mod}(\mathcal{C})$, for every object F in $\mathrm{Mod}(\mathcal{C})$ we can consider $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(F,A)$ for any i and for any object A in $E(\mathcal{S}(\mathcal{C}))$. This gives rise to an analogue of the usual Koszul duality functor $\phi\colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(E(\mathcal{S}(\mathcal{C})))$ given by

$$\phi(F) = \mathrm{Ext}^*_{\mathrm{Mod}(\mathcal{C})}(F,-) = \oplus_{i\geq 0} \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(F,-).$$

Indeed note that this is a contravariant functor. For related algebra results see [GM1, Proposition 5.5 (b)] for (b), [GM2, Theorem 5.2] for (e), [BGS, Theorem 2.10.2] or [GM1, Theorem 6.1] for (f), [BGS, Theorem 2.10.2] or [GM2, Theorem 2.3] for (g).

Theorem 3.3. *Let $\mathcal{C}$ be a Koszul K-category. Then the following assertions are true.*

(a) *If F is linear, then $\Omega^1_{\mathrm{Gr}(\mathcal{C})}(F)[1]$ is linear.*

(b) *If F is linear, then $\operatorname{rad}_{\mathcal{C}} F[1]$ is linear. In particular $\operatorname{rad}_{\mathcal{C}} F$ is finitely generated.*

(c) *Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in $\mathrm{Gr}(\mathcal{C})_0$ of functors generated in degree zero. Then $\operatorname{rad}_{\mathcal{C}} F_2 \cap F_1 = \operatorname{rad}_{\mathcal{C}} F_1$.*

(d) *If $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact in $\mathrm{Gr}(\mathcal{C})_0$ with F_1 and F_2 linear, then F_3 is linear.*

(e) *Let $\phi\colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$ be given by $\phi(F) = \operatorname{Ext}^*_{\mathrm{Mod}(\mathcal{C})}(F, -)$, where $\mathcal{S}(\mathcal{C})$ is the full additive subcategory generated by the simple functors in $\mathrm{Gr}(\mathcal{C})$. The functor ϕ restricts to a functor from the linear functors in $\mathrm{Gr}(\mathcal{C})$ to the linear functors in $\mathrm{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$.*

(f) *The graded K-category $\mathcal{C}' = E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}$ is Koszul.*

(g) *The graded K-categories $\mathcal{C}$ and $E(\mathcal{S}(\mathcal{C}'))^{\mathrm{op}}$ are equivalent.*

Proof. The claim in (a) is clear from the definition of linear functors. The statement in (b) follows in a similar way as for algebras using that if $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence in $\mathrm{Gr}(\mathcal{C})$ with degree zero homomorphisms and F_1 and F_2 linear, then F_3 is also linear.

(c) Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in $\mathrm{Gr}(\mathcal{C})_0$ of functors generated in degree zero. This exact sequence induces the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \cap \operatorname{rad}_{\mathcal{C}} F_2 & \longrightarrow & \operatorname{rad}_{\mathcal{C}} F_2 & \longrightarrow & \operatorname{rad}_{\mathcal{C}} F_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 0
\end{array}$$

Since F_2 is generated in degree zero, $\operatorname{rad}_{\mathcal{C}} F_2 = (F_2)_{\geq 1}$ and $\operatorname{rad}_{\mathcal{C}} F_2(X)_0 = (0)$ for all X in $\mathcal{C}$. Consequently $(F_1 \cap \operatorname{rad}_{\mathcal{C}} F_2)(X)_0 = (0)$ for all X. Similarly, $\operatorname{rad}_{\mathcal{C}} F_1 = (F_1)_{\geq 1}$ and $\operatorname{rad}_{\mathcal{C}} F_1(X)_i = F_1(X)_i$ for $i \geq 1$ and $\operatorname{rad}_{\mathcal{C}} F_1(X)_0 = (0)$ for all X. It follows that $F_1 \cap \operatorname{rad}_{\mathcal{C}} F_2 \subseteq (F_1)_{\geq 1} = \operatorname{rad}_{\mathcal{C}} F_1$. But in general $\operatorname{rad}_{\mathcal{C}} F_1 \subseteq F_1 \cap \operatorname{rad}_{\mathcal{C}} F_2$, hence $\operatorname{rad}_{\mathcal{C}} F_1 = F_1 \cap \operatorname{rad}_{\mathcal{C}} F_2$.

(d) Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be exact in $\mathrm{Gr}(\mathcal{C})_0$ with F_1 and F_2 linear. By (b) we infer that $0 \to F_1/\operatorname{rad}_{\mathcal{C}} F_1 \to F_2/\operatorname{rad}_{\mathcal{C}} F_2 \to F_3/\operatorname{rad}_{\mathcal{C}} F_3 \to 0$ is exact, and therefore $0 \to \Omega(F_1) \to \Omega(F_2) \to \Omega(F_3) \to 0$ is exact. Hence $\Omega(F_3)$ is generated in degree 0. By induction we infer that F_3 is linear.

(e) Given a linear functor F in $\mathrm{Gr}(\mathcal{C})$ there are exact sequences

$$0 \to \Omega^i(F) \to \Omega^i(F/\operatorname{rad}_{\mathcal{C}} F) \to \Omega^{i-1} \operatorname{rad}_{\mathcal{C}} F \to 0$$

for all $i \geq 1$. Similar arguments as for modules then show that there is an exact sequence

$$0 \to \operatorname{Ext}^*_{\mathrm{Mod}(\mathcal{C})}(\operatorname{rad}_{\mathcal{C}} F, -)[-1] \to \operatorname{Ext}^*_{\mathrm{Mod}(\mathcal{C})}(F/\operatorname{rad}_{\mathcal{C}} F, -) \to \operatorname{Ext}^*_{\mathrm{Mod}(\mathcal{C})}(F, -) \to 0,$$

where $F/\operatorname{rad}_{\mathcal{C}} F$ is a finite direct sum of simple functors. Since $\operatorname{rad}_{\mathcal{C}} F$ is shift of a linear functor, it follows by induction that $\phi(F)$ is a linear functor in $\mathrm{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$.

(f) The category $\mathcal{C}' = E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}$ is a positively graded Krull-Schmidt category, since $\mathcal{C}'$ is an additive K-category generated by the simple objects in $\mathrm{Gr}(\mathcal{C})$ and each simple object has a graded local endomorphism ring. It is locally finite, since all functors in $\mathcal{S}(\mathcal{C})$ are linear. The simple functors in $\operatorname{Gr}(\mathcal{C}')$ are given by $\widehat{S}_C = \operatorname{Ext}^*(S_C, -)/\operatorname{rad}\operatorname{Ext}^*(S_C, -) = \operatorname{Hom}(S_C, -)$ for some indecomposable object C in $\mathcal{C}$. It is easy to see that $\phi(\operatorname{Hom}_{\mathcal{C}}(-, C)) \simeq \widehat{S}_C$ for all indecomposable objects C in $\mathcal{C}$. By (e) we infer that $\mathcal{C}'$ is a Koszul K-category.

(g) Let $\mathcal{C}' = E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}$. Define $H\colon \mathcal{C} \to E(\mathcal{S}(\mathcal{C}'))^{\mathrm{op}}$ first on objects by letting for $C = \amalg_{i=1}^n C_i$ with C_i indecomposable in $\mathcal{C}$ for $i = 1, 2, \dots, n$

$$H(C) = \amalg_{i=1}^n \widehat{S}_{C_i}.$$

A morphism $f\colon D \to C$ between two indecomposable objects in degree i, that is, f is in $\operatorname{rad}^i_{\mathcal{C}}(D, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(D, C)$, gives rise to a morphism of functors $\operatorname{Hom}_{\mathcal{C}}(-, D) \to \operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C)$. Applying the functor ϕ to this morphism induces a morphism $\operatorname{Ext}^*(\operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C), -) \to \widehat{S}_D$. This we can interpret as an element $H(f)$ in $\operatorname{Ext}^i(\widehat{S}_C, \widehat{S}_D)$. This is linearly extended to any morphism in $\mathcal{C}$. This defines H on morphisms in $\mathcal{C}$, keeping in mind that our target category is $E(\mathcal{S}(\mathcal{C}'))^{\mathrm{op}}$. It is clear that H takes an identity morphism to an identity morphism.

Using the proof of (a) and (e), the simple functors $\widehat{S}_C = \operatorname{Hom}(S_C, -)$ in $\operatorname{Gr}(\mathcal{C}')$ have a minimal projective resolution given by

$$\begin{aligned}\cdots \to \operatorname{Ext}^*(\operatorname{rad}^2_{\mathcal{C}}(-, C)/\operatorname{rad}^3_{\mathcal{C}}(-, C), -)[-2] \to \\ \operatorname{Ext}^*(\operatorname{rad}_{\mathcal{C}}(-, C)/\operatorname{rad}^2_{\mathcal{C}}(-, C), -)[-1] \to \\ \operatorname{Ext}^*(\operatorname{Hom}(-, C)/\operatorname{rad}_{\mathcal{C}}(-, C), -) \to \widehat{S}_C \to 0\end{aligned}$$

We infer from this that

$$\begin{aligned}\operatorname{Ext}^i(\widehat{S}_C, \widehat{S}_D) &\simeq \operatorname{Hom}(\operatorname{Ext}^*(\operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C), -)[-i], \widehat{S}_D)\\ &\simeq \operatorname{Hom}(\operatorname{Hom}(\operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C), -)[-i], \widehat{S}_D)\\ &\simeq \widehat{S}_D(\operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C)[-i])\\ &= \operatorname{Hom}(S_D, \operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C)[-i])\\ &= \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-, D)/\operatorname{rad}_{\mathcal{C}}(-, D), \operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C)[-i])\\ &\simeq \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-, D), \operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C)[-i])\\ &\simeq \operatorname{rad}^i_{\mathcal{C}}(D, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(D, C)[-i]\\ &= \operatorname{Hom}_{\mathcal{C}}(D, C)_i\end{aligned}$$

Tracing through all these isomorphisms one can show that this is the morphism defining the functor H on morphisms in $\mathcal{C}$. Hence H is full and faithful.

Finally we need to show that H commutes with composition of maps. Let f be in $\operatorname{rad}^i_{\mathcal{C}}(D, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(D, C)$, and let g be in $\operatorname{rad}^i_{\mathcal{C}}(C, D')/\operatorname{rad}^{i+1}_{\mathcal{C}}(C, D')$. We want to show that $H(gf)$ is the Yoneda product of $H(f)$ and $H(g)$. In order to do so, we need to lift the morphism $\operatorname{Ext}^*(\operatorname{rad}^j_{\mathcal{C}}(-, C')/\operatorname{rad}^{j+1}_{\mathcal{C}}(-, C'), -) \to \operatorname{Ext}^*((-, C)/\operatorname{rad}_{\mathcal{C}}(-, C), -)$ induced by g, through a chain map to a morphism

$$\operatorname{Ext}^*(\operatorname{rad}^{i+j}_{\mathcal{C}}(-, C')/\operatorname{rad}^{i+j+1}_{\mathcal{C}}(-, C'), -) \to \operatorname{Ext}^*(\operatorname{rad}^i_{\mathcal{C}}(-, C)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C), -).$$

It is easily seen that this morphism also is induced by g, so that it follows that the Yoneda product of $H(f)$ and $H(g)$ is given by $H(gf)$. This shows that $H\colon \mathcal{C} \to E(\mathcal{S}(\mathcal{C}'))^{\mathrm{op}}$ is an equivalence of graded K-categories. □

In view of this result, for a Koszul K-category $\mathcal{C}$ we call the category $E(\mathcal{S}(\mathcal{C}))$ the *Koszul dual of* $\mathcal{C}$.

The prime example and the main application of the theory is related to the category of all additive functors from $(\operatorname{mod}\Lambda)^{\mathrm{op}}$ to vector spaces for a finite dimensional K-algebra Λ. This application will be presented in a forthcoming paper. As a further motivation for this paper and in particular for the next section we include a brief discussion of this application.

Let Λ be a finite dimensional algebra over a field K. Denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules, and by $\operatorname{Mod}(\operatorname{mod}\Lambda)$ the category of all additive functors $F\colon (\operatorname{mod}\Lambda)^{\mathrm{op}} \to \operatorname{Mod} K$. As we saw in Example 1.3 we can consider the associated graded category $\mathcal{A}_{\mathrm{gr}}(\operatorname{mod}\Lambda)$ of $\operatorname{Mod}(\operatorname{mod}\Lambda)$. It has the same objects as $\operatorname{mod}\Lambda$, while the morphisms are given by

$$\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\operatorname{mod}\Lambda)}(X,Y) = \amalg_{i\geq 0} \operatorname{rad}^i_{\mathcal{C}}(X,Y)/\operatorname{rad}^{i+1}_{\mathcal{C}}(X,Y).$$

The simple functors in $\operatorname{Mod}(\operatorname{mod}\Lambda)$ are of the form $S_C = (-,C)/\operatorname{rad}_{\mathcal{C}}(-,C)$ for some indecomposable C in $\operatorname{mod}\Lambda$. They have a minimal projective resolution given by

$$0 \to \operatorname{Hom}_\Lambda(-,\tau C) \to \operatorname{Hom}_\Lambda(-,E) \to \operatorname{Hom}_\Lambda(-,C) \to S_C \to 0$$

when C is non-projective with corresponding almost split sequence $0 \to \tau C \to E \to C \to 0$, and given by

$$0 \to \operatorname{Hom}_\Lambda(-,\mathfrak{r}P) \to \operatorname{Hom}_\Lambda(-,P) \to S_P \to 0$$

when P is an indecomposable projective Λ-module with radical $\mathfrak{r}P$.

Recalling the construction in subsection 1.4 we have a functor $G\colon \operatorname{Mod}(\operatorname{mod}\Lambda) \to \operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{mod}\Lambda))$ on objects and morphisms given for a functor F in $\operatorname{Mod}(\operatorname{mod}\Lambda)$ by

$$G(F) = \amalg_{i\geq 0} \operatorname{rad}^i F/\operatorname{rad}^{i+1} F.$$

The simple functors in $\operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{mod}\Lambda))$ are all of the form $G(S_C)$ for some C in $\operatorname{mod}\Lambda$, and they have a minimal projective resolutions given by

$$\begin{aligned} 0 \to \amalg_{i\geq 0} \operatorname{rad}^i(-,\tau C)/\operatorname{rad}^{i+1}(-,\tau C)[-2] &\to \\ \amalg_{i\geq 0} \operatorname{rad}^i(-,E)/\operatorname{rad}^{i+1}(-,E)[-1] &\to \\ \amalg_{i\geq 0} \operatorname{rad}^i(-,C)/\operatorname{rad}^{i+1}(-,C) &\to G(S_C) \to 0 \end{aligned}$$

in case C is non-projective, and given by

$$\begin{aligned} 0 \to \amalg_{i\geq 0} \operatorname{rad}^i(-,\mathfrak{r}P)/\operatorname{rad}^{i+1}(-,\mathfrak{r}P)[-1] &\to \\ \amalg_{i\geq 0} \operatorname{rad}^i(-,P)/\operatorname{rad}^{i+1}(-,P) &\to G(S_P) \to 0 \end{aligned}$$

when P is projective by [IT]. This shows that $\operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{mod}\Lambda))$ is a Koszul K-category. This illustrate the content of the next section. There we define weakly Koszul K-categories $\mathcal{C}$, which $\operatorname{Mod}(\operatorname{mod}\Lambda)$ is an example of. We study the relationship with $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ in general, and show that $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ always is a Koszul K-category.

The application is related to the representation theory of Λ and in particular to the Auslander-Reiten theory. The indecomposable modules in $\operatorname{mod}\Lambda$ is a disjoint union $\cup_{\sigma\in\Sigma} C_\sigma$, where each C_σ is an component of the Auslander-Reiten quiver of Λ. Furthermore, for X and Y in $\operatorname{add} C_\sigma$ and $\operatorname{add} C_{\sigma'}$ respectively with $\sigma \neq \sigma'$, we have

$$\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\operatorname{mod}\Lambda)}(X,Y) = \amalg_{i\geq 0} \operatorname{rad}^i(X,Y)/\operatorname{rad}^{i+1}(X,Y) = (0)$$

since any morphism from X to Y is in the infinite radical. Hence $\mathcal{A}_{\text{gr}}(\operatorname{mod}\Lambda)$ is a disjoint union $\cup_{\sigma\in\Sigma}\mathcal{A}_{\text{gr}}(\operatorname{add}\mathcal{C}_\sigma)$ of categories. As a consequence the category $\operatorname{Gr}(\mathcal{A}_{\text{gr}}(\operatorname{mod}\Lambda))$ is the product $\prod_{\sigma\in\Sigma}\operatorname{Gr}(\mathcal{A}_{\text{gr}}(\operatorname{add}\mathcal{C}_\sigma))$, which all are Koszul K-categories. We show in a forthcoming paper that properties of these categories reflect properties of the component $\mathcal{C}_\sigma$.

4. Weakly-Koszul categories

Here we introduce the non-graded analogue of Koszul categories similar as was done for algebras in [MVZ]. The primary example for us of a weakly Koszul category is the category of additive functors from $(\operatorname{mod}\Lambda)^{\text{op}}$ to vector spaces for a finite dimensional K-algebra Λ.

Let $\mathcal{C}$ be an additive Krull-Schmidt K-category. This in particular implies that $\operatorname{End}_{\mathcal{C}}(C)$ is semiperfect for all objects in C in $\mathcal{C}$. Consequently the category of finitely presented functors $\operatorname{mod}\mathcal{C}$ in $\operatorname{Mod}(\mathcal{C})$ has minimal projective presentations (compare Lemma 2.2). Also, similarly as in Lemma 2.4 this gives rise to one-to-one correspondence between the indecomposable objects in $\mathcal{C}$ and the simple objects in $\operatorname{Mod}(\mathcal{C})$, where an indecomposable object C in $\mathcal{C}$ gives rise to the simple object $S_C = \operatorname{Hom}_{\mathcal{C}}(-,C)/\operatorname{rad}_{\mathcal{C}}(-,C)$ in $\operatorname{Mod}(\mathcal{C})$.

We need a further finiteness condition to define weakly Koszul K-categories. A K-category $\mathcal{C}$ is called *locally radical finite* if $\operatorname{rad}^i_{\mathcal{C}}(A,B)/\operatorname{rad}^{i+1}_{\mathcal{C}}(A,B)$ is finite dimensional over K for all pairs of objects (A,B) in $\mathcal{C}$ and all $i\geq 0$. Throughout this section let $\mathcal{C}$ denote an additive Krull-Schmidt locally radical finite K-category. First we define a weakly Koszul K-category.

Definition 4.1. (i) A functor F in $\operatorname{Mod}(\mathcal{C})$ is *weakly Koszul* if F has a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to F \to 0$$

where P_i is a finitely generated projective object in $\operatorname{Mod}(\mathcal{C})$ for all $i\geq 0$ and $\operatorname{rad}^{i+1}_{\mathcal{C}}(P_j)\cap\Omega^{j+1}(F) = \operatorname{rad}^i_{\mathcal{C}}(\Omega^{j+1}(F))$ for all $j\geq 0$ and $i\geq 1$.

(ii) A category $\mathcal{C}$ is *weakly Koszul*, if $\mathcal{C}$ is an additive Krull-Schmidt locally radical finite K-category and every simple functor in $\operatorname{Mod}(\mathcal{C})$ is weakly Koszul.

Note that if a functor F is weakly Koszul, then $\Omega^i(F)$ is weakly Koszul for all $i\geq 0$.

For algebras the associated graded algebra, with respect to the Jacobson radical, of a weakly Koszul algebra is Koszul. We have the same for our weakly Koszul categories, as we show next. First recall from Example 1.3 and from subsection 1.4 how we from an additive K-category $\mathcal{C}$ constructed the associated positively graded K-category $\mathcal{A}_{\text{gr}}(\mathcal{C})$ and a functor $G\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Gr}(\mathcal{A}_{\text{gr}}(\mathcal{C}))$ on objects and morphisms, where for F in $\operatorname{Mod}(\mathcal{C})$ the functor G is given by $G(F) = \amalg_{i\geq 0}\operatorname{rad}^i_{\mathcal{C}} F/\operatorname{rad}^{i+1}_{\mathcal{C}} F$.

Proposition 4.2. *Suppose that $\mathcal{C}$ is an additive Krull-Schmidt locally radical finite K-category. Let F in $\operatorname{Mod}(\mathcal{C})$ be weakly Koszul. Then $G(F)$ in $\operatorname{Gr}(A_{\text{gr}}(\mathcal{C}))$ is Koszul. Moreover, if $\mathcal{C}$ is weakly Koszul, then $A_{\text{gr}}(\mathcal{C})$ is Koszul.*

Proof. The category $\mathcal{A}_{\text{gr}}(\mathcal{C})$ is clearly positively graded and locally finite, and it is Krull-Schmidt as the associated graded ring of a local ring is graded local.

Let

$$\cdots \to \operatorname{Hom}_{\mathcal{C}}(-,C_2) \to \operatorname{Hom}_{\mathcal{C}}(-,C_1) \to \operatorname{Hom}_{\mathcal{C}}(-,C_0) \xrightarrow{d} F \to 0$$

be a minimal projective resolution of F. By Lemma 1.7 (b) the map

$$d|_{\operatorname{rad}^i_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(-,C_0)} \colon \operatorname{rad}^i_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(-,C_0) \to \operatorname{rad}^i_{\mathcal{C}} F$$

is onto for all $i \geq 0$. The kernel of $d|_{\operatorname{rad}^i_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(-,C_0)}$ is $(\operatorname{rad}^i_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(-,C_0)) \cap \Omega(F)$, which is equal to $\operatorname{rad}^{i-1}_{\mathcal{C}} \Omega(F)$, since F is weakly Koszul. This gives rise to the exact sequences

$$0 \to \operatorname{rad}^i_{\mathcal{C}} \Omega(F) \to \operatorname{rad}^{i+1}_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(-,C_0) \to \operatorname{rad}^{i+1}_{\mathcal{C}} F \to 0$$

for all $i \geq 0$. Consequently there are exact sequences

$$0 \to \operatorname{rad}^{i-1}_{\mathcal{C}} \Omega(F)/\operatorname{rad}^i_{\mathcal{C}} \Omega(F) \to \operatorname{rad}^i_{\mathcal{C}}(-,C_0)/\operatorname{rad}^{i+1}_{\mathcal{C}}(-,C_0) \to \\ \operatorname{rad}^i_{\mathcal{C}} F/\operatorname{rad}^{i+1}_{\mathcal{C}} F \to 0$$

for all $i \geq 1$. Combining all these sequences we obtain the exact sequence

$$0 \to G(\Omega(F))[1] \to G(\operatorname{Hom}_{\mathcal{C}}(-,C_0)) \to G(F) \to 0$$

of functors. By induction we have an exact sequence

$$\cdots \to G(\operatorname{Hom}_{\mathcal{C}}(-,C_2))[2] \to G(\operatorname{Hom}_{\mathcal{C}}(-,C_0))[1] \to G(F) \to 0$$

We infer that $G(F)$ is a linear $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$-module.

Suppose that $\mathcal{C}$ is weakly Koszul. Since any simple $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$-module is of the form $G(S_C)$, it follows that $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is Koszul. □

In the following result and its corollary we show that weakly Koszul modules are closed under cokernels of monomorphisms and that the radical of a weakly Koszul modules again is weakly Koszul.

Proposition 4.3. *Suppose that $\mathcal{C}$ is weakly Koszul. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence in* $\operatorname{mod} \mathcal{C}$. *Assume that* $(\operatorname{rad}^k_{\mathcal{C}} F_2) \cap F_1 = \operatorname{rad}^k_{\mathcal{C}} F_1$ *for all* $k \geq 0$ *and that F_1 and F_2 are weakly Koszul. Then F_3 is weakly Koszul.*

Proof. By assumption the sequence

$$0 \to F_1/\operatorname{rad}_{\mathcal{C}} F_1 \to F_2/\operatorname{rad}_{\mathcal{C}} F_2 \to F_3/\operatorname{rad}_{\mathcal{C}} F_3 \to 0$$

is exact. Then we have an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega(F_1) & \to & \Omega(F_2) & \to & \Omega(F_3) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}_{\mathcal{C}}(-,C_1) & \to & \operatorname{Hom}_{\mathcal{C}}(-,C_2) & \to & \operatorname{Hom}_{\mathcal{C}}(-,C_3) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_1 & \to & F_2 & \to & F_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $\operatorname{Hom}_{\mathcal{C}}(-, C_i)/\operatorname{rad}_{\mathcal{C}}(-, C_i) \simeq F_i/\operatorname{rad} F_i$ for $i = 1, 2, 3$. Using the assumptions we have

$$\begin{aligned}(\operatorname{rad}^i_{\mathcal{C}} \Omega(F_2)) \cap \Omega(F_1) &= (\operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_2)) \cap \Omega(F_2) \cap \Omega(F_1)\\ &= \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_2) \cap \operatorname{Hom}_{\mathcal{C}}(-, C_1) \cap \Omega(F_1)\\ &= \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_1) \cap \Omega(F_1) = \operatorname{rad}^i_{\mathcal{C}} \Omega(F_1)\end{aligned}$$

for all $i \geq 1$. Hence we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{rad}^i_{\mathcal{C}} \Omega(F_1) & \to & \operatorname{rad}^i_{\mathcal{C}} \Omega(F_2) & \to & \operatorname{rad}^i_{\mathcal{C}} \Omega(F_3) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_1) & \to & \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_2) & \to & \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_3) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{rad}^{i+1}_{\mathcal{C}} F_1 & \to & \operatorname{rad}^{i+1}_{\mathcal{C}} F_2 & \to & \operatorname{rad}^{i+1}_{\mathcal{C}} F_3 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $\operatorname{rad}^i_{\mathcal{C}} \Omega(F_3) \subseteq \operatorname{rad}^{i+1}_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(-, C_3) \cap \Omega(F_3)$, the last column is a complex and the remaining columns and all the rows are exact. It follows by the Snake Lemma, that the rightmost column also is exact. Therefore

$$\operatorname{rad}^i_{\mathcal{C}} \Omega(F_3) = \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C_3) \cap \Omega(F_3).$$

By induction $\Omega(F_3)$ is weakly Koszul. Then F_3 is weakly Koszul, and this completes the proof. □

Corollary 4.4. *Let $\mathcal{C}$ be a weakly Koszul K-category. If F is weakly Koszul, then* $\operatorname{rad}_{\mathcal{C}} F$ *is weakly Koszul. In particular* $\operatorname{rad}_{\mathcal{C}} F$ *is finitely generated.*

Proof. The sequence, $0 \to \Omega(F) \to \operatorname{Hom}_{\mathcal{C}}(-, C) \to F \to 0$ induces an exact sequence $0 \to \Omega(F) \to \operatorname{rad}_{\mathcal{C}}(-, C) \to \operatorname{rad}_{\mathcal{C}} F \to 0$, satisfying the conditions of Proposition 4.3. Therefore $\operatorname{rad}_{\mathcal{C}} F$ is weakly Koszul. □

To further illuminate the relationship between linear objects and weakly Koszul objects for Koszul categories, we show that a weakly Koszul object generated in degree zero in a Koszul category is a linear object. This is true even more general. Recall that an object F in $\operatorname{Mod}(\mathcal{C})$ is *quasi-Koszul* if F has a finitely generated projective resolution

$$\cdots \to \operatorname{Hom}_{\mathcal{C}}(-, C_i) \to \cdots \to \operatorname{Hom}_{\mathcal{C}}(-, C_0) \to F \to 0$$

and that $\operatorname{rad}_{\mathcal{C}} \Omega^i(F) = \operatorname{rad}^2_{\mathcal{C}}(-, C_{i-1}) \cap \Omega^i(F)$ for all $i \geq 0$.

Lemma 4.5. *Let $\mathcal{C}$ be a Koszul algebra, and let F be in* $\operatorname{Gr}(\mathcal{C})$. *Assume that F is weakly Koszul (or weaker, quasi-Koszul) and generated in degree zero. Then F is linear.*

Proof. Since F is weakly Koszul (or quasi-Koszul), there is an exact sequence

$$0 \to \Omega(F) \to \operatorname{Hom}_{\mathcal{C}}(-, C) \to F \to 0$$

with $\operatorname{rad}_{\mathcal{C}} \Omega(F) = \operatorname{rad}^2_{\mathcal{C}}(-, C) \cap \Omega(F)$. By induction it is enough to prove that $\Omega(F)$ is generated in degree 1.

By the above observation we have the following exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(F) & \longrightarrow & \operatorname{rad}_{\mathcal{C}}(-, C) & \longrightarrow & \operatorname{rad}_{\mathcal{C}} F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega(F)/\operatorname{rad}_{\mathcal{C}} \Omega(F) & \longrightarrow & \operatorname{rad}_{\mathcal{C}}(-, C)/\operatorname{rad}^2_{\mathcal{C}}(-, C) & \longrightarrow & \operatorname{rad}_{\mathcal{C}} F/\operatorname{rad}^2_{\mathcal{C}} F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Hence $\Omega(F)/\operatorname{rad}_{\mathcal{C}} \Omega(F)$ is generated in degree 1. It follows that $\Omega(F)$ has a projective cover generated in degree 1. By induction F is linear. □

The next result indicates that for a weakly Koszul K-category $\mathcal{C}$ there is in addition to the naturally associated Koszul category $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$, a second associated category and possibly a Koszul category, namely $E(\mathcal{S}(\mathcal{C}))$. We shall later see that these categories are Koszul dual of each other.

Proposition 4.6. *Let $\mathcal{C}$ be a weakly Koszul K-category. Then the functor $\phi\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$ given by $\phi(F) = \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -)$ restricts to a functor from the category weakly Koszul modules $w\mathcal{K}(\mathcal{C})$ to the category of linear functors in $\operatorname{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$.*

Proof. It is clear that ϕ gives rise to a functor from $\operatorname{Mod}(\mathcal{C})$ to $\operatorname{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$. Next we show that $\operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -)$ is a linear $E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}$-module when F is a weakly Koszul $\mathcal{C}$-module.

Let F be weakly Koszul. Let $\operatorname{Hom}_{\mathcal{C}}(-, C) \to F$ be a projective cover. This gives rise to the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega(F) & \longrightarrow & \Omega(F/\operatorname{rad}_{\mathcal{C}} F) & \longrightarrow & \operatorname{rad}_{\mathcal{C}} F \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Hom}_{\mathcal{C}}(-, C) & = & \operatorname{Hom}_{\mathcal{C}}(-, C) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{rad}_{\mathcal{C}} F & \longrightarrow & F & \longrightarrow & F/\operatorname{rad}_{\mathcal{C}} F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where we have

$$\begin{aligned}
\operatorname{rad}^i_{\mathcal{C}} \Omega(F) &= \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C) \cap \Omega(F) \\
&= \operatorname{rad}^{i+1}_{\mathcal{C}}(-, C) \cap \Omega(F/\operatorname{rad}_{\mathcal{C}} F) \cap \Omega(F) \\
&= \operatorname{rad}^i_{\mathcal{C}} \Omega(F/\operatorname{rad}_{\mathcal{C}} F) \cap \Omega(F)
\end{aligned}$$

for all $i \geq 1$. It follows that there exist exact sequences

$$0 \to \Omega^i(F) \to \Omega^i(F/\operatorname{rad}_{\mathcal{C}} F) \to \Omega^{i-1}(\operatorname{rad}_{\mathcal{C}} F) \to 0$$

such that $\operatorname{rad}_{\mathcal{C}}^j \Omega^i(F/\operatorname{rad}_{\mathcal{C}} F) \cap \Omega^i(F) = \operatorname{rad}_{\mathcal{C}}^j \Omega^i(F)$ for all $j \geq 0$. This in turn gives the exact sequences

$$0 \to \operatorname{Hom}(\Omega^{i-1}(\operatorname{rad}_{\mathcal{C}} F), S_D) \to \operatorname{Hom}(\Omega^i(F/\operatorname{rad}_{\mathcal{C}} F), S_D) \to \operatorname{Hom}(\Omega^i(F), S_D) \to 0$$

for each simple functor S_D in $\operatorname{Mod}(\mathcal{C})$. Since $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(\Omega^j(G), S_D) \simeq \operatorname{Ext}^j_{\operatorname{Mod}(\mathcal{C})}(G, S_D)$, these sequences induce the exact sequence

$$0 \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(\operatorname{rad}_{\mathcal{C}} F, -)[-1] \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F/\operatorname{rad}_{\mathcal{C}} F, -) \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -) \to 0$$

of functors. By Corollary 4.4 $\operatorname{rad}_{\mathcal{C}} F$ is weakly Koszul, so that by induction there exists a long exact sequence

$$\cdots \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(\operatorname{rad}^2_{\mathcal{C}} F/\operatorname{rad}^3_{\mathcal{C}} F, -)[-2] \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(\operatorname{rad}_{\mathcal{C}} F/\operatorname{rad}^2_{\mathcal{C}} F, -)[-1] \to \\ \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F/\operatorname{rad}_{\mathcal{C}} F, -) \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -) \to 0$$

of functors. Note that by Corollary 4.4 the projectives occurring in this resolution are finitely generated. In particular $\phi(F) = \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -)$ is a linear module. □

The next result shows that for a weakly Koszul K-category $\mathcal{C}$ the two naturally associated categories $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ and $E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}$ are Koszul duals of each other.

Proposition 4.7. *Let $\mathcal{C}$ be a weakly Koszul K-category, and let $G\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ be given as before by $G(F) = \amalg_{i\geq 0} \operatorname{rad}^i_{\mathcal{C}} F/\operatorname{rad}^{i+1}_{\mathcal{C}} F$.*

(a) *For F in $w\mathcal{K}(\mathcal{C})$), then*

$$\operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -) \simeq \operatorname{Ext}^*_{\operatorname{Mod} \mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(F), -) \circ G$$

as objects in $\operatorname{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$.

(b) *The functor G induces an equivalence of the categories $E(\mathcal{S}(\mathcal{C}))$ and $E(\mathcal{S}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C})))$.*

Proof. (a) Let F be in $w\mathcal{K}(\mathcal{C})$, and let

$$\cdots \to \operatorname{Hom}_C(-, C_i) \to \operatorname{Hom}_{\mathcal{C}}(-, C_{i-1}) \to \cdots \to \operatorname{Hom}_{\mathcal{C}}(-, C_0) \to F \to 0$$

be a minimal projective resolution of F. By Proposition 4.2

$$\cdots \to G(\operatorname{Hom}_C(-, C_i)) \to G(\operatorname{Hom}_{\mathcal{C}}(-, C_{i-1})) \to \cdots \\ \to G(\operatorname{Hom}_{\mathcal{C}}(-, C_0)) \to G(F) \to 0$$

is a minimal projective resolution of $G(F)$. It follows from this that $G(\Omega^i(F)) \simeq \Omega^i(G(F))$.

For any pair of objects C and X in $\mathcal{C}$ with C indecomposable we have that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(\operatorname{Hom}_{\mathcal{C}}(-, X), S_C) \simeq \operatorname{Hom}_{\operatorname{Mod} \mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(\operatorname{Hom}_{\mathcal{C}}(-, X)), G(S_C))$, where the isomorphism is induced by G. It follows directly from this that for F in $w\mathcal{K}(\mathcal{C})$ we have an isomorphism of vector spaces

$$\operatorname{Ext}^i_{\operatorname{Mod}(\mathcal{C})}(F, S_C) \simeq \operatorname{Ext}^i_{\operatorname{Mod} \mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(F), G(S_C))$$

for any indecomposable object C in $\mathcal{C}$, where the isomorphism is induced by G.

Finally we need to see that the vector space isomorphism is a morphism of functors. Let $\theta\colon S_C \to S_D$ be a homogeneous morphism in $E(\mathcal{S}(\mathcal{C}))$, that is, let θ be some element in $\operatorname{Ext}^i_{\operatorname{Mod}(\mathcal{C})}(S_C, S_D)$ for some $i \geq 0$. Then

$$\operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, -)(\theta)\colon \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, S_C) \to \operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F, S_D)$$

is given by the Yoneda product with θ. And

$$\operatorname{Ext}^*_{\operatorname{Mod}\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(F),-)\circ G(\theta)\colon \operatorname{Ext}^*_{\operatorname{Mod}\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(F),G(S_C))\to \operatorname{Ext}^*_{\operatorname{Mod}\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(F),G(S_D))$$

is given by the Yoneda product with $G(\theta)$ in $\operatorname{Ext}^i_{\operatorname{Mod}\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(S_C),G(S_D))$. It is easy to see that vector space isomorphism commutes with these operations, so that $\operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F,-)$ and $\operatorname{Ext}^*_{\operatorname{Mod}\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(G(F),-)\circ G$ are isomorphic as objects in $\operatorname{Gr}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}})$.

(b) Since $\mathcal{C}$ is weakly Koszul, all simple functors are in $w\mathcal{K}(\mathcal{C})$. It then follows from (a) that $E(\mathcal{S}(\mathcal{C}))$ and $E(\mathcal{S}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C})))$ are equivalent categories, where the equivalence is induced by G. □

The above results can be summarized as having a commutative diagram

$$\begin{array}{ccc} w\mathcal{K}(\mathcal{C}) & \xrightarrow{\quad G \quad} & \mathcal{L}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C})) \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & \mathcal{L}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}) & \end{array}$$

where $\mathcal{L}(\mathcal{D})$ denotes the full subcategory consisting of the linear objects for a Koszul category $\mathcal{D}$, the functor $\phi(F)=\operatorname{Ext}^*_{\operatorname{Mod}(\mathcal{C})}(F,-)$ and the functor $\phi'(F')=\operatorname{Ext}^*_{\operatorname{Mod}\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(F',-)$. When $\mathcal{C}$ is weakly Koszul, then $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is Koszul by Proposition 4.2 and $E(\mathcal{S}(\mathcal{C}))$ is Koszul by Theorem 3.3. Furthermore, the functor ϕ' is a duality and $\operatorname{Gr}(E(\mathcal{S}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}))^{\mathrm{op}})$ is equivalent to $\operatorname{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$. In other words, we have the following.

Proposition 4.8. *Let $\mathcal{C}$ be a weakly Koszul K-category. Then the double Koszul dual $E(\mathcal{S}(E(\mathcal{S}(\mathcal{C}))^{\mathrm{op}}))^{\mathrm{op}}$ of the weakly Koszul K-category $\mathcal{C}$ is equivalent to the associated graded K-category $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$.*

We end this section with noting that when $\mathcal{C}$ is Koszul, then the associated graded category is equivalent to $\mathcal{C}$.

Proposition 4.9. *Let $\mathcal{C}$ be a Koszul K-category. Then $\mathcal{C}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ are equivalent graded K-categories.*

Proof. The objects in $\mathcal{C}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ are the same. The homomorphisms in $\mathcal{C}$ are graded vector spaces $\operatorname{Hom}_{\mathcal{C}}(C,D)=\amalg_{i\geq 0}\operatorname{Hom}_{\mathcal{C}}(C,D)_i$. By Lemma 3.2 we have $\operatorname{rad}^i_{\mathcal{C}}(C,D)=\amalg_{j\geq i}\operatorname{Hom}_{\mathcal{C}}(C,D)_j$ for any $i\geq 0$. In particular the natural morphism

$$\operatorname{Hom}_{\mathcal{C}}(C,D)_i\to \operatorname{rad}^i_{\mathcal{C}}(C,D)/\operatorname{rad}^{i+1}_{\mathcal{C}}(C,D)$$

is an isomorphism for all objects C and D in $\mathcal{C}$. This is easily seen to extend to an isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(C,D)\simeq \amalg_{i\geq 0}\operatorname{rad}^i_{\mathcal{C}}(C,D)/\operatorname{rad}^{i+1}_{\mathcal{C}}(C,D)$$

for all objects C and D in $\mathcal{C}$. Hence it follows that $\mathcal{C}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ are equivalent graded K-categories. □

5. Free tensor categories over a bimodule and Koszul duality

Let $\mathcal{C}$ be an additive Krull-Schmidt K-category, where $\operatorname{rad}_{\mathcal{C}}(-, C)$ is a finitely generated functor in $\operatorname{Mod}(\mathcal{C})$ for all C in $\mathcal{C}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is locally finite. In this section we define a free tensor category $T(\mathcal{C})$ associated to $\mathcal{C}$ over a bimodule, which is such that if $\mathcal{C}$ is weakly Koszul or Koszul, then $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is a quotient of $T(\mathcal{C})$ by an ideal $\mathcal{I}$ generated in degree 2. When $\mathcal{C}$ is weakly Koszul or Koszul, we show that the Koszul dual of $\mathcal{C}$ is given by $T(\mathcal{E}(\mathcal{C}))$ modulo the orthogonal relations of $\mathcal{I}_2$.

Let $\mathcal{C}$ be an additive Krull-Schmidt K-category, where $\operatorname{rad}_{\mathcal{C}}(-, C)$ is a finitely generated functor in $\operatorname{Mod}(\mathcal{C})$ for all C in $\mathcal{C}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is locally finite. Then we define the free tensor category $T(\mathcal{C})$ as follows. The objects in $T(\mathcal{C})$ are the same as the objects in $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$, and the morphisms in $T(\mathcal{C})$ are given as

$$\operatorname{Hom}_{T(\mathcal{C})}(X, Y) = \amalg_{i \geq 0} \operatorname{Hom}_{T(\mathcal{C})}(X, Y)_i,$$

where

$$\operatorname{Hom}_{T(\mathcal{C})}(X, Y)_i = \begin{cases} \operatorname{Hom}_{\mathcal{C}}(X, Y)/\operatorname{rad}_{\mathcal{C}}(X, Y), & \text{if } i = 0, \\ \amalg_{Y_1, \ldots, Y_{i-1} \in \mathcal{C}} \operatorname{rad}_{\mathcal{C}}(Y_{i-1}, Y)/\operatorname{rad}^2_{\mathcal{C}}(Y_{i-1}, Y) \otimes_{D_{Y_{i-1}}} & \\ \quad \cdots \otimes_{D_{Y_2}} \operatorname{rad}_{\mathcal{C}}(Y_1, Y_2)/\operatorname{rad}^2_{\mathcal{C}}(Y_1, Y_2) & \text{if } i \geq 1, \\ \quad \otimes_{D_{Y_1}} \operatorname{rad}_{\mathcal{C}}(X, Y_1)/\operatorname{rad}^2_{\mathcal{C}}(X, Y_1), & \end{cases}$$

viewed inside

$$\amalg_{Y_0, Y_1, \ldots, Y_{i-1}, Y_i \in \mathcal{C}} \operatorname{rad}_{\mathcal{C}}(Y_{i-1}, Y_i)/\operatorname{rad}^2_{\mathcal{C}}(Y_{i-1}, Y_i) \otimes_{D_{Y_{i-1}}} \cdots \otimes_{D_{Y_2}} \operatorname{rad}_{\mathcal{C}}(Y_1, Y_2)/\operatorname{rad}^2_{\mathcal{C}}(Y_1, Y_2) \otimes_{D_{Y_1}} \operatorname{rad}_{\mathcal{C}}(Y_0, Y_1)/\operatorname{rad}^2_{\mathcal{C}}(Y_0, Y_1),$$

where $D_Z = \operatorname{Hom}_{\mathcal{C}}(Z, Z)/\operatorname{rad}_{\mathcal{C}}(Z, Z)$ for Z in $\mathcal{C}$. The composition in $T(\mathcal{C})$ is given by

$$(f_m \otimes \cdots \otimes f_2 \otimes f_1) \circ (g_n \otimes \cdots \otimes g_2 \otimes g_1) = f_m \otimes \cdots \otimes f_2 \otimes f_1 \otimes g_n \otimes \cdots \otimes g_2 \otimes g_1.$$

With these definitions $T(\mathcal{C})$ is a locally finite graded K-category with radical given by

$$\operatorname{rad}_{T(\mathcal{C})}(X, Y) = \operatorname{Hom}_{T(\mathcal{C})}(X, Y)_{\geq 1}.$$

Furthermore, we have a full and dense functor $\phi \colon T(\mathcal{C}) \to \mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ given by

$$\phi(X) = X$$

for X in $T(\mathcal{C})$ and

$$\phi(f_m \otimes \cdots \otimes f_2 \otimes f_1) = \overline{f_m f_{m-1} \cdots f_2 f_1}$$

in $\operatorname{rad}^m_{\mathcal{C}}(X, Y)/\operatorname{rad}^{m+1}_{\mathcal{C}}(X, Y)$. This functor is full and dense, since $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is a graded category generated in the degrees 0 and 1. The kernel of the functor ϕ is an ideal $\mathcal{I}$ in $T(\mathcal{C})$ satisfying $\mathcal{I} \subseteq \operatorname{rad}^2_{T(\mathcal{C})}$. Hence, the categories $T(\mathcal{C})/\mathcal{I}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ are equivalent.

The next basic property of the construction of a free tensor category over a bimodule is the following. As for tensor algebras over a semisimple ring, the free tensor category $T(\mathcal{C})$ is hereditary, as we show next.

Lemma 5.1. *The category* $\operatorname{Gr}(T(\mathcal{C}))$ *is hereditary.*

Proof. The simple functors in $\mathrm{Gr}(T(\mathcal{C}))$ are given as $S_C = \mathrm{Hom}_{T(\mathcal{C})}(-,C)/\operatorname{rad}_{T(\mathcal{C})}(-,C)$ for an indecomposable object C in $T(\mathcal{C})$. Hence we have the exact sequence

$$0 \to \operatorname{rad}_{T(\mathcal{C})}(-,C) \to \mathrm{Hom}_{T(\mathcal{C})}(-,C) \to S_C \to 0.$$

We have that $\operatorname{rad}_{T(\mathcal{C})}(-,C)$ is given by

$$\begin{aligned}\amalg_{n\geq 1} \amalg_{Y_1,\ldots,Y_{n-1},Y_n\in\mathcal{C}} \operatorname{rad}_{\mathcal{C}}(Y_n,C)/\operatorname{rad}^2_{\mathcal{C}}(Y_n,C) \otimes_{D_{Y_n}} \\ \cdots \otimes_{D_{Y_2}} \operatorname{rad}_{\mathcal{C}}(Y_1,Y_2)/\operatorname{rad}^2_{\mathcal{C}}(Y_1,Y_2) \\ \otimes_{D_{Y_1}} \operatorname{rad}_{\mathcal{C}}(-,Y_1)/\operatorname{rad}^2_{\mathcal{C}}(-,Y_1),\end{aligned}$$

The functor $\operatorname{rad}_{T(\mathcal{C})}(-,C)/\operatorname{rad}^2_{T(\mathcal{C})}(-,C)$ in $\mathrm{Gr}(T(\mathcal{C}))$ is semisimple, so it is isomorphic to $\amalg_{i=1}^t S_{Z_i}$ for some indecomposable objects Z_i in $T(\mathcal{C})$ and

$$\amalg_{Y_n\in T(\mathcal{C})} \operatorname{rad}_{T(\mathcal{C})}(Y_n,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Y_n,C) \simeq \amalg_{i=1}^t \operatorname{rad}_{T(\mathcal{C})}(Z_i,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Z_i,C).$$

Hence we infer that

$$\operatorname{rad}_{T(\mathcal{C})}(-,C) = \amalg_{i=1}^t \operatorname{rad}_{T(\mathcal{C})}(Z_i,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} \mathrm{Hom}_{T(\mathcal{C})}(-,Z_i)$$

and conclude that $\operatorname{rad}_{T(\mathcal{C})}(-,C)$ is a projective object in $\mathrm{Gr}(T(\mathcal{C}))$. It follows from Theorem 2.9 that the category $\mathrm{Gr}(T(\mathcal{C}))$ is hereditary. □

As a consequence of the above considerations we obtain a generalization of the classical result for Koszul algebras that they are quadratic (see [BGS, Corollary 2.3.3] or [GM2, Corollary 7.3]).

Proposition 5.2. *Let $\mathcal{C}$ be a Koszul K-category. Then the category $\mathcal{C}$ is quadratic, that is, there exists an ideal $\mathcal{I}$ in $T(\mathcal{C})$ generated in degree 2 such that $\mathcal{C}$ and $T(\mathcal{C})/\mathcal{I}$ are equivalent categories.*

Proof. When $\mathcal{C}$ is a Koszul K-category, $\mathcal{C}$ and $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ are equivalent graded K-categories by Proposition 4.9. Suppose $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is equivalent to $T(\mathcal{C})/\mathcal{I}$. Then, for any simple functor S_C in $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$, we have the *Butler resolution* of S_C

$$\begin{aligned}0 \to \mathcal{I}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) \to \operatorname{rad}_{T(\mathcal{C})}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) \\ \to \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C) \to S_C \to 0,\end{aligned}$$

which is a start of a minimal projective resolution of S_C in $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$. Since $\mathrm{Gr}(T(\mathcal{C}))$ is hereditary, the functor $\operatorname{rad}_{T(\mathcal{C})}(-,C)$ is a projective functor and we have seen that it is isomorphic to

$$\amalg_{i=1}^t \operatorname{rad}_{T(\mathcal{C})}(Z_i,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} \mathrm{Hom}_{T(\mathcal{C})}(-,Z_i)$$

for some indecomposable objects Z_i in $T(\mathcal{C})$. This implies that we have the following

$$\begin{aligned}\operatorname{rad}_{T(\mathcal{C})}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) &\simeq \mathrm{Hom}_{T(\mathcal{C})}(-,-)/I(-,-) \otimes_{T(\mathcal{C})} \operatorname{rad}_{T(\mathcal{C})}(-,C) \\ &\simeq \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,-) \otimes_{T(\mathcal{C})} \operatorname{rad}_{T(\mathcal{C})}(-,C) \\ &\simeq \amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i),\end{aligned}$$

where $R^1_{T(\mathcal{C})}(Z_i,C) = \operatorname{rad}_{T(\mathcal{C})}(Z_i,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Z_i,C)$. Moreover we have that $\Omega^2(S_C) = \mathcal{I}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C)$ and

$$\operatorname{rad}_{\mathcal{C}} \Omega^2(S_C) = (\operatorname{rad}_{T(\mathcal{C})} \mathcal{I}(-,C) + \mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C))/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C).$$

From this we obtain that

$$\begin{aligned}\Omega^2(S_C)/\operatorname{rad}_{\mathcal{C}}\Omega^2(S_C) &\simeq \mathcal{I}(-,C)/(\operatorname{rad}_{T(\mathcal{C})}\mathcal{I}(-,C)+\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C))\\ &= \mathcal{I}_2(-,C)\end{aligned}$$

Since $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is a Koszul category, $\Omega^2(S_C)$ is generated in degree 2, which in turn implies that the ideal $\mathcal{I}$ is generated by $\mathcal{I}_2(-,-)$ as a two-sided ideal in $T(\mathcal{C})$. □

Given a Krull-Schmidt category $\mathcal{C}$, form the Ext-category $\mathcal{E}(\mathcal{C})$ as in Example 1.4 in Section 1. Then $\mathcal{E}(\mathcal{C})$ is a Krull-Schmidt category again, and we can form the free tensor category $T(\mathcal{E}(\mathcal{C}))$ of $\mathcal{E}(\mathcal{C})$ as above. The category $\mathcal{E}(\mathcal{C})$ has as indecomposable objects the simple functors S_C for C indecomposable in $\mathcal{C}$, and morphisms are given by

$$\operatorname{Hom}_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)=\amalg_{i\geq 0}\operatorname{Ext}^i_{\operatorname{Mod}(\mathcal{C})}(S_X,S_Y).$$

The radical in $\mathcal{E}(\mathcal{C})$ is given by $\operatorname{rad}_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)=\amalg_{i\geq 1}\operatorname{Ext}^i_{\operatorname{Mod}(\mathcal{C})}(S_X,S_Y)$, so that

$$\operatorname{rad}_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)/\operatorname{rad}^2_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)=\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(S_X,S_Y).$$

Since $\mathcal{E}(\mathcal{C})$ is a graded category with $\operatorname{Hom}_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)_i = \operatorname{rad}^i_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)/\operatorname{rad}^{i+1}_{\mathcal{E}(\mathcal{C})}(S_X,S_Y)$, the associated graded category of $\mathcal{E}(\mathcal{C})$ is the same as $\mathcal{E}(\mathcal{C})$. Therefore, as pointed out above, there is a full and dense functor $\phi\colon T(\mathcal{E}(\mathcal{C}))\to\mathcal{E}(\mathcal{C})$ with a kernel $\mathcal{I}'$ contained in $\operatorname{rad}^2_{T(\mathcal{E}(\mathcal{C}))}$. Furthermore, $T(\mathcal{E}(\mathcal{C}))/\mathcal{I}'$ and $\mathcal{E}(\mathcal{C})$ are equivalent graded categories. Our next aim is to show that the ideal $\mathcal{I}'$ of relations in $T(\mathcal{E}(\mathcal{C}))$ is obtained as "the orthogonal relations" of the relations for the presentation of $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ as a quotient $T(\mathcal{C})/\mathcal{I}$, whenever $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is a Koszul category. See [BGS, Corollary 2.3.3] or [GM1, Corollary 7.3] for corresponding result for algebras.

To justify the claim in the proposition below observe that for C and Z indecomposable in $\mathcal{C}$ the functor $U=D(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C)/\operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C))$ is isomorphic to

$$\begin{aligned}U &\simeq D(\operatorname{Hom}_{\mathcal{C}}(-,-)/\operatorname{rad}_{\mathcal{C}}(-,-)\otimes_{\mathcal{C}}\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C))\\ &\simeq \operatorname{Hom}(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C),D(\operatorname{Hom}_{\mathcal{C}}(-,-)/\operatorname{rad}_{\mathcal{C}}(-,-)))\\ &\simeq \operatorname{Hom}(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C),\operatorname{Hom}_{\mathcal{C}}(-,-)/\operatorname{rad}_{\mathcal{C}}(-,-))\\ &\simeq \operatorname{Hom}(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C)/\operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C),\operatorname{Hom}_{\mathcal{C}}(-,-)/\operatorname{rad}_{\mathcal{C}}(-,-)),\end{aligned}$$

and evaluating this isomorphism at Z we get

$$D(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(Z,C)/\operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(Z,C))\simeq\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(S_C,S_Z).$$

From this we infer that

$$D(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z)/\operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z))\otimes_{D_Z}D(\operatorname{rad}_{T(\mathcal{C})}(Z,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Z,C))$$

can be viewed as

$$\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(S_Z,S_{(-)})\otimes_{D_Z}\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(S_C,S_Z)$$

that is, contained in $\operatorname{rad}^2_{T(E(\mathcal{C}))}(S_C,-)$.

Proposition 5.3. *Let $\mathcal{C}$ be a Krull-Schmidt category, and assume that $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is Koszul. If $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is equivalent to $T(\mathcal{C})/\mathcal{I}$, then $\mathcal{E}(\mathcal{C})$ is equivalent to $T(\mathcal{E}(\mathcal{C}))/\mathcal{I}'$, where $\mathcal{I}' = \langle \mathcal{I}_2^\perp \rangle$ with $\mathcal{I}_2^\perp(-,C)$ being given as the kernel of the natural morphism*

$$\begin{array}{c}\amalg_{i=1}^t D(\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i)/\operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i)) \otimes_{D_{Z_i}} D(\operatorname{rad}_{T(\mathcal{C})}(Z_i,C)/\operatorname{rad}^2_{T(\mathcal{C})}(Z_i,C)) \\ \downarrow \\ D(\mathcal{I}_2(-,C)),\end{array}$$

for each indecomposable object C in $\mathcal{C}$ and some indecomposable objects Z_i in $\mathcal{C}$ for $i = 1, 2, \ldots, t$. Here $D_Z = \operatorname{End}_{\mathcal{C}}(Z)/\operatorname{rad}_{\mathcal{C}}(Z,Z)$.

Proof. There are long formulas in the proof of this result. So to save space we use the following short versions of the following spaces when convenient. Let $R^i_?(-,-) = \operatorname{rad}^i_?(-,-)/\operatorname{rad}^{i+1}_?(-,-)$, where ? denotes the category where the radical is given. Morphism spaces $\operatorname{Hom}_?(-,-)$ are shorten to $_?(-,-)$.

Suppose $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is equivalent to $T(\mathcal{C})/\mathcal{I}$. We want to describe the relations $\mathcal{I}'$ in $\mathcal{T}(\mathcal{E}(\mathcal{C}))$ such that $\mathcal{E}(\mathcal{C})$ is equivalent to $\mathcal{T}(\mathcal{E}(\mathcal{C}))/\mathcal{I}'$. To this end consider the following commutative diagram. The first column is obtained for an indecomposable object C in $\mathcal{C}$ as in the proof of Proposition 5.2.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{I}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) & \longrightarrow & \amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} \operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i) \\
\downarrow & & \downarrow \\
\amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} {}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i) & = & \amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} {}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i) \\
\downarrow & & \downarrow \\
\operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C) & \longrightarrow & R^1_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the middle horizontal morphism is the identity, the lower horizontal morphism is the natural projection, and the upper horizontal morphism is the induced inclusion. The Snake Lemma then gives rise to the exact sequence

$$\begin{aligned} 0 \to \mathcal{I}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) \to & \\ \amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} \operatorname{rad}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i) \to & \\ \operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C) \to 0 & \end{aligned}$$

Since $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is Koszul, we have that

$$\operatorname{rad}^2(\operatorname{rad}_{T(\mathcal{C})}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C)) \cap \Omega^2(S_C) = \operatorname{rad}\Omega^2(S_C),$$

hence

$$\begin{aligned}\Big(\amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} \operatorname{rad}^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i)\Big) \cap \mathcal{I}(-,C)/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) & \\ \simeq (\operatorname{rad}_{T(\mathcal{C})}\mathcal{I}(-,C) + \mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C))/\mathcal{I}\operatorname{rad}_{T(\mathcal{C})}(-,C) & \end{aligned}$$

From this we obtain the exact sequence

$$0 \to \mathcal{I}_2(-,C) \to \amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} R^1_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i) \to R^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C) \to 0$$

This is a sequence of semisimple functors, so that applying $\mathrm{Hom}(-,S_X)$ we get the following exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(R^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,C),,S_X) \to \\ \mathrm{Hom}(\Big(\amalg_{i=1}^t R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} R^1_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(-,Z_i)\Big), S_X) \to \\ \mathrm{Hom}(\mathcal{I}_2(-,C),S_X) \to 0 \end{aligned}$$

Using Proposition 2.6 (c), the fact that $D(S_X) \simeq S_X^{\mathrm{op}}$ and Lemma 1.9, this sequence is isomorphic to the sequence

$$(\ddagger) \quad \begin{aligned} 0 \to D(R^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(X,C)) \to \\ \amalg_{i=1}^t D(R^1_{T(\mathcal{C})}(Z_i,C) \otimes_{D_{Z_i}} R^1_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(X,Z_i)) \to \\ D(\mathcal{I}_2(X,C)) \to 0. \end{aligned}$$

The middle term of this sequence is isomorphic to

$$D(R^1_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(X,Z_i)) \otimes_{D_{Z_i}} D(R^1_{T(\mathcal{C})}(Z_i,C)).$$

As we observed before this proposition, we have

$$D(R^1_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(Z_i,C)) \simeq \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(S_C,S_{Z_i})$$

for all $i = 1,2,\ldots,t$ and furthermore

$$\begin{aligned} \mathrm{Ext}^2_{\mathrm{Mod}(\mathcal{C})}(S_C,S_X) &\simeq \mathrm{Hom}(\mathcal{I}(-,C)/\mathcal{I}\,\mathrm{rad}_{T(\mathcal{C})}(-,C),S_X) \\ &\simeq \mathrm{Hom}(\mathcal{I}_2(-,C),S_X) \\ &\simeq D(\mathcal{I}_2(X,C)). \end{aligned}$$

With these identifications we can rewrite the sequence (‡) as

$$\begin{aligned} 0 \to D(R^2_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(X,C)) \to \\ \amalg_{i=1}^t \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(S_{Z_i},S_X) \otimes_{D_{Z_i}} \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(S_C,S_{Z_i}) \to \\ \mathrm{Ext}^2_{\mathrm{Mod}(\mathcal{C})}(S_C,S_X) \to 0, \end{aligned}$$

and the non-zero epimorphism in the exact sequence (‡) corresponds to the Yoneda product in the latter exact sequence. This shows that $\mathcal{I}_2^{\perp} = \mathcal{I}'_2$, where $\mathcal{I}'$ is the kernel of the functor $T(\mathcal{E}(\mathcal{C})) \to \mathcal{E}(\mathcal{C})$. Since $\mathcal{E}(\mathcal{C})$ is Koszul, the ideal $\mathcal{I}'$ is generated in degree 2, we have $\mathcal{I}' = \langle \mathcal{I}'_2 \rangle = \langle \mathcal{I}_2^{\perp} \rangle$. □

References

[AF] Anderson, F., Fuller, K., *Rings and categories of modules*, Graduate Texts in Mathematics, Vol. 13. Springer-Verlag, New-York-Heidelberg, 1992.

[A] Auslander, M., *Representation theory of Artin algebras* I, Comm. Algebra 1 (1974), 177–268.

[AR1] Auslander, M., Reiten, I., *Representation theory of artin algebras* III. *Almost split sequences*, Communications in Algebra **3** (1975), 239–294.

[ARS] Auslander, M., Reiten, I., Smalø, S. O.; *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, (1995).

[BCS] Bautista, R., Colavita, L., Salmerón, L., *On adjoint functors in representation theory*, Representations of algebras (Puebla, 1980), pp. 9–25, Lecture Notes in Math. 903.

[BGS] Beilinson, A., Ginzburg, V., Soergel, W., *Koszul duality patterns in representation theory*, Journal of the American Mathematical Society, **9**, no. 2., (1996), 473–527.

[B] Berger, R., *Koszulity for non-quadratic algebras*, J. Algebra 239 (2001), no. 2, 705–734.

[CS] Cassidy, T., Shelton, B., *Generalizing the notion of Koszul algebra*, Math. Z. 260 (2008), no. 1, 93–114.

[GM1] Green, E. L., Martínez-Villa, R., *Koszul and Yoneda algebras*, Representation theory of algebras (Cocoyoc, 1994), 247–297, CMS Conf. Proc., **18**, Amer. Math. Soc., Providence, RI, (1996).

[GM2] Green, E. L., Martínez-Villa, R., *Koszul and Yoneda algebras II*, Algebras and modules, II (Geiranger, 1996), 227–244, CMS Conf. Proc., **24**, Amer. Math. Soc., Providence, RI, (1998).

[IT] Igusa, K., Todorov, G., *Radical layers of representable functors*, J. Algebra 89 (1984), no. 1, 105–147.

[MVZ] Martínez-Villa, R., Zacharia, D., *Approximations with modules having linear resolutions*, J. Algebra 266 (2003), no. 2, 671–697.

[MOS] Mazorchuk, V., Ovsienko, S., Stroppel, C., *Quadratic duals, Koszul dual functors and applications*, to appear in Trans. Amer. Math. Soc.

[M] Mitchel, B., *Rings with several objects*, Advances in Math. 8 (1972), 1–161.

[NvO] Năstăsescu, C., Van Oystaeyen, F., *Graded and Filtered Rings and Modules*, Lecture Notes in Mathematics, 758. Springer, Berlin, 1979.

[P] Priddy, S. B., *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970) 39–60.

Roberto Martínez-Villa, Instituto de Matemáticas, Universidad Nacional Autonoma de Mexico, Campus Morelia, Apartado Postal 27-3 (Xangari), C.P. 58089, Morelia, Michoacán, Mexico

E-mail address: mvilla@matmor.unam.mx

Øyvind Solberg, Institutt for matematiske fag, NTNU, N–7034 Trondheim, Norway

E-mail address: oyvinso@math.ntnu.no