Infinite Dimensional Representations of Canonical Algebras

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Dedicated to Vlastimil Dlab on the occasion of his 70th birthday.

ABSTRACT. The aim of this paper is to extend the structure theory for infinitely generated modules over tame hereditary algebras to the more general case of modules over concealed canonical algebras. Using tilting, we may assume that we deal with canonical algebras. The investigation is centered around the generic and the Prüfer modules, and how other modules are determined by these modules.

Introduction.

Let Λ be a finite dimensional algebra over a field k. Traditionally one mainly has considered the Λ-modules which are finitely generated. An early exception were papers by several authors dealing with modules over the Kronecker algebra. This was generalized by the second author [R1] to the case of a tame hereditary algebra Λ, an investigation which was based on the explicit knowledge of the finitely generated modules as presented in his joint work [DR] with Dlab. As it turned out, there are striking similarities between the category of all Λ-modules and the category of all abelian groups (or the category of all R-modules, where R is a Dedekind ring with infinitely many prime ideals). In particular, the so called generic module and the Prüfer modules play an important role, as they correspond to the indecomposable injective R-modules.

The aim of the present paper is to show that the core results of these old investigations only depend on the existence of a sincere stable separating tubular family, and not at all on the representation type of the algebra. Hence, in view of the characterization due to Lenzing and de la Peña in [LP], the natural setting is the class of concealed canonical algebras, which contains the class of tame hereditary algebras, but also many others. An important special class is the better known class of the canonical algebras, and actually, it is sufficient to deal with this class (with the tubular family considered to be given by the modules of defect zero), since it is easy to extend the results to the general class of concealed canonical algebras via a tilting procedure. Note that such a canonical algebra may be domestic, or non-domestic tame, or wild, but always we will obtain splitting results which are similar to those known for tame hereditary algebras. In the special case of a canonical algebra which is non-domestic tame (thus for all tubular algebras), there are countably many tubular families: any such family gives rise to corresponding split torsion pairs.

The key results are, as for tame hereditary algebras, centered around the explicit description of some modules, which are defined in a similar way as in the tame hereditary case: the generic module and the Prüfer modules. In some sense, all other modules are determined from these, via maps between them. In order to be more explicit, we need to introduce some notation and terminology.

Given a ring R, we consider usually left R-modules and call them just modules or also representations of R. The category of all R-modules will be denoted by $\operatorname{Mod} R$, the full subcategory of the finitely presented ones by $\operatorname{mod} R$. For any class $\mathcal{X}$ of R-modules, we denote by $\operatorname{add}\mathcal{X}$ its additive closure: it is the smallest full subcategory closed under isomorphisms, direct summands and finite direct sums. Similarly, $\operatorname{Add}\mathcal{X}$ is the smallest full subcategory closed under isomorphisms, direct summands and arbitrary direct sums, whereas $\operatorname{Prod}\mathcal{X}$ is the smallest full subcategory closed under isomorphisms, direct summands and arbitrary products. Given R-modules X, Y, we usually write $\operatorname{Hom}(X,Y)$ or $\operatorname{Ext}^1(X,Y)$ instead of $\operatorname{Hom}_R(X,Y)$ or $\operatorname{Ext}^1_R(X,Y)$. When dealing with classes $\mathcal{X}, \mathcal{Y}$ (or full subcategories) of R-modules, we write $\operatorname{Hom}(\mathcal{X},\mathcal{Y}) = 0$ in order to assert that $\operatorname{Hom}(X,Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and similarly for Ext^1. For any R-module M, we

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denote by $\operatorname{pd} M$ its projective dimension and by $\operatorname{id} M$ its injective dimension. When dealing with module classes (or full subcategories), two different types of notations will be used: the module classes denoted by script letters such as $\mathcal{X}, \mathcal{C}, \mathcal{Q}$ (or also ω and ω_0) will usually be closed under direct sums (often even infinite direct sums); in contrast, when dealing with an artin algebra, we will use small boldface letters such as $\mathbf{x}, \mathbf{p}, \mathbf{t}$ in order to denote classes consisting only of indecomposable modules of finite length.

Let Λ be an artin algebra, and assume that there exist classes $\mathbf{p}, \mathbf{t}, \mathbf{q}$ in $\operatorname{mod} \Lambda$ (a "trisection") with the following properties: $\mathbf{t}$ is a sincere stable separating tubular family and it separates $\mathbf{p}$ from $\mathbf{q}$ (see section 2). Note that an indecomposable Λ-module of finite length belongs to $\mathbf{p}$ or $\mathbf{t}$ if and only if it is cogenerated by $\mathbf{t}$. A crucial result of this paper will be the following: *Any (not necessarily finite dimensional) Λ-module M has a direct sum decomposition $M = M_0 \oplus M_1$, where M_0 is cogenerated by the direct limit closure $\mathcal{T}$ of $\mathbf{t}$ and M_1 is generated by $\mathbf{t}$;* in addition, we can assume that $\operatorname{Hom}(M_1, \mathcal{T}) = 0$, and then $\operatorname{Hom}(M_1, M_0) = 0$. We consider the class $\mathcal{C}$ of modules cogenerated by $\mathcal{T}$, thus a finite length module belongs to $\mathcal{C}$ if and only if it is cogenerated by $\mathbf{t}$. It follows that $\mathcal{C}$ is the torsionfree class of a split torsion pair in $\operatorname{Mod} \Lambda$. We will investigate in detail **all** the torsion pairs in $\operatorname{Mod} \Lambda$ with the property that a finite length module is torsionfree if and only if it is cogenerated by $\mathbf{t}$. As we will see, all these torsion pairs split. Now $\mathcal{C}$ is the largest possible torsionfree class of this kind. Also the largest possible torsion class $\mathcal{D}$ of such a torsion pair can be described easily: it is the class of all modules M with $\operatorname{Hom}(M, \mathbf{t}) = 0$. The category $\omega = \mathcal{C} \cap \mathcal{D}$ turns out to be of central importance. The main results of the paper can be expressed in terms of these categories $\mathcal{C}, \mathcal{D}$ and ω. The objects in ω can be completely classified: any object in ω is a direct sum of copies of the generic module G and of Prüfer modules. The class $\mathcal{C}$ is determined by ω as $\{C \mid \operatorname{Ext}^1(C, \omega) = 0\}$, and $\mathcal{D}$ is determined by ω as $\{D \mid \operatorname{Ext}^1(\omega, D) = 0\}$. Further there are exact sequences $0 \to C \to V \to V' \to 0$ with $V \in \omega$ and V' a direct sum of Prüfer modules, for C in $\mathcal{C}$, and $0 \to V' \to V \to D \to 0$, with $V' \in \operatorname{Add} G$ and $V \in \omega$, for D in $\mathcal{D}$. As a consequence, the modules in $\mathcal{C}$ can be characterized as the kernels of maps in ω, and similarly, the modules in $\mathcal{D}$ can be characterized as the cokernels of maps in ω. Thus any Λ-module M is obtained as a direct sum $M = M_0 \oplus M_1$, where M_0 is the kernel and M_1 the cokernel of suitable maps in ω: in this way, the category $\operatorname{Mod} \Lambda$ can be completely described in terms of ω.

When dealing with finite dimensional algebras, one may argue that it is the category of finite dimensional representations which is the primary object of interest. However, the relevance of infinite dimensional representations has been stressed at various occasions [R1,R8] and here we encounter again such a situation: it is the subcategory ω which plays the decisive role when studying the cut between $\mathbf{t}$ and $\mathbf{q}$ in $\operatorname{mod} \Lambda$, and as we have noted, ω does not contain a single non-zero finite-dimensional representation. We will denote by W the direct sum of all the indecomposables in ω, one from each isomorphism class. This module W allows to reconstruct ω (as $\operatorname{Add} W$), thus the whole category $\operatorname{Mod} \Lambda$. Clearly, W is a very valuable module! This can be phrased quite well in terms of tilting and cotilting theory. We will use the denomination inf-tilting and inf-cotilting when we deal with the general concepts without the restriction of dealing with finite dimensional modules, see section 11. Our results show that W is both an inf-tilting module of projective dimension one and an inf-cotilting module of injective dimension one (see [BS] for a different and independent approach to this for cotilting modules). It is also possible to perform tilting with respect to torsion pairs as in [HRS] to construct new hereditary categories where the objects in ω become enough projective or enough injective objects.

If we consider the special case of a tame hereditary algebra, most of the results presented here have been established in [R1], but for Proposition 4 (the classification of torsion pairs) we should refer to unpublished information by Assem and Kerner. It should be noted that Theorem 5 (the existence of the right ω-approximations) seems to be new even in this case. The proof is inspired by [AB].

We will follow quite closely the presentation given in [R1], using only the structure theory for finite dimensional representations, and not taking into account the large amount of information on infinite dimensional representations obtained in the meantime by various authors. In particular, we will construct the relevant "generic" module G from scratch. At the end we indicate a different approach using the available results. The reader should not mind that the text itself avoids all more sophisticated considerations, but this stubborn approach should make it quite transparent to trace in which way the structure of the category of finite dimensional representations determines that of all the representations.

The paper is organized as follows. In section 1 we give a criterion for a torsion pair to be split. In section 2 we recall basic properties of the central algebras in this paper; the canonical and concealed canonical algebras. In the next six sections (3-8) we deal with a canonical algebra Λ and the canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ of $\operatorname{mod}\Lambda$. We investigate the two extremal torsion pairs of $\operatorname{Mod}\Lambda$ mentioned above in section 3, and give the structure of the Prüfer modules. The left ω-approximation sequence is established in section 4, and the basic splitting result $\operatorname{Ext}^1(\mathcal{C}, \mathcal{D}) = 0$ is given in section 5. The structure of ω is presented in section 6, and the existence of the right ω-approximation sequences is deduced in section 7. The structure of ω is investigated more closely in section 8. In section 9 we outline that all these considerations are valid for any sincere stable separating tubular family, thus for any concealed canonical algebra. Of course, we use tilting functors in order to relate an arbitrary sincere stable separating tubular family with the canonical trisection of a canonical algebra. Connections with tilting theory are discussed in sections 10 and 11. In section 12 we provide further comments and indicate another approach to the results in this paper. A tubular algebra has a lot of sincere stable separating tubular families and as we will see in section 13, our considerations allow to attach a non-negative real number as a "slope" to any indecomposable infinite dimensional module.

The investigations presented here have for the most part been completed during a stay of the first author at Bielefeld in 1998 and she would like to thank the second author for his hospitality; unfortunately, the write-up of the results has been delayed for quite a while.

1. Torsion pairs.

The investigations presented in this paper are centered around various torsion pairs (or, as they are sometimes called, torsion "theories"). We are going to recall the relevant definitions and main properties, and we provide a general method for producing split torsion pairs.

Let R be a ring. For any class $\mathcal{Z}$ of R-modules, we denote by $l(\mathcal{Z})$ the class of all R-modules M with $\operatorname{Hom}(M, \mathcal{Z}) = 0$, and similarly, $r(\mathcal{Z})$ is the class of all R-modules M with $\operatorname{Hom}(\mathcal{Z}, M) = 0$ (let us stress that our notation $r(-)$ and $l(-)$ always refers to the complete category $\operatorname{Mod} R$ as ambient category, the only exception being section 10 where the ambient category is an arbitrary abelian category).

Lemma 1. *Let $\mathcal{F}, \mathcal{G}$ be classes of R-modules. The following conditions are equivalent:*
(i) $l(\mathcal{F}) = \mathcal{G}$ *and* $r(\mathcal{G}) = \mathcal{F}$.
(ii) $\operatorname{Hom}(\mathcal{G}, \mathcal{F}) = 0$ *and any module M has a submodule $M' \in \mathcal{G}$ such that $M/M' \in \mathcal{F}$.*

If these conditions are satisfied, the pair $(\mathcal{F}, \mathcal{G})$ is said to be a *torsion pair* with *torsionfree class* $\mathcal{F}$, and *torsion class* $\mathcal{G}$. The modules in $\mathcal{F}$ are called the *torsionfree*, those in $\mathcal{G}$ the *torsion* modules. It is straightforward to see that the submodule M' given in (ii) is uniquely determined by M (provided the torsion pair $(\mathcal{F}, \mathcal{G})$ is fixed).

Proof of the equivalence. (i) $\Longrightarrow$ (ii): Only the last assertion needs a proof. Thus, let M be an arbitrary R-module. Let M' be the sum of images of maps from a module in $\mathcal{G}$ to M. Since $\mathcal{G} = l(\mathcal{F})$, $\mathcal{G}$ is closed under factors and arbitrary sums, so that M' is in $\mathcal{G}$. Since $\mathcal{G}$ is also closed under extensions, we see that $\operatorname{Hom}(\mathcal{G}, M/M') = 0$, so that M/M' is in $\mathcal{F}$.

(ii) $\Longrightarrow$ (i). We show that $l(\mathcal{F}) \subseteq \mathcal{G}$. Let N belong to $l(\mathcal{F})$. According to (ii) there exists a submodule N' of N which belongs to $\mathcal{G}$ such that N/N' belongs to $\mathcal{F}$. But the assumption that $N \in l(\mathcal{F})$ implies that the projection map $N \to N/N'$ is the zero map, thus $N/N' = 0$ and therefore $N = N' \in \mathcal{G}$. Similarly, one shows that $r(\mathcal{G}) \subseteq \mathcal{F}$.

Some readers may wonder about the not quite usual sequence of naming the *torsionfree class* $\mathcal{F}$ first and the *torsion class* $\mathcal{G}$ second — this corresponds to the vision of drawing arrows and thus non-trivial maps from left to right (whenever possible): there usually will be many non-zero maps from the objects in $\mathcal{F}$ to the objects in $\mathcal{G}$ (but, by definition, none in the other direction), thus $\mathcal{F}$ may be considered as "situated to the left" of $\mathcal{G}$.

The torsion pair $(\mathcal{F}, \mathcal{G})$ is said to be *split* provided $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$, or, equivalently, provided every module is the direct sum of a module in $\mathcal{F}$ and a module in $\mathcal{G}$.

Any class $\mathcal{Z}$ of R-modules determines two torsion pairs, namely

$$(r(\mathcal{Z}), lr(\mathcal{Z})) \quad \text{and} \quad (rl(\mathcal{Z}), l(\mathcal{Z})).$$

Clearly, $lr(\mathcal{Z})$ is the smallest possible torsion class containing $\mathcal{Z}$, whereas $rl(\mathcal{Z})$ is the smallest possible torsionfree class containing $\mathcal{Z}$.

Lemma 2. *Let $\mathcal{Z}$ be any class of R-modules. Then an R-module M belongs to $lr(\mathcal{Z})$ if and only if the only submodule U of M with $M/U \in r(\mathcal{Z})$ is $U = M$. Similarly, an R-module M belongs to $rl(\mathcal{Z})$ if and only if the only submodule U of M with $U \in l(\mathcal{Z})$ is $U = 0$.*

Proof: If M belongs to $lr(\mathcal{Z})$ and U is a submodule of M with $M/U \in r(\mathcal{Z})$, then the projection map $M \to M/U$ has to be the zero map, thus $U = M$. Conversely, assume that M is an R-module such that the only submodule U with $M/U \in r(\mathcal{Z})$ is $U = M$. Since $(r(\mathcal{Z}), lr(\mathcal{Z}))$ is a torsion pair, the module M has a submodule M' which belongs to $lr(\mathcal{Z})$ such that M/M' belongs to $r(\mathcal{Z})$. Since M' is a submodule of M with $M/M' \in r(\mathcal{Z})$, we know by assumption that $M' = M$. But this shows that $M \in lr(\mathcal{Z})$. This proves the first equivalence. The second equivalence is shown in the same way.

Given a class $\mathcal{Z}$ of R-modules, we denote by $g(\mathcal{Z})$ the class of all R-modules generated by $\mathcal{Z}$ (these are just the factors of direct sums of modules in $\mathcal{Z}$, and by $c(\mathcal{Z})$ those cogenerated by $\mathcal{Z}$ (these are the submodules of products of modules in $\mathcal{Z}$). The following inclusions are trivial:

$$g(\mathcal{Z}) \subseteq lr(\mathcal{Z}) \quad \text{and} \quad c(\mathcal{Z}) \subseteq rl(\mathcal{Z}).$$

Lemma 3. *Let $\mathcal{Z}$ be a class of R-modules. Then $g(\mathcal{Z}) = lr(\mathcal{Z})$ if and only if $g(\mathcal{Z})$ is closed under extensions. Similarly, $c(\mathcal{Z}) = rl(\mathcal{Z})$ if and only if $c(\mathcal{Z})$ is closed under extensions.*

Proof: We show the first assertion (the second assertion is shown in the same way). Note that $lr(\mathcal{Z})$ is closed under extensions, thus the equality $g(\mathcal{Z}) = lr(\mathcal{Z})$ implies that $g(\mathcal{Z})$ is closed under extensions. Conversely, assume that $g(\mathcal{Z})$ is closed under extensions and let $M \in lr(\mathcal{Z})$. We have to show that M belongs to $g(\mathcal{Z})$. Let M' be the sum of all images of maps $Z \to M$ with $Z \in \mathcal{Z}$, thus M' is the maximal submodule of M generated by $\mathcal{Z}$. We claim that M/M' belongs to $r(\mathcal{Z})$. Namely, given a map $f\colon Z \to M/M'$ with $Z \in \mathcal{Z}$, let M''/M' be its image, where $M' \subseteq M'' \subseteq M$. Now M' and M''/M' are generated by $\mathcal{Z}$, thus, by assumption also M'' is generated by $\mathcal{Z}$. But this means that $M'' \subseteq M'$ and therefore $f = 0$. Since $M \in lr(\mathcal{Z})$ and $M/M' \in r(\mathcal{Z})$, the projection map $M \to M/M'$ is the zero map, thus $M = M'$ (of course, one also may refer to Lemma 2). This shows that M belongs to $g(\mathcal{Z})$.

It will be useful to know conditions so that a subcategory of the form $g(\mathcal{Z})$ is closed under extensions. From now on, we restrict to the case when $R = \Lambda$ is an artin algebra and we will denote the Auslander-Reiten translation in $\operatorname{mod}\Lambda$ by τ. Let us consider the case when $\mathcal{Z} = \mathbf{z}$ is a class of modules of finite length.

Lemma 4. *Let Λ be an artin algebra and $\mathbf{z}$ a class of Λ-modules of finite length. Assume that* $\operatorname{add}\mathbf{z}$ *is closed under extensions. If either* $\operatorname{add}\mathbf{z}$ *is also closed under factor modules or if* $\operatorname{pd} Z \le 1$ *for all* $Z \in \mathbf{z}$, *then* $g(\mathbf{z})$ *is closed under extensions.*

Proof. We first show the following: Under either assumption, given a finite length module Y and a submodule X of Y such that both X and Y/X are generated by $\mathbf{z}$, then also Y is generated by $\mathbf{z}$. Namely, if we assume that $\operatorname{add}\mathbf{z}$ is closed under factor modules, then both X and Y/X belong to $\operatorname{add}\mathbf{z}$, since they are factor modules of modules in $\operatorname{add}\mathbf{z}$. Thus also Y belongs to $\operatorname{add}\mathbf{z}$, since we assume that $\operatorname{add}\mathbf{z}$ is closed under extensions. Next, assume that $\operatorname{pd} Z \le 1$ for all $Z \in \mathbf{z}$. There are surjective maps $\pi\colon Z \to Y/X$ and $\pi'\colon Z' \to X$ where Z, Z' belong to $\operatorname{add}\mathbf{z}$. Starting from the exact sequence $0 \to X \to Y \to Y/X \to 0$, we can form the induced exact sequence with respect to π. Using now that $\operatorname{pd} Z \le 1$, and that π' is an epimorphism, we obtain a commutative diagram with exact rows of the following shape:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Y/X \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[u, equal] & Y' \arrow[r] \arrow[u, "f"'] & Z \arrow[r] \arrow[u, "\pi"'] & 0 \\
0 \arrow[r] & Z' \arrow[r] \arrow[u, "\pi'"'] & Y'' \arrow[r] \arrow[u, "f'"'] & Z \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

On one hand, the map ff' is surjective, on the other hand, Y'' belongs to $\operatorname{add}\mathbf{z}$, since $\operatorname{add}\mathbf{z}$ is closed under extensions. This shows that Y is generated by $\mathbf{z}$.

Now consider the general case of an arbitrary Λ-module Y and a submodule X of Y such that both X and Y/X are generated by $\mathbf{z}$. We have to show that Y is generated by $\mathbf{z}$. Write $Y = \sum_i Y_i$, where $X \subseteq Y_i$ and Y_i/X is isomorphic to a factor module of some module in $\mathbf{z}$. It is sufficient to show that all the Y_i belong to $g(\mathbf{z})$. Thus, without loss of generality, we may assume that Y/X is of finite length. Since Y/X is of finite length, there is a finite length submodule Y' of Y with $Y = X + Y'$. Now, X is the filtered union of submodules X_i of finite length generated by $\mathbf{z}$, thus there is some i with $X \cap Y' = X_i \cap Y'$. Thus $(X_i+Y')/X_i \simeq Y'/(X_i \cap Y') = Y'/(X\cap Y') \simeq (X+Y')/X = Y/X$. This shows that X_i+Y' is an extension of X_i by Y/X and both X_i and Y/X are finite length modules generated by $\mathbf{z}$. From our first considerations, we know that $X_i + Y'$ is generated by $\mathbf{z}$, thus also $Y = \bigcup_i (X_i + Y')$ is generated by $\mathbf{z}$.

Let $\mathbf{q}$ be a class of indecomposable Λ-modules of finite length. We want to find a criterion for $g(\mathbf{q})$ to be the torsion class of a split torsion pair in $\operatorname{Mod}\Lambda$. We denote by $K_0(\Lambda)$ the Grothendieck group of all finite length Λ-modules modulo exact sequences. In case $\delta\colon K_0(\Lambda) \to \mathbb{Z}$ is an additive map and M is a finite length module, we will write $\delta(M)$ for the value taken by δ on the equivalence class of M in $K_0(\Lambda)$.

We say that the class $\mathbf{q}$ of indecomposable Λ-modules of finite length is *closed under successors* provided given indecomposable Λ-modules M_1, M_2 of finite length with $\operatorname{Hom}(M_1, M_2) \neq 0$, then $M_1 \in \mathbf{q}$ implies $M_2 \in \mathbf{q}$.

We also consider the following finiteness condition (F): If N is a Λ-module with $\operatorname{Hom}(\mathbf{q}, N) = 0$ and has a submodule $U \subseteq N$ of finite length such that N/U is generated by $\mathbf{q}$, then N is of finite length.

Finally, let us say that $\mathbf{q}$ is *numerically determined* provided there exists a function $\delta\colon K_0(\Lambda) \to \mathbb{Z}$ such that an indecomposable Λ-module M of finite length belongs to $\mathbf{q}$ if and only if $\delta(M) > 0$.

Proposition 1. *Let Λ be an artin algebra. Let $\mathbf{q}$ be a class of indecomposable modules in $\operatorname{mod}\Lambda$ closed under successors.*

(a) *If $\mathbf{q}$ is numerically determined, then $\mathbf{q}$ satisfies the condition* (F)*.*

(b) *If $\mathbf{q}$ satisfies the condition* (F)*, then $g(\mathbf{q})$ is the torsion class of a split torsion pair in $\operatorname{Mod}\Lambda$. The corresponding torsionfree class is $r(\mathbf{q})$.*

Proof: (a) Assume that $\mathbf{q}$ is numerically determined with associated function δ. Let N be a Λ-module with $\operatorname{Hom}(\mathbf{q}, N) = 0$, and let U be a finite length submodule of N such that N/U is generated by $\mathbf{q}$. Then all submodules N' of N of finite length satisfy $\delta(N') \le 0$. In particular, we have $\delta(U) \le 0$ and we choose a finite length submodule U' of N with $U \subseteq U'$ such that $\delta(U')$ is maximal. We claim that $U' = N$. Otherwise, U'/U is a proper submodule of N/U, and since N/U is generated by $\mathbf{q}$, there is $Q \in \mathbf{q}$ and a map $f\colon Q \to N/U$ with image not contained in U'/U. Let $U''/U = U'/U + f(Q) \subseteq N/U$. In this way, we have found a submodule U'' of N with $U' \subset U''$ and such that U''/U' is a non-zero epimorphic image of a module in $\mathbf{q}$ and thus a non-zero direct sum of modules in $\mathbf{q}$. But the latter condition means that $\delta(U''/U') > 0$ and therefore $\delta(U') < \delta(U'')$, a contradiction to the choice of U'. Hence $U' = N$, and consequently N has finite length.

(b) Since $\operatorname{add}\mathbf{q}$ is closed both under extensions and under factor modules, Lemma 4 asserts that $g(\mathbf{q})$ is closed under extensions.

Denote as before by $r(\mathbf{q})$ the class of all Λ-modules L with $\operatorname{Hom}(\mathbf{q}, L) = 0$. We want to show that any exact sequence $0 \to X \to Y \to Z \to 0$ with $X \in g(\mathbf{q})$ and $Z \in r(\mathbf{q})$ splits.

First, consider the case when Z is of finite length. We may suppose that Z is indecomposable and also that Z is not projective. If the given map $Y \to Z$ is not split epi, we obtain a commutative diagram where the lower sequence is the almost split sequence ending in Z

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tau Z \arrow[r] & E \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

Note that τZ does not belong to $\mathbf{q}$, since $\mathbf{q}$ is closed under successors and Z is not in $\mathbf{q}$. But X is generated by $\mathbf{q}$, thus we see that f has to be the zero map. But this implies that the lower sequence splits, which is impossible.

In order to take care of the case of Z having arbitrary length, we show the following: Given a module Z and a chain of submodules U_i of Z with union $U = \bigcup_i U_i$, then, if all Z/U_i belong to $r(\mathbf{q})$, also Z/U belongs to $r(\mathbf{q})$. For the exact sequences $0 \to U_i \to Z \to Z/U_i \to 0$ give rise to the exact sequence $0 \to U \to Z \to \varinjlim Z/U_i \to 0$. Since $\operatorname{Hom}(\mathbf{q}, Z/U_i) = 0$ for all i, we have $\operatorname{Hom}(\mathbf{q}, \varinjlim Z/U_i) \simeq \varinjlim \operatorname{Hom}(\mathbf{q}, Z/U_i) = 0$, and hence $\operatorname{Hom}(\mathbf{q}, Z/U) = 0$.

Now, consider the case of Z being of arbitrary length. We may suppose that the map $X \to Y$ is an inclusion map. Let $\mathcal{U}$ be the set of submodules U of Y with $X \cap U = 0$ and $Y/(X+U) \in r(\mathbf{q})$. Since 0 belongs to $\mathcal{U}$, this set is non-empty. Given a chain $(U_i)_i$ of elements of $\mathcal{U}$, the union $U = \bigcup_i U_i$ belongs to $\mathcal{U}$; namely, it is clear that $X \cap U = 0$; and it follows from above that $Y/(X+U)$ belongs to $r(\mathbf{q})$, since all $Y/(X+U_i)$ belong to $r(\mathbf{q})$. As a consequence, we may choose a U maximal in $\mathcal{U}$. Assume $X+U$ is a proper submodule of Y. Let $X+U \subset Y' \subseteq Y$ with $Y'/(X+U)$ being simple. Let Y''/Y' be the largest submodule of Y/Y' generated by $\mathbf{q}$. Since $g(\mathbf{q})$ is closed under extensions, it follows that Y/Y'' belongs to $r(\mathbf{q})$. As a submodule of $Y/(X+U)$ the module $Y''/(X+U)$ belongs to $r(\mathbf{q})$. Condition (F) asserts that $Z' = Y''/(X+U)$ is of finite length. According to the first part of the proof, we know that $\operatorname{Ext}^1(Z', X) = 0$. Since the embedding $X \simeq (X+U)/U \subset Y''/U$ has cokernel Z', there exists a submodule U' of Y containing U with $(X+U) \cap U' = U$ and $(X+U)+U' = Y''$. We see that $X \cap U' = 0$ and that $Y/(X+U') = Y/Y''$ belongs to $r(\mathbf{q})$, thus U' belongs to $\mathcal{U}$, a contradiction to the maximality of U. Hence $Y = X+U = X \oplus U$, thus the sequence $0 \to X \to Y \to Y/X \to 0$ splits.

Remark. Let us note that these considerations can be extended as follows: We say that a class $\mathbf{q}$ of indecomposable modules of finite length is *numerically almost determined* provided there exists a function $\delta\colon K_0(\Lambda) \to \mathbb{Z}$ with the following properties: (i) If M belongs to $\mathbf{q}$, then $\delta(M) \ge 0$, and $\delta(M) > 0$ for all but a finite number of isomorphism classes of modules M in $\mathbf{q}$; (ii) any indecomposable Λ-module M in $\operatorname{mod}\Lambda$ with $\delta(M) > 0$ belongs to $\mathbf{q}$. We claim: *If* Λ *is an artin algebra and* $\mathbf{q}$ *is a class of indecomposable modules in* $\operatorname{mod}\Lambda$ *which is closed under successors and numerically almost determined, then* $\mathbf{q}$ *satisfies the condition* (F).

Proof: Assume that $\mathbf{q}$ is numerically almost determined with associated function δ. Let N be a Λ-module with $\operatorname{Hom}(\mathbf{q}, N) = 0$.

We first observe that for any submodule U' (of finite length) of N there is a bound b with the following property: If U'' is a submodule of N of finite length with $U' \subseteq U''$ such that U''/U' is generated by $\mathbf{q}$ and $\delta(U''/U') = 0$, then the length of U''/U' is bounded by b. Namely, let $Q_1, \dots, Q_m$ be the indecomposable modules in $\mathbf{q}$ (one from each isomorphism class) with $\delta(Q_i) = 0$ for $1 \le i \le m$, and assume that these modules Q_i are of length at most d. Let $\dim_k \operatorname{Ext}^1(Q_i, U') \le c$ for all i. Assume $U' \subseteq U'' \subseteq N$ is given with U'' of finite length, U''/U' generated by $\mathbf{q}$ and $\delta(U''/U') = 0$. If we write U''/U' as a direct sum of indecomposables, all these direct summands X belong to $\mathbf{q}$ (since $\mathbf{q}$ is closed under successors), thus $\delta(X) \ge 0$. But these numbers add up to zero, thus we have $\delta(X) = 0$. This shows that $U''/U' \simeq \bigoplus_i Q_i^{t_i}$, for some natural numbers t_i. Now if $t_i > c$ for some i, then $\dim_k \operatorname{Ext}^1(Q_i, U') \le c$ implies that U'' has a submodule isomorphic to Q_i, in contrast to the fact that $\operatorname{Hom}(Q_i, N) = 0$. Altogether we see that the length of U''/U' is bounded by $b = cd$.

Now, let U be a finite length submodule of N such that N/U is generated by $\mathbf{q}$ and choose (as in the proof of Proposition 1) a finite length submodule U' of N with $U \subseteq U'$ such that $\delta(U')$ is maximal. Consider chains $U' = U_0 \subseteq U_1 \subseteq U_2 \subseteq \dots$ of finite length modules such that the factors U_i/U_{i-1} are generated by $\mathbf{q}$ and satisfy $\delta(U_i/U_{i-1}) = 0$ for all i. We claim that such a sequence stabilizes. For U_i/U_0 is generated by $\mathbf{q}$ and $\delta(U_i/U_0) = 0$, thus, as we have seen, U_i/U_0 is of bounded length, with a bound only depending on U_0. It has the following consequence: Replacing U' if necessary by a larger submodule, we may assume in addition that any finite length submodule U'' of N with $U' \subseteq U''$ which is generated by $\mathbf{q}$ satisfies $\delta(U''/U') > 0$. This then allows to complete the proof as above.

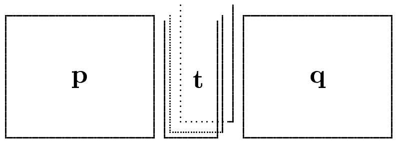
2. (Concealed) canonical algebras and separating tubular families.

In this section we recall some background material on canonical and concealed canonical algebras.

Given a class $\mathbf{x}$ of indecomposable modules of finite length, we say that an indecomposable module M of finite length is a *proper predecessor of* $\mathbf{x}$ provided it does not belong to $\mathbf{x}$, but there is a sequence of indecomposables $M = M_0, M_1, \dots, M_n$ with $\mathrm{Hom}(M_{i-1}, M_i) \neq 0$ for all $1 \le i \le n$ such that M_n belongs to $\mathbf{x}$. Similarly, M is said to be *a proper successor of* $\mathbf{x}$ provided it does not belong to $\mathbf{x}$, but there is a sequence of indecomposables $M_0, M_1, \dots, M_n = M$ with $\mathrm{Hom}(M_{i-1}, M_i) \neq 0$ for all $1 \le i \le n$ such that M_0 belongs to $\mathbf{x}$.

Separating tubular families (see [R2,R4,LP,RS]). Let Λ be an artin algebra, and let $\mathbf{t}$ be a sincere stable separating tubular family. Recall that this means the following: a *tubular family* consists of all the indecomposables belonging to a set of tubes in the Auslander-Reiten quiver of Λ (in particular, all the modules in $\mathbf{t}$ are of finite length). Such a tubular family is said to be *stable* provided all the tubes are stable, thus provided it does not contain any indecomposable module which is projective or injective. A family of modules is said to be *sincere* provided every simple Λ-module occurs as the composition factor of at least one of the given modules. Finally, let us say that the tubular family $\mathbf{t}$ is *separating* provided it is standard, there are no indecomposable modules M of finite length which are both proper predecessors of $\mathbf{t}$ and proper successors of $\mathbf{t}$, and any map from a proper predecessor of $\mathbf{t}$ to a proper successor of $\mathbf{t}$ factors through any of the tubes in $\mathbf{t}$.

Now let $\mathbf{t}$ be a separating tubular family. We denote by $\mathbf{p}$ the class of indecomposables of finite length which are proper predecessors of $\mathbf{t}$, and by $\mathbf{q}$ the class of indecomposables of finite length which are proper successors of $\mathbf{t}$. Then any indecomposable module of finite length belongs either to $\mathbf{p}$, $\mathbf{t}$ or $\mathbf{q}$,



and one says that $\mathbf{t}$ *separates* $\mathbf{p}$ *from* $\mathbf{q}$. Note that there are no maps "backwards":

$$\mathrm{Hom}(\mathbf{t}, \mathbf{p}) = \mathrm{Hom}(\mathbf{q}, \mathbf{p}) = \mathrm{Hom}(\mathbf{q}, \mathbf{t}) = 0$$

and any map from a module in $\mathbf{p}$ to a module in $\mathbf{q}$ can be factored through a module in $\mathbf{t}$ (even through one lying in a prescribed tube inside $\mathbf{t}$). In case $\mathbf{t}$ is in addition sincere and stable, then all the indecomposable projective modules belong to $\mathbf{p}$, the indecomposable injective modules belong to $\mathbf{q}$. As a consequence, in this case the modules which belong to $\mathbf{p}$ or $\mathbf{t}$ have projective dimension at most 1, those which belong to $\mathbf{t}$ or $\mathbf{q}$ have injective dimension at most 1. Also note that a stable separating tubular family $\mathbf{t}$ always is an exact abelian subcategory of $\mathrm{mod}\,\Lambda$ (and all the indecomposables in $\mathbf{t}$ are serial when considered as objects in this subcategory).

The algebras Λ with a sincere stable separating tubular family are the *concealed canonical algebras.* They have been studied in [LM], [LP] and [RS], and we are going to review the main steps of the construction at the end of this section. The essential ingredient for many of our considerations are the defect functions on the Grothendieck groups $K_0(\Lambda)$.

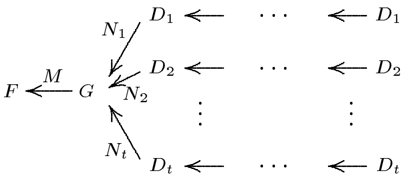
The construction of the canonical algebras (see [R4]). Let k be a field. We start with a tame bimodule ${}_FM_G$, thus F, G are division rings having k as central subfield and being finite-dimensional over k, and M is an F-G-bimodule with $\dim {}_FM \cdot \dim M_G = 4$ and such that k operates centrally on M. This means that $\Lambda_0 = \left[\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right]$ is a finite-dimensional tame hereditary k-algebra with precisely two simple modules, and, up to Morita equivalence, all finite-dimensional tame hereditary k-algebras with precisely two simple modules are obtained in this way. A non-zero Λ_0-module N is called *simple regular,* provided $\tau N \simeq N$ and $\mathrm{End}(N)$ is a division ring. It is well-known that there are many simple regular Λ_0-modules, the number of isomorphism classes is $\max(\aleph_0, |k|)$.

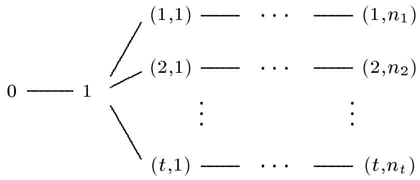
If R is any ring, N any R-module with endomorphism ring D^{op} and $n \ge 1$ a natural number, we denote by $R[N, n]$ the n-point extension of R by N, it is the matrix ring

$$R[N, n] \;=\; \begin{bmatrix} R & N & \cdots & N \\ 0 & D & \cdots & D \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & D \end{bmatrix}.$$

Since any R-module may be considered (in a natural way) as an $R[N,n]$-module, we may iterate this procedure: given a finite sequence $N_1,\dots,N_t$ of R-modules and natural numbers $n_1,\dots,n_t$, we may form $R[N_1,n_1]\cdots[N_t,n_t]$.

Let us return to Λ_0. Choose t pairwise non-isomorphic simple regular Λ_0-modules $N_1,\dots,N_t$ (with endomorphism rings D_i^{op}) and natural numbers $n_1,\dots,n_t$ and consider $\Lambda' = \Lambda_0[N_1,n_1]\cdots[N_t,n_t]$, a so called *squid algebra*; its quiver (or better species) is of the form as shown to the left:





As shown to the right, we label the two vertices of Λ_0 by 0 and 1, and the extension vertices of the i-th branch by $(i,1),\dots,(i,n_i)$, always from left to right. It is not difficult to see that $I(0)\oplus I(1)\oplus\bigoplus_{i,j}\tau^j I(i,j)$ is a cotilting module; its endomorphism ring is denoted by Λ and the algebras Λ obtained in this way are the *canonical* algebras.

Given the canonical algebra Λ, let us write down the (canonical) defect function $\delta\colon K_0(\Lambda)\to\mathbb{Z}$: Note that Λ has a unique simple projective module S and a unique simple injective module S'. The defect $\delta(M)$ of a Λ-module M is calculated in terms of the Jordan-Hölder multiplicities $[M:S]$ and $[M:S']$, as follows:

$$\delta(M)=\begin{cases} [M:S']-[M:S] & \\ 2[M:S']-[M:S] & \text{in case} \\ [M:S']-2[M:S] & \end{cases}\qquad \begin{matrix}\dim_k S'=\dim_k S\\ \dim_k S'>\dim_k S\\ \dim_k S'<\dim_k S\end{matrix}$$

If we denote by $\mathbf{t}$ the class of all indecomposable Λ-modules M with $\delta(M)=0$, then $\mathbf{t}$ is a stable separating tubular family, separating the class $\mathbf{p}$ of all indecomposable modules M with $\delta(M)<0$ from the class $\mathbf{q}$ of all indecomposable modules M with $\delta(M)>0$. We call this triple $(\mathbf{p},\mathbf{t},\mathbf{q})$ the *canonical trisection* of $\operatorname{mod}\Lambda$.

The construction of the concealed canonical algebras. Let Λ be a canonical algebra, with canonical trisection $(\mathbf{p},\mathbf{t},\mathbf{q})$. Let T be a tilting module which belongs to $\operatorname{add}\mathbf{p}$ (since all the modules in $\mathbf{p}$ have projective dimension at most 1, to be a tilting module means in addition that $\operatorname{Ext}^1(T,T)=0$ and that there is an exact sequence $0\to{}_\Lambda\Lambda\to T'\to T''\to 0$ with $T',T''\in\operatorname{add}T$). Then, by definition, $\Lambda'=\operatorname{End}(T)^{\mathrm{op}}$ is a *concealed canonical* algebra. We note that the tilting functor $F=\operatorname{Hom}(T,-)$ sends $\mathbf{t}$ to a sincere stable separating tubular family $\mathbf{t}'$ in $\operatorname{mod}\Lambda'$, and as it has been shown in [LM] and [LP] all sincere stable separating tubular families are obtained in this way.

3. Two extremal torsion pairs in $\operatorname{Mod}\Lambda$

In the next sections, let Λ be a canonical algebra and $(\mathbf{p},\mathbf{t},\mathbf{q})$ its canonical trisection. In this section we introduce two torsion pairs in $\operatorname{Mod}\Lambda$ which will turn out to be split, both having the property that the indecomposable torsion modules of finite length are just the modules in $\mathbf{q}$. We also introduce and investigate the class of Prüfer modules.

The torsion pair $(\mathcal{C},\mathcal{Q})$. As we have mentioned, the category $\mathbf{q}$ is closed under successors. Since it is also numerically determined, we are able to apply Proposition 1. Thus, if we denote $\mathcal{C}=r(\mathbf{q})$ and $\mathcal{Q}=g(\mathbf{q})$, then $(\mathcal{C},\mathcal{Q})$ is a split torsion pair in $\operatorname{Mod}\Lambda$.

The torsion pair $(\mathcal{R},\mathcal{D})$. Let $\mathcal{D}=l(\mathbf{t})$; note that $\mathcal{D}$ can also be described as $\mathcal{D}=\{M\mid \operatorname{Ext}^1(\mathbf{t},M)=0\}$. Namely, the objects T in $\mathbf{t}$ have projective dimension 1, thus we have $\operatorname{Ext}^1(T,M)\cong D\operatorname{Hom}(M,\tau T)$ (here, $D=\operatorname{Hom}_k(-,k)$ is the duality with respect to the base field k). Since $\mathbf{t}$ consists of stable tubes, the Auslander-Reiten translation is bijective on the isomorphism classes in $\mathbf{t}$.

Since $\mathcal{D}=l(\mathbf{t})$, it is the torsion class of a torsion pair in $\operatorname{Mod}\Lambda$, namely of $(\mathcal{R},\mathcal{D})$, where $\mathcal{R}=r(\mathcal{D})=rl(\mathbf{t})$ is the smallest torsionfree class containing the class $\mathbf{t}$. As we have mentioned above, we may describe

$\mathcal{R}$ also as follows: A module M belongs to $\mathcal{R}$ if and only if the only submodule U of M with $\mathrm{Hom}(U, \mathbf{t}) = 0$ is $U = 0$. (We later will see that also this torsion pair is split.)

The two torsion pairs $(\mathcal{C}, \mathcal{Q})$ and $(\mathcal{R}, \mathcal{D})$ are related as follows:

$$\mathcal{R} \subseteq \mathcal{C} \quad \text{and} \quad \mathcal{Q} \subseteq \mathcal{D}.$$

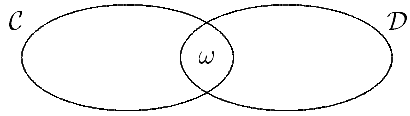
These two assertions are equivalent, thus it is sufficient to verify one of them. But actually, both follow directly from the assertion $\mathrm{Hom}(\mathbf{q}, \mathbf{t}) = 0$.

There is the following straightforward characterization of these two torsion pairs as the extremal ones when dealing with all the torsion pairs of $\mathrm{Mod}\,\Lambda$ with prescribed distribution of the finite dimensional modules: we deal with the torsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathbf{t} \subset \mathcal{X}$ and $\mathbf{q} \subset \mathcal{Y}$.

The two torsion pairs $(\mathcal{R}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{Q})$ have the property that the finite dimensional indecomposable torsionfree modules are those in $\mathbf{p}$ and $\mathbf{t}$, whereas the finite-dimensional indecomposable torsion modules are those in $\mathbf{q}$. If $(\mathcal{X}, \mathcal{Y})$ is an arbitrary torsion pair in $\mathrm{Mod}\,\Lambda$ such that the finite dimensional indecomposable torsionfree modules are those in $\mathbf{p}$ and $\mathbf{t}$, or, equivalently, such that the finite dimensional indecomposable torsion modules are those in $\mathbf{q}$, then

$$\mathcal{R} \subseteq \mathcal{X} \subseteq \mathcal{C} \quad \text{and} \quad \mathcal{Q} \subseteq \mathcal{Y} \subseteq \mathcal{D}.$$

The intersection ω of $\mathcal{C}$ and $\mathcal{D}$. Let $\omega = \mathcal{C} \cap \mathcal{D}$.



A complete description of ω and its relation to $\mathcal{C}$ and $\mathcal{D}$ is one of the aims of the paper.

The torsion modules $\mathcal{T}$. Let $\mathcal{T} = \mathcal{C} \cap g(\mathbf{t})$, these are the modules in $\mathcal{C}$ generated by $\mathbf{t}$. We stress that *every module in $\mathcal{T}$ is the union of modules in* $\mathrm{add}\,\mathbf{t}$ *and that* $\mathcal{T} = \varinjlim \mathbf{t}$, the direct limit closure of $\mathrm{add}\,\mathbf{t}$. Namely, if $M = \varinjlim M_i$ is the direct limit of a directed system of modules M_i in $\mathrm{add}\,\mathbf{t}$, then M obviously belongs to $g(\mathbf{t})$; since $\mathrm{Hom}(Y, M) = \varinjlim \mathrm{Hom}(Y, M_i)$ for any finitely generated module Y, we also see that $\mathrm{Hom}(Y, M) = 0$ for $Y \in \mathbf{q}$, thus M also belongs to $r(\mathbf{q}) = \mathcal{C}$. And conversely, if we assume that M is generated by $\mathbf{t}$, then M is the union of all its submodules M_i which are images of maps from modules in $\mathrm{add}\,\mathbf{t}$ to M, and this is a directed system. As a factor module of a module in $\mathrm{add}\,\mathbf{t}$, any M_i is a direct sum of modules in $\mathbf{t}$ and in $\mathbf{q}$. If we assume in addition that M belongs to $r(\mathbf{q})$, then all the M_i belong to $\mathrm{add}\,\mathbf{t}$. This shows that M is the union of a directed system of submodules which belong to $\mathrm{add}\,\mathbf{t}$.

Since $\mathbf{t}$ is an exact abelian subcategory of $\mathrm{mod}\,\Lambda$, it follows that $\mathcal{T}$ *is an exact abelian subcategory of* $\mathrm{Mod}\,\Lambda$. In particular, $\mathcal{T}$ is closed under kernels, images and cokernels, and also under direct sums.

The Prüfer modules. Of special interest are the so called *Prüfer modules.* They are constructed as follows: The full subcategory $\mathrm{add}\,\mathbf{t}$ given by the finite direct sums of modules in $\mathbf{t}$ is an abelian length category. Every module in $\mathbf{t}$ has a unique composition series when considered inside this subcategory $\mathrm{add}\,\mathbf{t}$; its length is called the *regular length*; the modules in $\mathbf{t}$ of regular length 1 are just the simple objects of $\mathrm{add}\,\mathbf{t}$ (we call them the *simple objects* of $\mathbf{t}$).

The isomorphism classes in $\mathbf{t}$ are indexed by pairs consisting of a natural number r and (the isomorphism class of) a simple object S in $\mathbf{t}$. We will denote the corresponding Λ-module by $S[r]$. It is the unique module in $\mathbf{t}$ with regular length r and having S as a submodule. For any simple object S in $\mathbf{t}$, there is a sequence of inclusion maps

$$S = S[1] \to S[2] \to \cdots \to S[r] \to \cdots,$$

and we denote by $S[\infty]$ the direct limit of this sequence. This is the *Prüfer module with regular socle S.*

We note the following: As an object in $\mathcal{T}$, any Prüfer module $S[\infty]$ belongs to $\mathcal{C}$. Since $S[\infty]$ is injective in $\mathcal{T}$, we have in particular $\mathrm{Ext}^1(T, S[\infty]) = 0$ for any object T in $\mathbf{t}$, thus $S[\infty]$ also belongs to $\mathcal{D}$. This shows: *The Prüfer modules belong to ω.* We can strengthen this assertion as follows: *The direct sums of Prüfer modules are just the injective objects of the abelian category $\mathcal{T}$, and every object in $\mathcal{T}$ has an injective envelope* (see [R1]). Let us denote $\omega_0 = \mathcal{T} \cap \mathcal{D}$, then this is the full subcategory of all injective objects of $\mathcal{T}$. Thus ω_0 is the full subcategory of all direct sums of Prüfer modules.

Given a module M, we denote by tM the maximal submodule of M generated by $\mathbf{t}$, thus $t(M/tM) = 0$ for any module M. We use Lemma 4 in order to see that the class of modules $g(\mathbf{t})$ generated by $\mathbf{t}$ is closed under extensions. As a consequence, the pair $(\mathcal{F}, g(\mathbf{t}))$ where $\mathcal{F} = r(\mathbf{t})$, is a torsion pair (but this is a torsion pair which is not split).

The analogy. The notation introduced above should remind the reader of the analogous situation when dealing with the category $\mathrm{Mod}\, R$, where R is a Dedekind ring, say a Dedekind ring with infinitely many maximal ideals. The torsion pair in $\mathrm{Mod}\, R$ which we have in mind is the usual one: the torsion modules are those R-modules M where every element is annihilated by some non-zero ideal, the torsionfree R-modules are those with zero torsion submodule. In our situation of dealing with a canonical algebra Λ, we consider the torsion pair $(\mathcal{F}, g(\mathbf{t}))$; note that it is the subcategory $\mathcal{C}$ of $\mathrm{Mod}\,\Lambda$ which shows strong similarity to $\mathrm{Mod}\, R$, thus we reserve the symbol $\mathcal{T}$ for the intersection of $g(\mathbf{t})$ with $\mathcal{C}$. The module class $\mathcal{D}$ should be interpreted as the "divisible" modules, the module class $\mathcal{R}$ as the "reduced" ones.

4. The ω-coresolution of the modules in $\mathcal{C}$.

In this section we show that there are ω-coresolutions for the modules in $\mathcal{C}$, with $\omega = \mathcal{C} \cap \mathcal{D}$ as before.

Theorem 1. *For every Λ-module M, there exists a minimal left ω-approximation, $M \to M_\omega$, and its cokernel belongs to ω_0. This minimal left ω-approximation is injective if and only if M belongs to $\mathcal{C}$.*
If M belongs to $\mathcal{F}$, then M_ω belongs to $\mathcal{F}$. If M belongs to $\mathcal{T}$, then M_ω belongs to $\mathcal{T}$.

Part of the theorem may be reformulated as follows: *For any $M \in \mathcal{C}$, there is an exact sequence*

$$0 \to M \xrightarrow{f} M_\omega \to T \to 0$$

with $M_\omega \in \omega$ and $T \in \omega_0$, such that f is a minimal left ω-approximation. In this way, one obtains a characterization of the modules in $\mathcal{C}$ as follows: *The modules in $\mathcal{C}$ are the kernels of epimorphisms in ω.*

Since $(\mathcal{C}, \mathcal{Q})$ is a split torsion pair, the module M is a direct sum of a module in $\mathcal{C}$ and a module in $\mathcal{Q}$. For $M \in \mathcal{Q}$, the minimal left ω-approximation M_ω has to be zero, since $\mathrm{Hom}(\mathbf{q}, \mathcal{C}) = 0$. For the proof of Theorem 1, it is sufficient to construct a minimal left ω-approximation for the modules in $\mathcal{C}$. First, we construct an exact sequence

$$0 \to M \to M_\omega \to T \to 0$$

where M_ω is in ω and $T \in \omega_0$.

For the proof we will need a splitting result which later will be incorporated into our basic splitting theorem (Theorem 2):

Lemma 5.

$$\mathrm{Ext}^1(\mathcal{T}, \mathcal{D}) = 0.$$

This is an immediate consequence of the fact that $\mathcal{T} = \varinjlim \mathbf{t}$.

Proof of Theorem 1. As we have mentioned, we can assume that M belongs to $\mathcal{C}$. Take a universal extension

$$0 \to M \xrightarrow{\mu'} M' \xrightarrow{\pi'} T' \to 0,$$

with T' a direct sum of simple objects in $\mathbf{t}$ and let ϵ be its equivalence class in $\operatorname{Ext}^1(T', M)$. The universality means the following: given a simple object S in $\mathbf{t}$, then first, any element of $\operatorname{Ext}^1(S, M)$ is induced from ϵ by a map $S \to T'$, and second, that $\pi' f = 0$ for any map $f\colon S \to M'$. Note that the first of these conditions can be reformulated as saying that $\operatorname{Ext}^1(S, \mu') = 0$.

Take an injective envelope $u\colon T' \to T$ in the abelian category $\mathcal{T}$, thus $T \in \omega_0$. The cokernel of u belongs to $\mathcal{T}$, thus it has projective dimension at most 1 since it is a direct limit of finite length modules of projective dimension 1. It follows that the map $\operatorname{Ext}^1(u, M)$ is surjective, thus there exists a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mu'"] \arrow[d, equal] & M' \arrow[r, "\pi'"] \arrow[d, "u'"] & T' \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & M \arrow[r, "\mu"] & M_\omega \arrow[r, "\pi"] & T \arrow[r] & 0 \end{tikzcd}$$

Since M and T belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under extensions, we see that M_ω belongs to $\mathcal{C}$.

In order to show that M_ω belongs to $\mathcal{D}$, it is enough to show that $\operatorname{Ext}^1(S, M_\omega) = 0$ for all simple objects in $\mathbf{t}$ (then clearly $\operatorname{Ext}^1(T'', M_\omega) = 0$ for any object T'' in $\mathbf{t}$). The maps μ, π yield an exact sequence

$$\operatorname{Ext}^1(S, M) \xrightarrow{\operatorname{Ext}^1(S,\mu)} \operatorname{Ext}^1(S, M_\omega) \xrightarrow{\operatorname{Ext}^1(S,\pi)} \operatorname{Ext}^1(S, T),$$

and the last term is zero, since T is injective in $\mathcal{T}$. Thus the map $\operatorname{Ext}^1(S, \mu)$ is surjective. However, this map $\operatorname{Ext}^1(S, \mu)$ factors through $\operatorname{Ext}^1(S, \mu') = 0$. Thus we conclude that $\operatorname{Ext}^1(S, M_\omega) = 0$.

It remains to be seen that the map μ is a minimal left ω-approximation. According to Lemma 5, we have $\operatorname{Ext}^1(T, \omega) = 0$, and thus μ is a left ω-approximation.

In order to show that μ is left minimal, we first show that for a direct sum decomposition $M_\omega = N \oplus N'$ with $\mu(M) \subseteq N$ we must have $N' = 0$. Thus, consider such a direct sum decomposition $M_\omega = N \oplus N'$ with $\mu(M) \subseteq N$. The cokernel T of μ is isomorphic to $N/\mu(M) \oplus N'$. Assume N' is non-zero. Since T and therefore N' is isomorphic to a direct sum of Prüfer modules, there is a monomorphism $f\colon S \to N'$ with S a simple object of $\mathbf{t}$. The image of f has to lie in the image of u', thus there is $f'\colon S \to M'$ with $f = u'f'$. By construction of the universal extension ϵ, the composition $\pi' f'$ is zero, thus $f' = \mu' f''$ for some $f''\colon S \to M$. But this implies that the image of $f = u'f' = u'\mu' f'' = \mu f''$ lies in $\mu(M) \subseteq N$ and not in N'. This contradiction shows that $N' = 0$.

Now consider a map $g\colon M_\omega \to M_\omega$ with $g\mu = \mu$. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mu"] \arrow[d, equal] & M_\omega \arrow[r, "\pi"] \arrow[d, "g"] & T \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r, "\mu"] & M_\omega \arrow[r, "\pi"] & T \arrow[r] & 0 \end{tikzcd}$$

Note that π induces an isomorphism between the kernel of g and the kernel of g'. In order to show that g is injective, let us assume to the contrary that $\operatorname{Ker} g \simeq \operatorname{Ker} g'$ is non-zero. Note that the kernel $\operatorname{Ker} g'$ of g' belongs to $\mathcal{T}$, thus there is a simple object S of $\mathcal{T}$ which is contained in $\operatorname{Ker} g'$ and hence $S \subseteq T'$. The isomorphism of kernels $\operatorname{Ker} g \simeq \operatorname{Ker} g'$ shows that this S may be considered as a submodule of M' with non-zero composition $S \to M' \to T'$, but this is a contradiction. Thus g is a monomorphism.

To see that g is an epimorphism, denote by L its cokernel $\operatorname{Cok} g$. Also for the cokernels, π induces an isomorphism $\operatorname{Cok} g \to \operatorname{Cok} g'$. Since $g'\colon T \to T$ is a split monomorphism, and $T \in \omega_0$, we see that $L \in \omega_0$, thus belongs to ω. Using $\operatorname{Ext}^1(\omega, \omega) = 0$, we see that g is a split monomorphism. But according to the previous considerations this implies that g is surjective.

Of course, if M belongs to $\mathcal{T}$, then also M_ω belongs to $\mathcal{T}$, since $\mathcal{T}$ is closed under extensions. Thus, finally, consider the case when $\operatorname{Hom}(\mathbf{t}, M) = 0$. In order to show that $\operatorname{Hom}(\mathbf{t}, M_\omega) = 0$, it is sufficient to show that $\operatorname{Hom}(S, M_\omega) = 0$ for any simple object S in $\mathbf{t}$. Thus, take a nonzero map $f\colon S \to M_\omega$. Its composition with π goes to the socle of T in $\mathcal{T}$, thus $f = u'f'$ for some $f'\colon S \to M'$. But by construction $\pi' f' = 0$, thus $f' = \mu' f''$ for some $f''\colon S \to M$. Since we assume that there are no non-zero maps $S \to M$, it follows that $f = 0$. Hence M_ω is in $\mathcal{F}$.

Lemma 6. *Assume that M belongs to $\mathcal{C}$ and is of finite length. If we write M_ω/M as a direct sum of Prüfer modules, then any Prüfer module occurs with finite multiplicity.*

Proof: We may assume that M is indecomposable. If M belongs to $\mathbf{t}$, then M_ω and M_ω/M are Prüfer modules themselves.

Thus, we may assume that M belongs to $\mathbf{p}$ and therefore to $\mathcal{F}$. Let S be a simple object of $\mathbf{t}$ and $S[\infty]$ the corresponding Prüfer module. Let $n = \dim_k \operatorname{Ext}^1(S, M)$. We claim that $S[\infty]$ occurs in M_ω/M with multiplicity at most n. Otherwise, M_ω/M has a submodule of the form S^{n+1}, say U/M, where U is a submodule of M_ω with $M \subseteq U$. Since $\dim_k \operatorname{Ext}^1(S, M) = n$, it follows that U has a submodule isomorphic to S. This is impossible, since M_ω belongs to $\mathcal{F}$.

5. The basic splitting result.

The aim of this section is to prove the basic splitting result $\operatorname{Ext}^1(\mathcal{C}, \mathcal{D}) = 0$, which will also have as a consequence that the torsion pair determined by $\mathcal{D}$ splits.

Let Λ be a canonical algebra with canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ of $\operatorname{mod}\Lambda$ and defect function δ. We say that an indecomposable projective Λ-module P is called a *peg* (with respect to δ) provided $\delta(P) = -1$. If Λ is a canonical algebra, then it is easy to see that a peg exists: if the simple projective module is not a peg, then the sincere indecomposable projective module turns out to be a peg. More precisely, let P be the simple projective module and P' the sincere indecomposable projective module. Then P is a peg if and only if $\dim \operatorname{End} P \le \dim \operatorname{End} P'$, and P' is a peg if and only if $\dim \operatorname{End} P \ge \dim \operatorname{End} P'$.

Lemma 7. *Let M belong to ω and let P be a peg. Then M has a submodule U which is a direct sum of copies of P such that M/U belongs to ω_0.*

Proof: Let $\mathcal{U}$ be the set of submodules U' of M which are direct sums of copies of P such that M/U' is in $\mathcal{C}$. We consider this set as being partially ordered with respect to split embeddings. Given a chain inside $\mathcal{U}$, it is not difficult to see that the union is again in $\mathcal{U}$. Thus we can apply Zorn's lemma in order to obtain a maximal member U of $\mathcal{U}$. We show that M/U belongs to $\mathcal{T}$. Let $U \subseteq V \subseteq M$ such that $V/U = t(M/U)$. Thus we have to show that $V = M$. The module M/V belongs to $\mathcal{F}$, and also to $\mathcal{D}$, thus to ω. The non-zero modules in ω are sincere. Thus there is a non-zero homomorphism $f\colon P \to M/V$. Note that the kernel K of f has to be zero, since otherwise $\delta(K) \le \delta(P)$ and therefore $\delta(P/K) = \delta(P) - \delta(K) \ge 0$. However since M/V is a module in $\mathcal{F}$, every non-zero submodule of M/V of finite length has negative defect. Since P is projective, we can lift the homomorphism $f\colon P \to M/V$ to a homomorphism $f'\colon P \to M$ such that $f = pf'$, where $p\colon M \to M/V$ is the canonical map. The image P' of f' is a submodule of M isomorphic to P and $P' \cap V = 0$. In particular, we also have $P' \cap U = 0$. Let $U' = P' \oplus U$. Then this is a submodule of M which is a direct sum of copies of P. In order to see that the factor module M/U' belongs to $\mathcal{C}$, one observes that M/U' is an extension of the modules V/U and $M/(P'+V)$. The module V/U is a submodule of M/U, thus it belongs to $\mathcal{C}$. If $M/(P'+V)$ would contain a module from $\mathbf{q}$ as a submodule, then its inverse image under the projection $M/V \to M/(P'+V)$ would have non-negative defect (being an extension of the module $(P'+V)/V \simeq P$ of defect -1 by a module of positive defect). But this is impossible, since M/V belongs to $\mathcal{F}$. Altogether, we see that U' belongs to $\mathcal{U}$. Since U is a direct summand of U', we obtain a contradiction to the maximality of U. This shows that M/U belongs to $\mathcal{T}$. As a factor module of $M \in \mathcal{D}$, the module M/U also belongs to $\mathcal{D}$, thus to $\omega_0 = \mathcal{D} \cap \mathcal{T}$.

Theorem 2 (Basic splitting result):

$$\operatorname{Ext}^1(\mathcal{C}, \mathcal{D}) = 0.$$

Proof. Let $C \in \mathcal{C}$, $D \in \mathcal{D}$. Since $(\mathcal{C}, \mathcal{Q})$ is a split torsion pair, we can write $D = D' \oplus D''$ with $D' \in \mathcal{C}$ and $D'' \in \mathcal{Q}$. Also, since $(\mathcal{C}, \mathcal{Q})$ is a split torsion pair, we have $\operatorname{Ext}^1(\mathcal{C}, \mathcal{Q}) = 0$. Thus it is sufficient to show that $\operatorname{Ext}^1(\mathcal{C}, D') = 0$. Note that D' as a direct summand of D belongs to $\mathcal{D}$, thus to ω. This shows that we have to show $\operatorname{Ext}^1(\mathcal{C}, \omega) = 0$.

First, we show $\mathrm{Ext}^1(\omega,\omega)=0$. Start with a module $M\in\omega$. According to Lemma 7, there is a submodule U which is a direct sum $\bigoplus_I P$ of copies of a peg P such that M/U belongs to $\mathcal{T}$. Let N be a second module in ω. On one hand, we have

$$\mathrm{Ext}^1(U,N)=\mathrm{Ext}^1(\bigoplus\nolimits_I P,N)\cong\prod\nolimits_I \mathrm{Ext}^1(P,N)=0.$$

On the other hand, Lemma 5 asserts that $\mathrm{Ext}^1(M/U,N)=0$. Altogether we conclude that $\mathrm{Ext}^1(M,N)=0$.

Now take an arbitrary module M in $\mathcal{C}$, and consider the minimal left ω-approximation given by Theorem 1:

$$0\to M\to M_\omega\to T\to 0$$

with M_ω in ω and $T\in\mathcal{T}$. Applying the long exact sequence with respect to $\mathrm{Hom}(-,N)$, where $N\in\omega$, we get the exact sequence

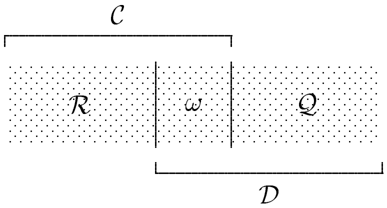
$$\mathrm{Ext}^1(M_\omega,N)\to\mathrm{Ext}^1(M,N)\to\mathrm{Ext}^2(T,N).$$

Since the projective dimension of T is at most one, the last term vanishes. Since M_ω and N both belong to ω, also the first term is zero. Thus $\mathrm{Ext}^1(M,N)=0$. In this way, we have shown that $\mathrm{Ext}^1(\mathcal{C},\omega)=0$, as required. This concludes the proof.

Corollary 1. *The torsion pair $(\mathcal{R},\mathcal{D})$ splits.*

This follows immediately from the inclusion $\mathcal{R}\subseteq\mathcal{C}$.

The category $\mathrm{Mod}\,\Lambda$ consists of three parts:



and this means the following: *any Λ-module M is a direct sum $M=M_1\oplus M_2\oplus M_3$ with $M_1\in\mathcal{R}, M_2\in\omega, M_3\in\mathcal{Q}$, there are no maps "backwards":*

$$\mathrm{Hom}(\omega,\mathcal{R})=\mathrm{Hom}(\mathcal{Q},\mathcal{R})=\mathrm{Hom}(\mathcal{Q},\omega)=0.$$

and any map from $\mathcal{R}$ to $\mathcal{Q}$ factors through a module in ω. Also the last assertion is an immediate consequence of previous results: Given a map $h\colon M\to N$ with $M\in\mathcal{R}$ and $N\in\mathcal{Q}$, choose a minimal left ω-approximation $f\colon M\to M_\omega$. According to Theorem 1, the map f is injective and its cokernel T belongs to ω_0. Since $\mathrm{Ext}^1(T,N)=0$ (by Theorem 2 or already Lemma 5), we conclude that h factors through f.

Corollary 2. *We have the following.*

(a) $\mathrm{pd}\ C\le 1$ *for C in $\mathcal{C}$.*
(b) $\mathrm{id}\ D\le 1$ *for D in $\mathcal{D}$.*

Proof (a): For X in $\mathrm{Mod}\,\Lambda$ we have $\mathrm{Ext}^2(C,X)\simeq\mathrm{Ext}^1(C,\Omega^{-1}X)$. Now $\Omega^{-1}X$ is generated by injective modules, thus belongs to $\mathcal{D}$. Now $\mathrm{Ext}^1(\mathcal{C},\mathcal{D})=0$ shows that $\mathrm{Ext}^2(C,X)=0$, hence $\mathrm{pd}\ C\le 1$.

(b) This follows similarly, using that all the projective modules belong to $\mathcal{C}$. If X is any Λ-module, then ΩX belongs to $\mathcal{C}$, thus $\mathrm{Ext}^2(X,D)\simeq\mathrm{Ext}^1(\Omega X,D)=0$.

6. The structure of ω.

In this section we give the structure of the modules in ω.

Theorem 3. *Let $P\to P_\omega$ be the minimal left ω-approximation of a peg P and let E be the endomorphism ring of P_ω. Then E is a division ring.*

We will denote P_ω by G and call it the *canonical generic* module (or just the generic module). It will turn out to be independent of the choice of the peg P and we will characterize G (up to isomorphism) as the only module in ω with endomorphism ring a division ring.

Proof. Let $G = P_\omega$. Note that G contains P as a submodule and we denote by $p\colon G \to T = G/P$ the projection map. We know that T is a direct sum of Prüfer modules, each occurring with finite multiplicity, see Lemma 6.

Let $f\colon G \to G$ be a nonzero endomorphism. The restriction of f to P is also non-zero, since otherwise f would yield a non-zero map $T \to G$. However G/P is a direct sum of Prüfer modules and G belongs to $\mathcal{F}$. Since $\delta(P) = -1$, we conclude that the restriction of f to P must be a monomorphism, using that the kernel $\operatorname{Ker} f$ and the image $f(P)$ are submodules of G.

Since $P + f(P)$ is a finitely generated submodule of G, there exists a submodule P' of G with $P + f(P) \subseteq P'$ such that P'/P belongs to $\mathbf{t}$. Note that we have $\delta(P') = \delta(P) + \delta(P'/P) = -1$. Since P' is a submodule of G, it has no non-zero submodules of non-negative defect, thus P' has to be indecomposable.

We claim that $P'/f(P)$ belongs to $\mathbf{t}$. The module $P'/f(P)$ has zero defect. If we assume that $P'/f(P)$ does not belong to $\mathbf{t}$, then $P'/f(P)$ has a submodule of positive defect and its inverse image in P' would yield a non-zero submodule of non-negative defect, impossible.

Let $T' = G/P'$, with projection map $p'\colon G \to T'$. The map f induces maps $f'\colon P \to P'$ and $f''\colon T \to T'$, thus we deal with the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "f'"] & G \arrow[r, "p"] \arrow[d, "f"] & T \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & P' \arrow[r] & G \arrow[r, "p'"] & T' \arrow[r] & 0
\end{tikzcd}$$

and the snake lemma yields an exact sequence

$$\operatorname{Ker} f' \to \operatorname{Ker} f \to \operatorname{Ker} f'' \to \operatorname{Cok} f' \to \operatorname{Cok} f \to \operatorname{Cok} f'' \to 0.$$

As we have noted, $\operatorname{Ker} f' = 0$. Since f'' is a map inside $\mathcal{T}$, its kernel and cokernel both belong to $\mathcal{T}$. Also, we have shown that the cokernel of f' belongs to $\mathcal{T}$. This implies that $\operatorname{Ker} f$ and $\operatorname{Cok} f$ belong to $\mathcal{T}$. However G is in $\mathcal{F}$, thus $\operatorname{Ker} f = 0$. This already shows that f is injective. Also, we see that $\operatorname{Ker} f''$ is a submodule of $\operatorname{Cok} f'$, thus of finite length.

We claim that f'' is surjective. Note that $\operatorname{add}\mathbf{t}$ is a direct sum of serial categories $\operatorname{add}\mathbf{t}(\lambda)$, with λ in some index set Ω, and each subcategory $\mathbf{t}(\lambda)$ contains only finitely many isomorphism classes of simple objects. Of course, $\mathcal{T}$ is a corresponding direct sum of categories denoted by $\mathcal{T}(\lambda)$ with $\lambda \in \Omega$. If we decompose T and T' accordingly, we obtain direct summands T_λ of T and T'_λ of T' and f'' maps T_λ into T'_λ. On the one hand, T'_λ is obtained from T_λ by factoring out a subobject in $\mathcal{T}(\lambda)$ of finite length, thus if we write T_λ and T'_λ as direct sums of Prüfer modules, the numbers of direct summands are equal. On the other hand, f'' induces a map $T_\lambda \to T'_\lambda$ with kernel of finite length. Altogether this implies that f'' maps T_λ onto T'_λ. Thus f'' is surjective.

But $\operatorname{Cok} f'' = 0$ implies that $\operatorname{Cok} f$ is of finite length and therefore in $\mathbf{t}$. Since $\operatorname{Hom}(G, \mathbf{t}) = 0$, we see that f itself is surjective.

In this way, we have shown that any non-zero endomorphism of G is invertible, thus the endomorphism ring of G is a division ring.

Corollary 3. *The generic module G is embeddable into a direct sum of Prüfer modules.*

Proof: Let P be a peg. Choose any Prüfer module, say $S[\infty]$ (where S is a simple object of $\mathbf{t}$). We claim: *The module P is embeddable into $S[\infty]$.* Let $f\colon P \to S[\infty]$ be a non-zero homomorphism. If the kernel P' of f is non-zero, then $\delta(P') \le -1$ and $\delta(P) = -1$. Since P/P' is an subobject of some module $S[m]$ in $\mathbf{t}$, we have $\delta(P/P') \le 0$, thus $\delta(P/P') = 0$. Now P/P' is indecomposable, thus an object in $\mathbf{t}$. It follows that P/P' is a subobject of $S[t]$, where t is the τ-period of S. But $\operatorname{Hom}(P, S[t])$ is finite dimensional, whereas $\operatorname{Hom}(P, S[\infty])$ is infinite dimensional. This shows that there are many monomorphisms $P \to S[\infty]$.

Starting with a minimal left ω-approximation of P and a monomorphism $f\colon P \to S[\infty]$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & G & \longrightarrow & T & \longrightarrow & 0 \\
 & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & S[\infty] & \longrightarrow & N & \longrightarrow & T & \longrightarrow & 0
\end{array}$$

Here T and also $S[\infty]$ belong to ω_0. It follows that the lower sequence splits and that $N = T \oplus S[\infty]$ belongs to ω_0. With f also f' is injective, thus G embeds into an object of ω.

Remark. This statement is quite surprising already in the case of the Kronecker algebra Λ (this is the path algebra of the quiver with two vertices, say a and b and two arrows starting at b and ending in a) where the module category $\operatorname{Mod}\Lambda$ shows a strong resemblance to the category of all abelian groups (or better the category of all $k[T]$-modules), with the canonical generic Λ-module G corresponding to $\mathbb{Q}$ and the Prüfer modules corresponding to the Prüfer groups. Of course, in sharp contrast to the embedding of G into a direct sum of Prüfer modules, there does not exist any embedding of $\mathbb{Q}$ into a direct sum of Prüfer groups!

Theorem 4. *Any module in ω is a direct sum of Prüfer modules and of copies of the generic module.*

Proof: Let M be a module in ω. According to Lemma 7, there is a submodule U which is a direct sum of copies of P such that M/U is a direct sum of Prüfer modules. Let $U \to U_\omega$ be the minimal left ω-approximation of U. We obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & U_\omega & \longrightarrow & T & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & U & \longrightarrow & M & \longrightarrow & M/U & \longrightarrow & 0
\end{array}$$

The snake lemma yields an isomorphism of the kernel of f and the kernel of g, as well as an isomorphism of the cokernel of f and the cokernel of g. Since $g\colon T \to M/U$ is a map in the abelian subcategory $\mathcal{T}$, the kernel and the cokernel of g both belong to $\mathcal{T}$.

Since U belongs to $\mathcal{F}$, also U_ω belongs to $\mathcal{F}$ by Theorem 1. Thus the only subobject of U_ω which belongs to $\mathcal{T}$ is the zero module. This shows that both f and g are monomorphisms.

Any factor module of a module in ω belongs to $\mathcal{D}$. Thus the cokernel N of g belongs to $\mathcal{T} \cap \mathcal{D} \subset \omega$, and is again a direct sum of Prüfer modules. It follows that $\operatorname{Ext}^1(N, U_\omega) = 0$, and thus f splits. This shows that M is the direct sum of U_ω and N. Since U is a direct sum of copies of P, we see that U_ω is a direct sum of copies of the generic module $G = P_\omega$. Thus M is a direct sum of Prüfer modules and of copies of G.

Note that all the indecomposable modules in ω have local endomorphism rings: the endomorphism ring E of the canonical generic module is a division ring (Theorem 3), the endomorphism ring of a Prüfer module is a (not necessarily commutative) discrete valuation ring. As a consequence, the Theorem of Krull-Remak-Schmidt-Azumaya can be used: the direct sum decompositions provided in Theorem 4 are unique up to isomorphisms.

Corollary 4. *The modules in $\mathcal{C}$ are precisely the modules cogenerated by $\mathcal{T}$ and also precisely those modules which can be embedded into a module in ω_0.*

Proof: If a module can be embedded into a module in ω, then it is cogenerated by Prüfer modules, thus cogenerated by $\mathcal{T}$. The modules in $\mathcal{T}$ belong to $\mathcal{C}$ and $\mathcal{C}$ is the torsionfree class of a torsion pair, thus any module cogenerated by modules in $\mathcal{C}$ belongs to $\mathcal{C}$. It remains to be shown that any module M in $\mathcal{C}$ can be embedded into a module from ω_0. Now according to Theorem 1, the module M embeds into $M_\omega \in \omega$, and according to Theorem 4, we know that M_ω is a direct sum of Prüfer modules and of copies of G. We have seen in Corollary 3 that G itself can be embedded into a direct sum of Prüfer modules, thus M can be embedded into a direct sum of Prüfer modules.

Corollary 5. *If M belongs to $\mathcal{F}$, then M_ω is a direct sum of copies of G. If M belongs to $\mathcal{F}$ and has finite length, then M_ω is a finite direct sum of copies of G.*

Only the last assertion has to be shown. However, given a left minimal map $X \to Y$ where X is finitely generated and $Y = \bigoplus_{i\in I} Y_i$ is a direct sum of non-zero modules Y_i, one immediately sees that the index set I has to be finite.

Corollary 6. *The module G has finite length as an E^{op}-module, where E is its endomorphism ring.*

Proof: We may identify G as a vector space with the vector space $\mathrm{Hom}({}_\Lambda\Lambda, G)$, and this is an identification of E^{op}-modules. Let us denote the minimal left ω-approximation of ${}_\Lambda\Lambda$ by N. Then this is a finite direct sum of copies of G, say $N \simeq G^n$ for some n. The approximation map ${}_\Lambda\Lambda \to N$ yields an isomorphism

$$\mathrm{Hom}({}_\Lambda\Lambda, G) \simeq \mathrm{Hom}(N, G) \simeq \mathrm{Hom}(G^n, G) \simeq E^n,$$

and all these isomorphisms are isomorphisms of E^{op}-modules.

7. The ω-resolution of the modules in $\mathcal{D}$.

Using the previous results we can now obtain ω-resolutions for the modules in $\mathcal{D}$.

Theorem 5. *For every Λ-module M, there exists a minimal right ω-approximation $M^\omega \to M$. Its kernel is a direct sum of copies of the generic module. This minimal right ω-approximation is surjective if and only if M belongs to $\mathcal{D}$. If M belongs to $\mathcal{Q}$, then M^ω belongs to ω_0.*

Again, we may reformulate the essential part of the theorem: *For any $M \in \mathcal{Q}$, there is an exact sequence*

$$0 \to V \to M^\omega \xrightarrow{g} M \to 0$$

with $M^\omega \in \omega_0$ and V a direct sum of copies of the generic module, such that g is a minimal right ω-approximation. And we obtain a characterization of the modules in $\mathcal{D}$ as follows: *The modules in $\mathcal{D}$ are the cokernels of monomorphisms in ω.*

Proof. If M belongs to $\mathcal{R}$, then $\mathrm{Hom}(\omega, M) = 0$, thus $0 \to M$ is a minimal right ω-approximation. If M belongs to ω, then the identity map $M \to M$ is a minimal right ω-approximation.

Thus we may restrict to the case where M belongs to $\mathcal{Q}$. We claim that in this case M is generated by a direct sum of Prüfer modules. It is sufficient to show this for a module $Q \in \mathbf{q}$. Since the projective cover of Q belongs to $\mathbf{p}$ and any map from $\mathbf{p}$ to $\mathbf{q}$ factors through $\mathbf{t}$, we only have to show that for S a simple object in $\mathbf{t}$ and any natural number r any map $S[r] \to Q$ can be extended to $S[r+1]$. However, this follows directly from the fact that $S[r+1]/S[r]$ belongs to $\mathbf{t}$ and $\mathrm{Ext}^1(\mathbf{t}, \mathbf{q}) = 0$.

Thus there exists an exact sequence

$$0 \to K \xrightarrow{f} N \xrightarrow{g} M \to 0$$

with $N \in \omega_0$. Let us show that there exists such a sequence where K belongs to $\mathcal{F}$. Without loss of generality, we can assume that the map $f\colon K \to N$ is an inclusion map. Let tK be the maximal submodule of K generated by $\mathcal{T}$. Since it is the image of a map from a module in $\mathcal{T}$ to $N \in \mathcal{T}$, it follows that tK belongs to $\mathcal{T}$. If we factor out tK from K as well as from N, we obtain an exact sequence

$$0 \to K/tK \to N/tK \to M \to 0,$$

where K/tK belongs to $\mathcal{F}$. As a factor module of N inside $\mathcal{T}$, the module N/tK belongs to ω_0. So we can assume that K is in $\mathcal{F}$.

Next, we claim that we even can assume that K is a direct sum of copies of the generic module. Let $h\colon K \to K_\omega$ be the minimal left ω-approximation of K and form the induced exact sequence with respect to h. We obtain a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "h"] & N \arrow[r] \arrow[d, "h'"] & M \arrow[r, "g"] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_\omega \arrow[r, "f'"] & N' \arrow[r, "g'"] & M \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The cokernels of h and h' coincide. Since by Theorem 1, the cokernel N'' of h' belongs to ω_0, then N' is an extension of N and N'' (indeed: a split extension), thus it belongs to ω_0.

Thus, consider now an exact sequence

$$0 \to K \to N \xrightarrow{g} M \to 0$$

where K is a direct sum of copies of the generic module and $N \in \omega_0$. Then the map $N \to M$ is a right ω-approximation, since $\mathrm{Ext}^1(M,K) = 0$ due to the basic splitting theorem. In order to see that g is right minimal, let $\eta\colon N \to N$ be an endomorphism with $g\eta = \eta$, thus we deal with the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & N & \xrightarrow{g} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & N & \xrightarrow{g} & M & \longrightarrow & 0
\end{array}$$

with exact rows. Since K belongs to $\mathrm{Add}\, G$, the same is true for the kernel and the cokernel of η'. Since N belongs to $\mathcal{T}$, the same is true for the kernel and the cokernel of η. Thus $\mathrm{Ker}\,\eta' \simeq \mathrm{Ker}\,\eta$ belongs to $\mathrm{Add}\, G \cap \mathcal{T} = 0$, and also $\mathrm{Cok}\,\eta' \simeq \mathrm{Cok}\,\eta$ belongs to $\mathrm{Add}\, G \cap \mathcal{T} = 0$. This shows that η is an isomorphism.

Corollary 7. *Assume that M belongs to $\mathcal{Q}$ and has finite length. If $g\colon M^\omega \to M$ is a minimal right ω-approximation with kernel V, then V is a finite direct sum of copies of G.*

Proof. It is sufficient to show the following: for every finite length module M, the left E^{op}-module $\mathrm{Ext}^1(M,G)$ has finite length, here E is the endomorphism ring of G. Let $\Omega(M)$ be the kernel of a projective cover of M. Then $\mathrm{Ext}^1(M,G)$ is an epimorphic image of $\mathrm{Hom}(\Omega(M),G)$, thus we want to show that $\mathrm{Hom}(N,G)$ is of finite length as an E^{op}-module, for every Λ-module N of finite length. Choose a free module $F = {}_\Lambda\Lambda^n$ of finite length which maps onto N. Such a map induces an inclusion of $\mathrm{Hom}(N,G)$ into $\mathrm{Hom}(F,G)$, and $\mathrm{Hom}(F,G)$ is isomorphic to G^n as an E^{op}-module. Now use Corollary 6.

As a direct consequence of Theorem 5 we get the following description of $\mathcal{C}$ and $\mathcal{D}$ in terms of ω.

Proposition 2.

$$\mathcal{C} = \{M \mid \mathrm{Ext}^1(M,\omega) = 0\},$$
$$\mathcal{D} = \{M \mid \mathrm{Ext}^1(\omega,M) = 0\},$$

Proof: We show the first equality: The inclusion $\subseteq$ follows from the basic splitting result. For the inclusion $\supseteq$, let $\mathrm{Ext}^1(M,\omega) = 0$. Write $M = N' \oplus N$ with $N' \in \mathcal{C}$ and $N \in \mathcal{Q}$. The minimal right ω-approximation yields an exact sequence $0 \to K \to N^\omega \to N \to 0$ where also K is in ω. Since $\mathrm{Ext}^1(N,\omega) = 0$, we see that N is a direct summand of N^ω, thus N is in $\omega \subseteq \mathcal{C}$.

The proof of the second assertion uses the corresponding (dual) arguments.

Proposition 3. *The class ω consists of the relative injective objects inside $\mathcal{C}$ and of the relative projective objects inside $\mathcal{D}$.* This means:

$$\omega = \{M \in \mathcal{C} \mid \mathrm{Ext}^1(\mathcal{C},M) = 0\} = \{M \in \mathcal{D} \mid \mathrm{Ext}^1(M,\mathcal{D}) = 0\}.$$

Proof: That the modules in ω are relative injective in $\mathcal{C}$ and relative projective in $\mathcal{D}$ follows again from the basic splitting result. Conversely, if $M \in \mathcal{C}$ satisfies $\mathrm{Ext}^1(\mathcal{C},M) = 0$, then we have in particular $\mathrm{Ext}^1(\omega,M) = 0$ and therefore $M \in \mathcal{D}$. But $\mathcal{C} \cap \mathcal{D} = \omega$. In a similar way, one obtains the second equality.

Remark. We should stress that there is an important difference between the ω-coresolutions and the ω-resolutions. As we know, any module M can be written as $M = M_0 \oplus M_1 \oplus M_2$ with $M_0 \in \mathcal{R}$, $M_1 \in \omega$ and M_2 in $\mathcal{Q}$. Non-trivial left ω-approximations do exist for modules in $\mathcal{R}$, non-trivial right ω-approximations for modules in $\mathcal{Q}$. Whereas the minimal left ω-approximation M_ω of a module $M \in \mathcal{R}$ may be an arbitrary module in ω, the minimal right ω-approximation of a module $M \in \mathcal{Q}$ will be a direct sum of Prüfer modules alone.

There is another substantial difference: let us compare the possible minimal left ω-approximations M_ω and minimal right ω-approximations M^ω of finite dimensional modules M. Of course, if M belongs to $\mathbf{t}$, then M_ω is a Prüfer module and $M^\omega = 0$. Consider the remaining indecomposable modules M of finite length. If M belongs to $\mathbf{p}$, then M_ω is a **finite** direct sum of copies of G and $M^\omega = 0$. If M belongs to $\mathbf{q}$, then M^ω is an **infinite** direct sum of Prüfer modules and $M_\omega = 0$. If we take into account the cokernel of the monomorphism $M \to M_\omega$ for $M \in \mathbf{p}$ and the kernel of the epimorphism $M^\omega \to M$ for $M \in \mathbf{q}$, then this strict dichotomy pertains: the cokernel of the monomorphism $M \to M_\omega$ will be an **infinite** direct sum of Prüfer modules, the kernel of the epimorphism $M^\omega \to M$ will be a **finite** direct sum of copies of G. But actually, looking at maps we encounter some astonishing parallelity: it turns out that both the ω-coresolutions of the modules in $\mathbf{p}$ as well as the ω-resolutions of the modules in $\mathbf{q}$, are maps $X \to Y$, where X is a finite direct sum of copies of G and Y is an infinite direct sum of Prüfer modules. For a module M in $\mathbf{p}$, we need an epimorphism of this kind, and M will be the kernel. For a module M in $\mathbf{q}$, we need a monomorphism of this kind, and M will be the cokernel.

Let us consider in detail one special example. Let P be a peg, thus $P_\omega = G$. Consider the ω-coresolution of P

$$0 \to P \to G \to Y \to 0,$$

here $Y = G/P$ is an infinite direct sum of Prüfer modules. Such an embedding of P into G will remind anyone of the embedding of $\mathbb{Z}$ in $\mathbb{Q}$, with $\mathbb{Q}/\mathbb{Z}$ being an infinite direct sum of Prüfer groups. But note that there does not exist any embedding of $\mathbb{Q}$ into a direct sum of Prüfer groups. In contrast, let M be an indecomposable Λ-module in $\mathbf{q}$ with projective dimension $\operatorname{pd} M = 1$ and defect $\delta(M) = -1$. Then it is easy to see (see the proof of Corollary 7) that $\operatorname{Ext}^1(M, G)$ is one-dimensional as an $E^{\rm op}$-space and therefore the ω-resolution of M is of the form

$$0 \to G \to Y' \to M \to 0$$

with Y' a direct sum of Prüfer modules. Of course, Y' has to be an infinite direct sum of Prüfer modules. Thus we obtain an embedding of G into an infinite direct sum of Prüfer modules, and the cokernel is indecomposable and of finite length.

Given an abelian category $\mathcal{A}$, it is quite customary to form the quotient category $\mathcal{A}/\mathcal{A}_0$, where $\mathcal{A}_0$ is the subcategory of all modules of finite length. In our case, we look at the quotient category $\operatorname{Mod}\Lambda/\operatorname{mod}\Lambda$. Note that any finite dimensional module M in $\mathbf{p}$, say with ω-coresolution $M_\omega \to M_\omega/M$, yields an isomorphism between M_ω and M_ω/M in the quotient category $\operatorname{Mod}\Lambda/\operatorname{mod}\Lambda$. Similarly, any finite dimensional module M in $\mathbf{q}$, say with ω-resolution $V \to M^\omega$, yields an isomorphism between V and M^ω in the quotient category $\operatorname{Mod}\Lambda/\operatorname{mod}\Lambda$. In particular, we obtain in this way isomorphisms in the quotient category $\operatorname{Mod}\Lambda/\operatorname{mod}\Lambda$ between G and infinite direct sums of Prüfer modules.

8. Further structure of ω.

In this section we investigate the maps inside ω and the torsion classes in $\operatorname{Mod}\Lambda$ with the property that the indecomposable torsion modules of finite length are just the modules in $\mathbf{t}$.

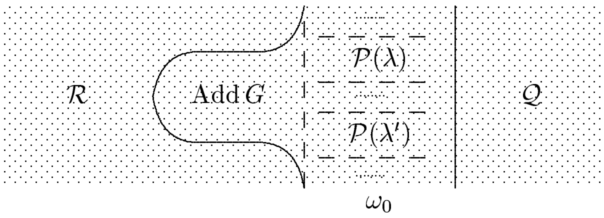
There are no non-zero maps from a Prüfer module to G, since a Prüfer module belongs to $\mathcal{T}$ whereas G belongs to $\mathcal{F}$. On the other hand, *any Prüfer module is generated by G.* Namely, let S be a simple object of $\mathbf{t}$ and $p\colon M \to S[\infty]$ a projective cover. Since M belongs to $\mathcal{F}$, its minimal left ω-approximation M_ω is in $\mathcal{F} \cap \omega = \operatorname{Add} G$. If we factor p through M_ω, we obtain a surjective map $M_\omega \to S[\infty]$. (Actually, the construction of G shows that G maps onto $(\tau^t S)[\infty]$ for some t, but any $(\tau^t S)[\infty]$ maps onto $S[\infty]$.) In this way, we obtain a further characterization of $\mathcal{D}$.

Corollary 8. *We have $\mathcal{D} = g(G)$.*

Proof: Since G belongs to $\mathcal{D}$ and $\mathcal{D}$ is closed under direct sums and factor modules, we see that $g(G) \subseteq \mathcal{D}$. On the other hand, the minimal right ω-approximation of any module in $\mathcal{D}$ is surjective, according to Theorem 5. Thus $\mathcal{D} \subseteq g(\omega)$. But any module in ω is a direct sum of Prüfer modules and copies of G, thus generated by G.

Also, note that $\operatorname{add} \mathbf{t}$ is a direct sum of infinitely many serial categories $\operatorname{add} \mathbf{t}(\lambda)$, with λ in some index set Ω of cardinality $\max(\aleph_0, |k|)$. Of course, $\mathcal{T}$ is a corresponding direct sum of categories denoted $\mathcal{T}(\lambda)$ with $\lambda \in \Omega$. Let us denote by $\mathcal{P}(\lambda)$ the full subcategory of all direct sums of copies of the Prüfer modules belonging to $\mathcal{T}(\lambda)$, for any $\lambda \in \Omega$. Note that for all λ, $\mathcal{T}(\lambda)$ contains only finitely many isomorphism classes of Prüfer modules, and for all but a finite number of λ only one.

Our three part visualization of $\operatorname{Mod} \Lambda$ can be refined accordingly: Recall that the modules in ω are direct sums of a module in $\operatorname{Add} G$ (the full subcategory of all direct sums of copies of G) and a module in ω_0, and we divide ω_0 further into the various full subcategories $\mathcal{P}(\lambda)$.



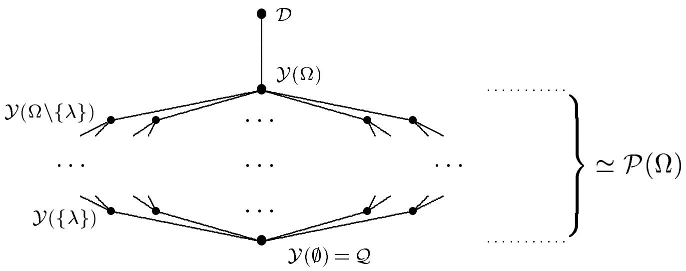
Note that the full subcategory ω_0 is separating in the following sense: First of all, the groups $\operatorname{Hom}(\omega_0, \mathcal{R})$, $\operatorname{Hom}(\omega_0, \operatorname{Add} G)$, $\operatorname{Hom}(\mathcal{Q}, \mathcal{R})$, $\operatorname{Hom}(\mathcal{Q}, \operatorname{Add} G)$, $\operatorname{Hom}(\mathcal{Q}, \omega_0)$ all are zero, and second, any map $h \colon N \to M$ from a module N in $\mathcal{R}$ or $\operatorname{Add} G$ to a module M in $\mathcal{Q}$ factors through a module in ω_0 (namely, take a minimal ω-resolution $0 \to V \to M^\omega \to M \to 0$; since $\operatorname{Ext}^1(N, V) = 0$, the map h factors through M^ω, but M^ω belongs to ω_0). The rather strange shape which we use in order to depict the $\operatorname{Add} G$ part of ω should stress that in contrast to ω_0 which is separating, the subcategory $\operatorname{Add} G$ does not have a corresponding property.

Using any decomposition of Ω as the disjoint union of two subsets Ω_1, Ω_2, we can write $\operatorname{add} \mathbf{t}$ as a product of two categories, and this yields a corresponding decomposition of $\mathcal{T} = \mathcal{T}(\Omega_1) \times \mathcal{T}(\Omega_2)$ as a product of two categories. Any set-theoretical decomposition $\Omega = \Omega_1 \cup \Omega_2$ therefore gives rise to a split torsion pair $(\mathcal{X}(\Omega_1), \mathcal{Y}(\Omega_2))$ in $\operatorname{Mod} \Lambda$: the modules in $\mathcal{X}(\Omega_1)$ are the direct sums $M_1 \oplus M_2$, where M_1 is a module in $\mathcal{R}$ and M_2 is the direct sum of copies of G and of Prüfer modules belonging to $\mathcal{T}(\Omega_1)$, whereas the modules in $\mathcal{Y}$ are the direct sums $M_3 \oplus M_4$, where M_3 is a direct sum of Prüfer modules in $\mathcal{T}(\Omega_2)$ and M_4 is a module in $\mathcal{Q}$, We have the following information on torsion pairs.

Proposition 4. *The only torsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathbf{t} \subset \mathcal{X}$ and $\mathbf{q} \subset \mathcal{Y}$ are $(\mathcal{R}, \mathcal{D})$ and those of the form $(\mathcal{X}(\Omega_1), \mathcal{Y}(\Omega_2))$, where Ω is the disjoint union of Ω_1 and Ω_2. In particular, all torsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathbf{t} \subset \mathcal{X}$ and $\mathbf{q} \subset \mathcal{Y}$ split.*

Proof: Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair with $\mathbf{t} \subset \mathcal{X}$ and $\mathbf{q} \subset \mathcal{Y}$. In case the generic module G belongs to $\mathcal{Y}$, all the Prüfer modules $S[\infty]$ belong to $\mathcal{Y}$, since they are factor modules of G, thus $\mathcal{Y} \supseteq \mathcal{D}$, but then $\mathcal{X} = \mathcal{R}$ and $\mathcal{Y} = \mathcal{D}$. Now, let us assume that G does not belong to $\mathcal{Y}$. Denote by Ω_2 the set of all $\lambda \in \Omega$ such that $\mathcal{T}(\lambda) \cap \mathcal{Y}$ contains a non-zero module, and let $\Omega_1 = \Omega \setminus \Omega_2$. If $\lambda \in \Omega_2$, then $\mathcal{Y}$ contains at least one and thus all the Prüfer modules from $\mathcal{T}(\lambda)$, since all of them are epimorphic images of a given one. It follows that $\mathcal{Y} = \mathcal{Y}(\Omega_2)$ and thus $\mathcal{X} = \mathcal{X}(\Omega_1)$.

The lattice of the subcategories $\mathcal{Y}$, where $(\mathcal{X}, \mathcal{Y})$ is a torsion pair with $\mathbf{t} \subset \mathcal{X}$ and $\mathbf{q} \subset \mathcal{Y}$, looks as follows, where the lower part is order isomorphic to the power set $\mathcal{P}(\Omega)$:



Let us add the following observations which will be useful in section 10.

Lemma 8. *Let Y be in ω. Any monomorphism $X \to Y$ with $X \in \omega_0$ splits. Any epimorphism $Y \to Z$ with $Z \in \operatorname{Add} G$ splits.*

Proof: Decompose $Y = Y_1 \oplus Y_0$ with $Y_1 \in \operatorname{Add} G$ and $Y_0 \in \omega_0$. Now $\operatorname{Hom}(X, Y_1) = 0$, thus any map $X \to Y$ maps into Y_0. But clearly any monomorphism in ω_0 splits. Similarly, $\operatorname{Hom}(Y_0, Z) = 0$, thus any map $Y \to Z$ vanishes on Y_0 and induces an epimorphism $Y_1 \to Z$. But any epimorphism in $\operatorname{Add} G$ splits. This completes the proof.

Lemma 9. *The subcategories $\mathcal{C}, \mathcal{D}$ and ω are closed under products.*

Proof: Since $\omega = \mathcal{C} \cap \mathcal{D}$, we only have to consider $\mathcal{C}$ and $\mathcal{D}$. It is clear that $\mathcal{C}$ is closed under products, since $\mathcal{C}$ is a torsionfree class. For $\mathcal{D}$, we use the description $\mathcal{D} = \{M \mid \operatorname{Ext}^1(\mathbf{t}, M) = 0\}$ in order to see it.

9. Concealed canonical algebras.

Let us outline why the results presented above for the canonical algebras immediately extend to the more general class of concealed canonical algebras using tilting. As we have mentioned above, according to [LP] this takes care of any sincere stable separating tubular family, thus in particular of all the stable separating tubular families of a tubular algebra.

Recall that the concealed canonical algebras are obtained in the following way: we start with a canonical algebra Λ with the canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ and we consider the subcategories $\mathcal{C}, \mathcal{D}, \omega$ as defined above. Let T be a tilting module which belongs to $\mathbf{p}$ and consider $\Lambda' = \operatorname{End}(T)^{\mathrm{op}}$. Then, by definition, Λ' is a concealed canonical algebra. The tilting functor $F = \operatorname{Hom}(T, -)$ sends $\mathbf{t}$ to a sincere stable separating tubular family $\mathbf{t}'$ in $\operatorname{mod} \Lambda'$, and all sincere stable separating tubular families are obtained in this way. The tubular family $\mathbf{t}'$ separates say $\mathbf{p}'$ from $\mathbf{q}'$, here $\mathbf{p}'$ is the image of $\mathbf{p}$ under F, whereas the modules M' in $\mathbf{q}'$ are extensions of the form

$$0 \to F'(M_1) \to M' \to F(M_2) \to 0$$

with $M_1 \in \operatorname{add} \mathbf{p}$, $M_2 \in \operatorname{add} \mathbf{q}$, and $F' = \operatorname{Ext}^1(T, -)$.

As above, we may define

$$\mathcal{C}' = r(\mathbf{q}'),\ \mathcal{Q}' = g(\mathbf{q}'), \quad \text{and} \quad \mathcal{R}' = rl(\mathbf{t}'),\ \mathcal{D}' = l(\mathbf{t}').$$

Of course, we also put $\omega' = \mathcal{C}' \cap \mathcal{D}'$. The generic module G' and the Prüfer modules for Λ' are defined as the images of the corresponding Λ-modules under F. But we may define the Prüfer modules for Λ' also directly as unions of chains of indecomposable modules in $\mathbf{t}'$. And we denote by ω_0' the full subcategory of all direct sums of Prüfer modules.

We claim that the following assertions hold:

(1) *The categories ω and ω' are equivalent categories,* an equivalence being given by the restriction of F, thus any object in ω' is a direct sum of indecomposable objects and the indecomposables in ω' are a generic module and Prüfer modules. And this equivalence yields an equivalence of ω_0 and ω_0'.

(2) *Any Λ'-module is a direct sum of a module in $\mathcal{R}'$, a module in ω' and a module in $\mathcal{Q}'$.*

(3) *For each module C' in $\mathcal{C}'$ there is an exact sequence*

$$0 \to C' \xrightarrow{f} X' \to Y' \to 0$$

with $Y' \in \omega_0'$ and $X' \in \omega'$, where $f \colon C' \to X'$ is a minimal left ω-approximation.

(4) *For each module D' in $\mathcal{D}'$ there is an exact sequence*

$$0 \to X' \to Y' \xrightarrow{g} D' \to 0$$

with X' a direct sum of copies of G' and $Y' \in \omega'$, where $g \colon Y' \to D'$ is a minimal right ω-approximation.

(5) $\operatorname{Ext}^1(\mathcal{C}', \mathcal{D}') = 0$.

(6) $\mathcal{C}' = \{C' \mid \operatorname{Ext}^1(C', \omega') = 0\}$ *and* $\mathcal{D}' = \{D' \mid \operatorname{Ext}^1(\omega', D') = 0\}$ *and* $\omega' = \{B' \in \mathcal{C}' \mid \operatorname{Ext}^1(\mathcal{C}', B') = 0\} = \{A' \in \mathcal{D}' \mid \operatorname{Ext}^1(A', \mathcal{D}') = 0\}$.
(7) $\operatorname{pd} C' \leq 1$ *for any* $C' \in \mathcal{C}'$ *and* $\operatorname{id} D' \leq 1$ *for any* $D' \in \mathcal{D}'$.
(8) *Let* Y' *be in* ω'. *Any monomorphism* $X' \to Y'$ *with* $X' \in \omega'_0$ *splits. Any epimorphism* $Y' \to Z'$ *with* $Z' \in \operatorname{Add} G'$ *splits.*

Proof: We recall that we denote by $r(T)$ the full subcategory of $\operatorname{Mod}\Lambda$ given by the modules M with $\operatorname{Hom}(T, M) = 0$, thus $(r(T), g(T))$ is the torsion pair in $\operatorname{Mod}\Lambda$ attached to the tilting module T. As usual, we also need the cotilting module $T' = F(D\Lambda)$ in $\operatorname{mod}\Lambda'$ and the full subcategory $l(T')$ of all Λ'-modules N with $\operatorname{Hom}(N, T') = 0$, so that $(c(T'), l(T'))$ is the torsion pair attached to the cotilting module T'. Tilting theory asserts that F yields an equivalence between $g(T)$ and $c(T')$ and that F' yields an equivalence between $r(T)$ and $l(T')$, see [CF].

Let us look at the eight assertions. Assertion (1) is obvious, since ω is contained in $g(T)$. In order to verify (2), the essential observation is the following vanishing result

$$\operatorname{Ext}^1(F(\mathcal{C}), F'(r(T))) = 0.$$

This formula may be shown directly, but one also may prefer to work in the bounded derived category $D^b(\operatorname{Mod}\Lambda)$, see [R3]. It follows from the formula that any Λ'-module is the direct sum of a module in $F(\mathcal{C})$ and a module N which has a submodule N' in $F'(r(T))$ such that N/N' belongs to $F(\mathcal{Q})$. Then it is easy to see that $\operatorname{Hom}(N', \mathbf{t}') = 0 = \operatorname{Hom}(N/N', \mathbf{t}')$, so that N' and N/N' are in $\mathcal{D}'$, but clearly not in ω'. Since $\mathbf{q}'$ is numerically determined, $(\mathcal{C}', \mathcal{Q}')$ is a split torsion pair and hence N is in $\mathcal{Q}'$. It follows that $F(\mathcal{C}) = \mathcal{C}'$ by Proposition 1. It remains to show that $F(\mathcal{R}) \subseteq \mathcal{R}'$, or, in other words $\operatorname{Hom}(\mathbf{q}', F(\mathcal{R})) = 0$. Now $\operatorname{Hom}(F(\mathbf{q}), F(\mathcal{R})) = 0$, since $\operatorname{Hom}(\mathbf{q}, \mathcal{R}) = 0$ and also $\operatorname{Hom}(F'(M_1), F(\mathcal{R})) = 0$, for $M_1 \in \operatorname{add}\mathbf{p}$.

For (3) we use that $F\colon \operatorname{Mod}\Lambda \to \operatorname{Mod}\Lambda'$ restricts to an equivalence from $\mathcal{C} \cap g(T)$ to $\mathcal{C}'$, and from ω to ω', together with the corresponding result for $\operatorname{Mod}\Lambda$.

The proof of Theorem 5 generalizes to Λ'-modules. We need only to observe that $\operatorname{Ext}^1(\omega', \omega') = 0$. This follows from $\operatorname{Ext}^1(\omega, \omega) = 0$ by using the inverse equivalence from ω' to ω induced by $T \otimes_{\Lambda'}\colon \operatorname{Mod}\Lambda' \to \operatorname{Mod}\Lambda$. Hence we get (4).

We have already pointed out that the torsion pair $(\mathcal{C}', \mathcal{Q}')$ splits, and it follows from (2) that the pair $(\mathcal{R}', \mathcal{D}')$ splits. Hence we have $\operatorname{Ext}^1(\mathcal{C}', \mathcal{Q}') = 0 = \operatorname{Ext}^1(\mathcal{R}', \mathcal{D}')$. We have also seen that $\operatorname{Ext}^1(\omega', \omega') = 0$.

We claim that the modules in ω' have projective and injective dimension at most one. The Prüfer modules have projective and injective dimension one, since they are direct limits of modules in $\mathbf{t}'$ which have projective and injective dimension one. Applying the exact sequences in (3) and (4) to $R' = \Lambda$ and $Q' = D\Lambda$, we see that also the generic module G' has projective and injective dimension one. Hence the functors $\operatorname{Ext}^1(M', -)$ and $\operatorname{Ext}^1(-, M')$ are right exact for $M' \in \omega'$. Since $\mathcal{C}' = c(\omega')$ by (3) and $\mathcal{D}' = g(\omega')$ by (4), it follows that $\operatorname{Ext}^1(\mathcal{C}', \omega') = 0 = \operatorname{Ext}^1(\omega', \mathcal{D}')$. This proves (5), and (6) now follows easily (see Propositions 2 and 3). For (7), see the proof of Corollary 2, for (8) that of Lemma 8.

10. Inf-tilting and inf-cotilting modules.

In this section we discuss connections with tilting theory, for concealed canonical algebras. A usual (finite length) tilting module yields a torsion pair, but not all torsion pairs are obtained in this way. We are going to show that in our situation some generalization of the concept of a tilting module which allows a tilting module to be of infinite length is very helpful. In order to distinguish this generalization from the traditional notion we refer to these modules as "inf-tilting" and "inf-cotilting" modules.

Up to now, the torsion pairs which we have considered explicitly were torsion pairs in a complete module category $\operatorname{Mod} R$, where R is any ring. Of course, implicitly, we also dealt with torsion pairs in categories of the form $\operatorname{mod}\Lambda$ with Λ an artin algebra. Indeed, the general concept of a torsion pair $(\mathcal{F}, \mathcal{G})$ is defined in an arbitrary abelian category $\mathcal{A}$; one requires that $\operatorname{Hom}(G, F) = 0$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, that $\mathcal{F}$ and $\mathcal{G}$ are closed under isomorphisms and that for every object $A \in \mathcal{A}$ there exists a short exact sequence $0 \to A' \to A \to A'' \to 0$ with $A' \in \mathcal{G}$ and $A'' \in \mathcal{F}$. From now on, we will use the operators $r(-)$, $l(-)$, $g(-)$, $c(-)$ in this more general setting and we hope that this will not lead to any confusion.

Given an artin algebra Λ, torsion pairs in the category $\operatorname{mod}\Lambda$ of modules of finite length occur frequently as being related to a tilting or a cotilting module. Given a tilting module T of projective dimension at most one, the pair $(r(T), g(T))$ in $\operatorname{mod}\Lambda$ is a torsion pair. We say that it is *associated* with the tilting module T. Similarly, given a cotilting module T of injective dimension at most one, the pair $(c(T), l(T))$ in $\operatorname{mod}\Lambda$ is a torsion pair. We say that it is *associated* with the cotilting module T. Starting with tilting or cotilting modules, we obtain in this way many torsion pairs, but there are also interesting torsion pairs $(\mathcal{Y}, \mathcal{X})$ in $\operatorname{mod}\Lambda$ which do not appear in this way. Especially interesting are those where $\mathcal{X}$ contains the injective modules or $\mathcal{Y}$ contains the projective modules. In [HRS], these are called tilting and cotilting torsion pairs respectively, and it is possible to imitate the usual tilting procedure passing from Λ to $\Gamma = \operatorname{End}(T)^{\mathrm{op}}$ by performing tilting with respect to such a torsion pair inside the bounded derived category.

An example of a tilting and cotilting torsion pair in $\operatorname{mod}\Lambda$ not associated with a tilting or cotilting Λ-module is $(\operatorname{add}(\mathbf{p}\cup\mathbf{t}),\ \operatorname{add}\mathbf{q})$, where Λ is a canonical algebra with canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ (or more generally any concealed canonical algebra ...). Let us turn our attention to arbitrary, not necessarily finitely generated modules, and consider the two extremal torsion pairs $(\mathcal{R}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{Q})$ which are extensions to $\operatorname{Mod}\Lambda$ of the torsion pair $(\operatorname{add}(\mathbf{p}\cup\mathbf{t}),\ \operatorname{add}\mathbf{q})$ in $\operatorname{mod}\Lambda$. We claim that these torsion pairs are associated to something like tilting and cotilting modules respectively: we need to work with a generalization of the concept of a tilting or a cotilting module which allows to deal with infinitely generated modules. Let us refer here to Colpi-Trlifaj [CP] where these inf-tilting modules of projective dimension at most 1 have been considered and to Colpi-D'Este-Tonolo [CET] for an investigation of inf-cotilting modules of injective dimension at most 1, but also to [AC] and [ATT].

Definition: Let R be any ring. An R-module W of projective dimension at most one will be called an *inf-tilting* module, provided it satisfies the following two properties: We have $\operatorname{Ext}^1(W, \operatorname{Add} W) = 0$ and there is an exact sequence $0 \to \Lambda \to X \to Y \to 0$ with X and Y in $\operatorname{Add} W$. Dually, an R-module W of injective dimension at most one will be called an *inf-cotilting* module provided $\operatorname{Ext}^1(\operatorname{Prod} W, W) = 0$ and there is an exact sequence $0 \to X' \to Y' \to D\Lambda \to 0$ with X' and Y' in $\operatorname{Prod} W$. Here, $D\Lambda$ denotes the dual of Λ_Λ, this is the minimal injective cogenerator (at least in case Λ is basic).

Now assume again that Λ is an artin algebra and let $\mathbf{t}$ be a sincere stable tubular family in $\operatorname{mod}\Lambda$ separating $\mathbf{p}$ from $\mathbf{q}$ (in particular, Λ is concealed canonical). Let $\mathcal{C} = r(\mathbf{q})$ and $\mathcal{D} = l(\mathbf{t})$. The crucial subcategory to be considered is $\omega = \mathcal{C}\cap\mathcal{D}$. Let W_0 be the direct sum of all the Prüfer modules in ω, one copy from each isomorphism class, and let $W = G \oplus W_0$, where G is the generic module in ω.

Proposition 5. *The module W is an inf-tilting module of projective dimension one, the modules W and W_0 are inf-cotilting modules of injective dimension one and*

$$\omega = \operatorname{Add} W = \operatorname{Prod} W = \operatorname{Prod} W_0.$$

Proof: The following references are all to section 9. The fact that W and W_0 have projective dimension one and injective dimension one follows from (7), since both modules belong to $\omega = \mathcal{C}\cap\mathcal{D}$ (and are neither projective nor injective).

Clearly, $\omega = \operatorname{Add} W$. According to Lemma 9, ω is closed under products, thus we have $\operatorname{Prod} W_0 \subseteq \operatorname{Prod} W \subseteq \omega$. In order to show $\omega \subseteq \operatorname{Prod} W_0$, it is sufficient to verify that any direct sum $\bigoplus_I W$ with I an infinite index set belongs to $\operatorname{Prod} W_0$. Thus, let I be an infinite index set and consider the product $Y = \prod_I W_0$ of I copies of W_0. We know that Y belongs to ω, thus it is a direct sum of copies of the generic module and the Prüfer modules. It is sufficient to show that in such a direct sum decomposition, any indecomposable module occurs with multiplicity at least I. First, consider a Prüfer module P. We have obvious inclusion maps $\bigoplus_I P \to \prod_I P \to \prod_I W_0 = Y$. According to (8), this monomorphism splits. Also, as Krause ([K], see also [R6]) has shown, $\prod_I P$ contains $\bigoplus_I G$ as a direct summand, thus we obtain an epimorphism $Y = \prod_I W_0 \to \prod_I P \to \bigoplus_I G$. We use the second part of (8) in order to conclude that $\bigoplus_I G$ is a direct summand of Y. Altogether we see that $\bigoplus_I W$ is a direct summand of Y.

The remaining assertions now follow easily: Since $\operatorname{Add} W = \omega = \operatorname{Prod} W$, it follows from the basic splitting theorem that $\operatorname{Ext}^1(W, \operatorname{Add} W) = 0 = \operatorname{Ext}^1(\operatorname{Prod} W, W)$ and of course also $\operatorname{Ext}^1(\operatorname{Prod} W_0, W_0) = 0$. Since ${}_\Lambda\Lambda$ belongs to $\mathbf{p} \subseteq \mathcal{C}$, its minimal left ω-approximation yields an exact sequence $0 \to \Lambda \to X \to Y \to 0$ with X and Y in $\omega = \operatorname{Add} W$, see (3). This shows that W is an inf-tilting module of projective dimension one. Dually, the module $D\Lambda$ belongs to $\mathbf{q} \subseteq \mathcal{D}$, thus its minimal right ω-approximation yields an exact sequence $0 \to X' \to Y' \to D\Lambda \to 0$ with X' and Y' in $\omega = \operatorname{Prod} W_0 = \operatorname{Prod} W$, see (4). Thus both W and W_0 are inf-cotilting modules of injective dimension one.

Note that the torsion pair $(\mathcal{R}, \mathcal{D})$ is associated with the inf-tilting module W, since $\mathcal{D} = g(W)$, and $(\mathcal{C}, \mathcal{Q})$ is associated with the inf-cotilting modules W and W_0, since $\mathcal{C} = c(W) = c(W_0)$.

Remark. Note that the torsion pair $(\mathcal{R}, \mathcal{D})$ does not seem to be associated with something like a cotilting module, but all the torsion pairs $(\mathcal{X}(\complement\Omega'), \mathcal{Y}(\Omega'))$ are, where Ω' is a subset of Ω and $\complement\Omega'$ is its complement inside Ω. Namely, *define $T(\Omega')$ as the direct sum of the generic module G, the Prüfer modules $S[\infty]$ with $S \in \mathcal{T}(\Omega')$ and the adic modules $\widehat{S}$ with $S \in \mathcal{T}(\complement\Omega')$. Then $T(\Omega')$ is an inf-cotilting module and* $\mathcal{X}(\complement\Omega') = c(T(\Omega'))$. (The adic module $\widehat{S}$ is the inverse limit of a chain of epimorphisms

$$\cdots \to [n]S \to [n-1]S \to \cdots \to [2]S \to S,$$

where $[n]S$ is the (uniquely determined) module in $\mathbf{t}$ of regular length n which has S as a factor module, see for example [R5].) In the case of a tame hereditary algebra, we may refer to [BK] for a description of all the pure injective cotilting modules.

11. Derived equivalent categories.

In this section we outline the effect of tilting with respect to some of the torsion pairs considered above inside the derived category $D^b(\operatorname{Mod}\Lambda)$.

If R is any ring, let $D^b(\operatorname{Mod} R)$ be its bounded derived category (with shift automorphisms $X \mapsto X[n]$ for all $n \in \mathbb{Z}$ and homology functors $H^n\colon D^b(\operatorname{Mod} R) \to \operatorname{Mod} R$). We always will identify $\operatorname{Mod} R$ with the full subcategory of all objects X in $D^b(\operatorname{Mod} R)$ with $H^i(X) = 0$ for $i \neq 0$. Given a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{Mod} R$, there is an inf-tilting procedure inside $D^b(\operatorname{Mod} R)$ with respect to this torsion pair. It yields a new abelian category $\mathcal{A}$ which is contained in $D^b(\operatorname{Mod} R)$, as follows: $\mathcal{A} = \mathcal{A}(\mathcal{X}, \mathcal{Y})$ is the full subcategory of all objects A of $D^b(\operatorname{Mod} R)$ such that

$$H^{-1}(A) \in \mathcal{Y}, \quad H^0(A) \in \mathcal{X} \quad \text{and} \quad H^i(A) = 0 \text{ for } i \notin \{-1, 0\}$$

(see [HRS]). Under the condition that $\mathcal{X}$ contains all projective R-modules or that $\mathcal{Y}$ contains all injective R-modules, it follows that $\mathcal{A}$ is derived equivalent to $\operatorname{Mod} R$ and that $(\mathcal{Y}[-1], \mathcal{X})$ is a torsion pair in $\mathcal{A}$. In case the torsion pair $(\mathcal{X}, \mathcal{Y})$ is split with $\operatorname{pd} X \leq 1$ for X in $\mathcal{X}$, and if $\mathcal{X}$ contains all the projectives, then the new abelian category is hereditary (see [HR] and [HRS]).

Let us now assume again that Λ is a canonical algebra with a stable tubular family $\mathbf{t}$ separating $\mathbf{p}$ from $\mathbf{q}$. We consider the subcategories $\mathcal{C}, \mathcal{D}, \omega$, and so on, as defined above, relative to $\mathbf{t}$. In particular, let us look at some of the torsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathbf{t} \subset \mathcal{X}$ and $\mathbf{q} \subset \mathcal{Y}$. It is interesting to observe that some of the exact sequences in $\operatorname{Mod}\Lambda$ which have been discussed in this paper, can be interpreted as injective or projective resolutions in the new abelian category $\mathcal{A}$, depending on the choice of the torsion pair $(\mathcal{X}, \mathcal{Y})$ and that the generic module G and the Prüfer modules yield enough injective or projective objects in $\mathcal{A}$.

Proposition 6. *The category $\mathcal{A} = \mathcal{A}(\mathcal{C}, \mathcal{Q})$ is a hereditary abelian category derived equivalent to* $\operatorname{Mod}\Lambda$. *The pair $(\mathcal{Q}[-1], \mathcal{C})$ is a torsion pair in $\mathcal{A}$. The subcategory ω is the class of all injective objects in $\mathcal{A}$ and $\mathcal{A}$ has sufficiently many injective objects.*

Proposition 7. *The category $\mathcal{A}' = \mathcal{A}(\mathcal{R}, \mathcal{D})$ is a hereditary abelian category derived equivalent to* $\operatorname{Mod}\Lambda$. *The pair $(\mathcal{D}[-1], \mathcal{R})$ is a torsion pair in $\mathcal{A}'$. The subcategory $\omega[-1]$ is the class of all projective objects in $\mathcal{A}'$ and $\mathcal{A}'$ has sufficiently many projective objects.*

Proposition 8. *The category $\mathcal{A}'' = \mathcal{A}(\mathcal{X}(\emptyset), \mathcal{Y}(\Omega))$ is a hereditary abelian category derived equivalent to* $\operatorname{Mod}\Lambda$. *The pair $(\mathcal{Y}(\Omega)[-1], \mathcal{X}(\emptyset))$ is a torsion pair in $\mathcal{A}''$. If P is a Prüfer module in* $\operatorname{Mod}\Lambda$, *then $P[-1]$ is an indecomposable projective object of $\mathcal{A}$, whereas the generic module G, considered as an object in $\mathcal{A}$ is simple injective. Thus, $\mathcal{A}''$ has non-zero projective and non-zero injective objects, but neither sufficiently many projective objects nor sufficiently many injective objects.*

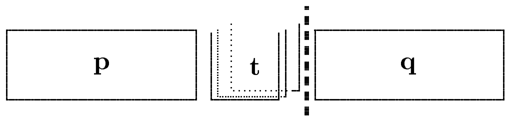
The proofs of these propositions follow quite easily from the above remarks and the properties of torsion pairs in question which have been established in previous sections.

Note that in the opposite direction it is shown in [L] that generic modules over canonical algebras can be investigated by first considering generic sheaves in the category of quasicoherent sheaves over weighted projective lines.

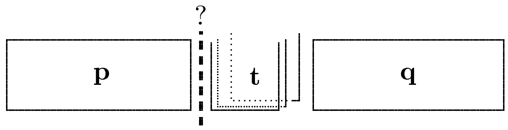
12. Additional comments.

In this section we discuss the relationship between split torsion pairs of $\operatorname{mod}\Lambda$ and $\operatorname{Mod}\Lambda$. We also indicate briefly a different approach to the study of $\operatorname{Mod}\Lambda$ when Λ is a canonical algebra, using tame bimodules.

In this paper we have considered in detail the cut of $\operatorname{mod}\Lambda$ between $\mathbf{t}$ and $\mathbf{q}$, where $\mathbf{t}$ is a sincere stable tubular family separating $\mathbf{p}$ from $\mathbf{q}$:



One may try to look also at the dual cut between $\mathbf{p}$ and $\mathbf{t}$:



The situation seems to be similar, but it is not! There is a dual cut only when dealing with finite dimensional representations — the behavior of the infinite dimensional modules in this part of the category $\operatorname{Mod}\Lambda$ is far more complicated: indeed, there do exist many torsion pairs $(\mathcal{X},\mathcal{Y})$ in $\operatorname{Mod}\Lambda$ with $\mathbf{p}\subset\mathcal{X}$ and $\mathbf{t}\subset\mathcal{Y}$ which do not split (indeed, we do not know any one which splits).

In order to provide at least one example, let us consider again the special case where Λ is the Kronecker algebra, thus we consider the representations of the quiver

$$a \circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ\, b$$

We consider the torsion pair $(\mathcal{X},\mathcal{Y})$, where $\mathcal{X}=r(\mathbf{t})$ and $\mathcal{Y}=g(\mathbf{t})$. Note that the full subcategory $\mathcal{M}$ of all representations V with $V(\beta)$ being an identity map is isomorphic to the category of $k[T]$-modules (an isomorphism is obtained by sending the $k[T]$-module M to the representation V_M with $V_M(a)=V_M(b)=M$, such that $V_M(\alpha)$ is the multiplication by T and $V_M(\beta)=1_M$). It is quite obvious that the restriction of the torsion pair $(\mathcal{X},\mathcal{Y})$ to $\mathcal{M}$ is just the usual pair of torsionfree and torsion $k[T]$-modules and it is well-known that there do exist many $k[T]$-modules whose torsion submodule does not split off (see [F], Chapter XIV).

Given such a trisection $(\mathbf{p},\mathbf{t},\mathbf{q})$ of $\operatorname{mod}\Lambda$, the difference of the two cuts between $\mathbf{p}$ and $\mathbf{t}$ on one hand and between $\mathbf{t}$ and $\mathbf{q}$ on the other hand should not prevent a detailed study of what lies inbetween $\mathbf{p}$ and $\mathbf{q}$ in $\operatorname{Mod}\Lambda$, namely the subcategory $\mathcal{M}(\mathbf{t})=l(\mathbf{p})\cap r(\mathbf{q})$. For the case of a tubular algebra Λ and its various trisections $(\mathbf{p}_w,\mathbf{t}_w,\mathbf{q}_w)$, with $w\in Q_0^\infty$, the subcategory $\mathcal{M}(\mathbf{t}_w)$ can be denoted just by $\mathcal{M}(w)$. This subcategory, as well as corresponding subcategories $\mathcal{M}(w)$ for $w\in\mathbb{R}_0^\infty\setminus Q_0^\infty$ will be studied in the final section of the paper.

Since most of the results in this paper have been shown in [R1] for tame hereditary algebras, it would seem reasonable to establish the general case by reducing the investigation of the module theory for canonical algebras to that of tame bimodules, where the corresponding assertions are already known. This definitely can be done. Let $P_0=S$ be the simple projective module and P_∞ the projective cover of the simple injective module S'. If $P=P_0\oplus P_\infty$, then basic properties of the functor $F=\operatorname{Hom}(P,\ ):\operatorname{Mod}\Lambda\to\operatorname{Mod}\Lambda_0$ for $\Lambda_1=\operatorname{End}(P)^{\mathrm{op}}$ were investigated in [R4] for finitely generated modules and can be generalized without problems to arbitrary modules. One considers the modules C in $\operatorname{Mod}\Lambda$ with $F(C') \neq 0$ for each nonzero summand C' of C, and one can compare crucial properties for C and $F(C)$ in this case. This allows to use results for $\operatorname{Mod}\Lambda_1$ and transfer them to $\operatorname{Mod}\Lambda$. Note that Λ_1 is given by a tame bimodule, namely by $\operatorname{Hom}(P_0, P_\infty)$, so that all the relevant properties of the category $\operatorname{Mod}\Lambda_1$ are known for a long time. In particular, in this way the canonical generic module does not have to be constructed from scratch for all the canonical algebras, but only for the tame bimodules involved. On the other hand, we hope that the direct approach presented in this paper helps to trace in which way the structure of the category of finite dimensional representations determines that of all the modules.

13. Tubular algebras

Let Λ be a canonical algebra with canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$. Let S be the simple projective module and S' the simple injective module. We denote by Λ_0 the factor algebra of Λ so that $\operatorname{mod}\Lambda_0$ is the full subcategory of all Λ-modules M with $[M : S'] = 0$. Similarly, Λ_∞ is the factor algebra of Λ with the property that $\operatorname{mod}\Lambda_\infty$ is the full subcategory of all Λ-modules M with $[M : S] = 0$. Note that both Λ_0 and Λ_∞ are hereditary algebras. The representation types of Λ_0 and of Λ_∞ coincide and determine the representation type of Λ. Let us review the different cases.

If Λ_0 and Λ_∞ are of finite representation type, then Λ is a tame concealed algebra, $\mathbf{p}$ is a preprojective component, $\mathbf{q}$ a preinjective component. This is essentially the case which has been studied in detail in [R1]. In particular, it has been shown there that $\mathcal{Q} = \operatorname{Add}\mathbf{q}$, whereas $\mathcal{R}$ is a wild category (see also [R7]). In this case, the asymmetry between $\mathcal{R}$ and $\mathcal{Q}$ is best visible.

If Λ_0 and Λ_∞ are of wild representation type, then Λ is of course also of wild representation type. In this case, not much is known even for the finite dimensional Λ-modules, but it should be worthwhile to study this case in more detail in future.

It remains to consider the case where both Λ_0 and Λ_∞ are of tame representation type, in this case Λ is said to a *tubular canonical* algebra. More generally, we may consider an arbitrary *tubular* algebra, these are the concealed canonical algebras obtained from a tubular canonical algebra by tilting.

¿From now on, let Λ be a tubular algebra. The structure of $\operatorname{mod}\Lambda$ is known in detail (see [R2] and [LP]). There is a preprojective component $\mathbf{p}_0$ and a preinjective component $\mathbf{q}_\infty$. We denote by I_0 the ideal which is maximal with the property that it annihilates all the modules in $\mathbf{p}_0$ and by I_∞ the ideal which is maximal with the property that it annihilates all the modules in $\mathbf{q}_\infty$. Then we obtain factor algebras $\Lambda_0 = \Lambda/I_0$ and $\Lambda_\infty = \Lambda/I_\infty$ which both are tame concealed algebras (in the case of a tubular canonical algebra, we recover the factor algebras already introduced). Let $\mathbf{t}_0$ be the Auslander-Reiten components of $\operatorname{mod}\Lambda$ which contain regular Λ_0-modules, and $\mathbf{t}_\infty$ those which contain regular Λ_∞-modules. Then both $\mathbf{t}_0$ and $\mathbf{t}_\infty$ are sincere separating tubular families, but both are not stable ($\mathbf{t}_0$ will contain indecomposable projective modules, $\mathbf{t}_\infty$ indecomposable injective ones). If we denote by $\mathbf{q}_0$ the indecomposable modules in $\operatorname{mod}\Lambda$ which do not belong to $\mathbf{p}_0$ or $\mathbf{t}_0$, then $\mathbf{t}_0$ separates $\mathbf{p}_0$ from $\mathbf{q}_0$. If we denote by $\mathbf{p}_\infty$ the indecomposable modules in $\operatorname{mod}\Lambda$ which do not belong to $\mathbf{t}_\infty$ or $\mathbf{q}_\infty$, then $\mathbf{t}_\infty$ separates $\mathbf{p}_\infty$ from $\mathbf{q}_\infty$. The modules in $\mathbf{q}_0 \cap \mathbf{p}_\infty$ fall into a countable number of sincere stable separating tubular families $\mathbf{t}_\alpha$, indexed by $\alpha \in \mathbb{Q}^+$, such that for $\alpha < \beta$ in $\mathbb{Q}^+$ the class $\mathbf{t}_\alpha$ generates $\mathbf{t}_\beta$, and also $\mathbf{t}_\alpha$ is cogenerated by $\mathbf{t}_\beta$. More generally, this generation and cogeneration property holds for all $\alpha < \beta$ in $\mathbb{Q}_0^\infty = \mathbb{Q}^+ \cup \{0, \infty\}$.

Let $\mathbb{R}_0^\infty = \mathbb{R}^+ \cup \{0, \infty\}$. For any $w \in \mathbb{R}_0^\infty$, we denote by $\mathbf{p}_w$ the modules which belong to $\mathbf{p}_0$ or to some $\mathbf{t}_\alpha$ with $\alpha < w$, and we denote by $\mathbf{q}_w$ the modules which belong to $\mathbf{t}_\gamma$ with $w < \gamma$ or to $\mathbf{q}_\infty$ (here, α, γ belong to $\mathbb{Q}_0^\infty$). For $\beta \in \mathbb{Q}_0^\infty$ we obtain in this way a trisection $(\mathbf{p}_\beta, \mathbf{t}_\beta, \mathbf{q}_\beta)$ of $\operatorname{mod}\Lambda$, with $\mathbf{t}_\beta$ a tubular family which separates $\mathbf{p}_\beta$ from $\mathbf{q}_\beta$, and $\mathbf{t}_\beta$ is stable provided $0 < \beta < \infty$. For $w \in \mathbb{R}_0^\infty \setminus \mathbb{Q}_0^\infty$, the two module classes $\mathbf{p}_w$ and $\mathbf{q}_w$ comprise all the indecomposables from $\operatorname{mod}\Lambda$.

Let us turn our attention now to arbitrary, not necessarily finite dimensional modules. For any $w \in \mathbb{R}_0^\infty$, let $\mathcal{C}_w = r(\mathbf{q}_w)$ and $\mathcal{B}_w = l(\mathbf{p}_w)$. The subcategories we are interested in are those of the form

$$\mathcal{M}(w) = \mathcal{C}_w \cap \mathcal{B}_w = r(\mathbf{q}_w) \cap l(\mathbf{p}_w),$$

defined for any $w \in \mathbb{R}_0^\infty$. The modules in $\mathcal{M}(w)$ are said to have *slope* w. Of course, for $\alpha \in \mathbb{Q}_0^\infty$ the modules in $\mathbf{t}_\alpha$ as well as those in ω_α have slope α. For non-rational w, examples of modules in $\mathcal{M}(w)$ will be presented at the end of the section.

Theorem 6. *Any indecomposable Λ-module which does not belong to $\mathbf{p}_0$ or $\mathbf{q}_\infty$ has a slope. For $0 \le w < w' \le \infty$, we have* $\mathrm{Hom}(\mathcal{M}(w'), \mathcal{M}(w)) = 0$.

Note that the second assertion immediately implies that $\mathcal{M}(w) \cap \mathcal{M}(w') = 0$, thus *if a module has a slope, its slope is a well-defined element of $\mathbb{R}_0^\infty$.*

Before we start with the proof, let us analyze the two module classes $\mathcal{C}_w$ and $\mathcal{B}_w$, as well as related ones.

The torsion pair $(\mathcal{C}_w, \mathcal{Q}_w)$. First, we consider $\mathcal{C}_w = r(\mathbf{q}_w)$. Let $\mathcal{Q}_w = g(\mathbf{q}_w)$. Note that $\mathbf{q}_w$ is always closed under successors, thus $(\mathcal{C}_w, \mathcal{Q}_w)$ is a torsion pair, according to Lemma 3 and Lemma 4. *For $\beta \in \mathbb{Q}_0^\infty$, the torsion pair $(\mathcal{C}_\beta, \mathcal{Q}_\beta)$ is split.* Proof: For $\beta \in \mathbb{Q}^+$ and for $\beta = \infty$, the module class $\mathbf{q}_\beta$ is numerically determined, thus we can use Proposition 1. The class $\mathbf{q}_0$ is never numerically determined, but it is at least numerically almost determined (the corresponding function δ vanishes precisely on those modules in $\mathbf{q}_0$ which do not have any simple Λ_0-module as composition factor, but there are only finitely many isomorphism classes of indecomposable modules of this kind). Thus we can use the Remark at the end of section 1.

The subcategories ω_β for $\beta \in \mathbb{Q}^+$. For $\beta \in \mathbb{Q}^+$, the trisection $(\mathbf{p}_\beta, \mathbf{t}_\beta, \mathbf{q}_\beta)$ allows us to use all the previous results of the paper. In particular, there is a corresponding subcategory ω_β containing a generic module G_β as well as Prüfer modules. Actually, there are generic modules G_β also for $\beta \in \{0, \infty\}$, thus for all $\beta \in \mathbb{Q}_0^\infty$; namely, G_0 is the generic module of Λ_0, and similarly, G_∞ is the generic module of Λ_∞. According to Corollary 8, we have $l(\mathbf{t}_\beta) = g(G_\beta)$, for $\beta \in \mathbb{Q}^+$, and this also holds for $\beta = \infty$ (but not for $\beta = 0$; in order to show that $l(\mathbf{t}_\infty) = g(G_\infty)$, one first should notice that both classes are contained in $\mathrm{Mod}\,\Lambda_\infty$ and then use Corollary 8 for the unique separating tubular family of $\mathrm{mod}\,\Lambda_\infty$).

Lemma 10. *Let $\alpha < \beta$ in $\mathbb{Q}_0^\infty$. Then $\mathbf{t}_\alpha$ generates G_β. If in addition $0 < \alpha$, then G_α generates $\mathbf{t}_\beta$.*

Proof. In order to show that $\mathbf{t}_\alpha$ generates G_β for all $0 \le \alpha < \beta \le \infty$, consider first the case $0 < \alpha$.

First, we claim that G_β cannot belong to $\mathcal{C}_\alpha$. Choose γ with $\alpha < \gamma < \beta$ and take an indecomposable module M in $\mathbf{t}_\gamma$. If $\beta < \infty$, consider the left ω_β-approximation $M \to M_{\omega_\beta}$. Since M_{ω_β} is a non-zero direct sum of copies of G_β, there are non-zero maps $M \to G_\beta$. Consider now the case of $\beta = \infty$. Since Λ is a coray coextension of the tame concealed algebra Λ_∞, the trisection $(\mathbf{p}_\infty, \mathbf{t}_\infty, \mathbf{q}_\infty)$ is obtained as follows: $\mathbf{q}_\infty$ consists of the preinjective Λ_∞-modules, whereas $\mathbf{p}_\infty$ consists of all those indecomposable Λ-modules N whose restriction $N^{(\infty)}$ to Λ_∞ is a direct sum of preprojective Λ_∞-modules. Of course, $N^{(\infty)}$ is the maximal factor module of N which is a Λ_∞-module. Note that $N^{(\infty)} = 0$ only for finitely many Λ-modules and all of them belong to $\mathbf{p}_0$. Since the module M belongs to $\mathbf{t}_\gamma$, and $0 < \gamma < \infty$, we see that $M^{(\infty)}$ is a non-zero preprojective Λ_∞-module and therefore $\mathrm{Hom}(M^{(\infty)}, G_\infty) \neq 0$. Since $M^{(\infty)}$ is a factor module of M, we conclude that $\mathrm{Hom}(M, G_\infty) \neq 0$. Always, $M \in \mathbf{q}_\alpha$, thus we see that G_β can not belong to $\mathcal{C}_\alpha$.

Since G_β cannot belong to $\mathcal{C}_\alpha$, and G_β is indecomposable, it belongs to $\mathcal{Q}_\alpha$. This shows that there is a direct sum $\bigoplus_{i \in I} M_i$ of modules $M_i \in \mathbf{q}_\alpha$ which maps onto G_β. However, the projective cover $P(M_i) \to M_i$ factors through $\mathrm{add}\,\mathbf{t}_\alpha$, thus we see that any M_i is generated by $\mathbf{t}_\alpha$. This shows that G_β is generated by $\mathbf{t}_\alpha$.

If $\alpha = 0$, then choose $0 < \alpha' < \beta$. By the previous considerations, G_β is generated by $\mathbf{t}_{\alpha'}$. Since $\mathbf{t}_{\alpha'}$ is generated by $\mathbf{t}_0$, we conclude that G_β is generated by $\mathbf{t}_0$.

In order to show the second assertion, note that we deal with $0 < \alpha < \beta \le \infty$. Now $\mathbf{t}_\beta \subset l(\mathbf{t}_\alpha) = g(G_\alpha)$.

Remarks. The first assertion of Lemma 10 can be strengthened as follows: *If $\alpha < \beta$ in $\mathbb{Q}_0^\infty$ and $\lambda \in \Omega_\alpha$, then the class $\mathbf{t}_\alpha(\lambda)$ generates G_β* (here Ω_α is the index set for the tubular family $\mathbf{t}_\alpha$). This follows from the proof, but can be derived also from the statement itself: Let $\alpha < \alpha' < \beta$. Then Lemma 10 asserts that $\mathbf{t}_{\alpha'}$ generates G_β, but it is well-known that any $\mathbf{t}_\alpha(\lambda)$ generates $\mathbf{t}_{\alpha'}$.

Also, let us stress that G_0 does not generate $\mathbf{t}_\beta$ for any $\beta \in \mathbb{Q}_0^\infty$, since G_0 is a Λ_0-module, whereas all the $\mathbf{t}_\beta$ contain modules which are not Λ_0-modules. If we denote by P the direct sum of all indecomposable projective modules in $\mathbf{t}_0$, then $G_0 \oplus P$ generates $\mathbf{t}_\beta$, for any $0 < \beta$.

The module class $\mathcal{B}_w$. For any $w \in \mathbb{R}_0^\infty$, we have defined $\mathcal{B}_w = l(\mathbf{p}_w)$. By definition, this is the torsion class of a torsion pair, the corresponding torsionfree class is $r(\mathcal{B}_w) = rl(\mathbf{p}_w)$. For $w = 0$, the module class $\mathcal{B}_w$ consists of all the modules M which do not have an indecomposable direct summand in $\mathbf{p}_0$.

Lemma 11. *Let $w \in \mathbb{R}^+ \cup \{\infty\}$, then*

$$\begin{aligned}\mathcal{B}_w &= \bigcap\nolimits_{v<w} \mathcal{Q}_v \\ &= \{M \mid M \text{ is generated by } t_\alpha \text{ for any } \alpha \in \mathbb{Q} \text{ with } 0 < \alpha < w\} \\ &= \{M \mid M \text{ is generated by } G_\alpha \text{ for any } \alpha \in \mathbb{Q} \text{ with } 0 < \alpha < w\},\end{aligned}$$

here the v are non-negative real numbers, but it is sufficient to form the intersection using just a sequence of real numbers $v < w$ which converges to w; similarly, in the last two descriptions, it is sufficient to consider a sequence of rational numbers $\alpha < w$ which converges to w.

Proof: The second equality of these different descriptions of $\mathcal{B}_w$ is straight forward, the last one follows immediately from Lemma 10. Let us show that $l(\mathbf{p}_w) = \bigcap_{v<w} \mathcal{Q}_v$. First, assume that M belongs to the intersection. Let N be in $\mathbf{p}_w$. We want to show that $\operatorname{Hom}(M, N) = 0$. There is a rational α with $0 < \alpha < w$ such that N belongs to $\mathbf{p}_\alpha$. Since M is generated by $\mathbf{t}_\alpha$ and $\operatorname{Hom}(\mathbf{t}_\alpha, \mathbf{p}_\alpha) = 0$, it follows that $\operatorname{Hom}(M, N) = 0$. Conversely, assume that M belongs to $l(\mathbf{p}_w)$. Take a rational α with $0 < \alpha < w$. We want to show that M is generated by $\mathbf{t}_\alpha$. Choose β rational with $\alpha < \beta < w$. Then $\mathbf{t}_\beta \subset \mathbf{p}_w$, thus $l(\mathbf{p}_w) \subseteq l(\mathbf{t}_\beta) = g(G_\beta)$. And $g(G_\beta) \subset g(\mathbf{t}_\alpha)$, according to Lemma 10. This shows that M is generated by $\mathbf{t}_\alpha$.

Proof of Theorem 6. For the second assertion, we only note that $\mathcal{M}(w) \subseteq \mathcal{C}_w$ and that $\mathcal{M}(w') \subseteq \mathcal{B}_{w'} \subseteq \mathcal{Q}_w$, since $w < w'$.

For the first assertion, let M be any indecomposable module which does not belong to $\mathbf{p}_0$ or $\mathbf{q}_\infty$. Since M is indecomposable and does not belong to $\mathbf{q}_\infty$, we have $\operatorname{Hom}(\mathbf{q}_\infty, M) = 0$. Let w be the infimum of all $\alpha \in \mathbb{Q}_0^\infty$ such that $\operatorname{Hom}(\mathbf{q}_\alpha, M) = 0$. Since $\mathbf{q}_w = \bigcup_{w<\alpha} \mathbf{q}_\alpha$, it follows that $\operatorname{Hom}(\mathbf{q}_w, M) = 0$, thus M belongs to $\mathcal{C}_w$. It remains to be shown that M also belongs to $\mathcal{B}_w$. For $w = 0$, this follows immediately from our assumption that M is indecomposable and does not belong to $\mathbf{p}_0$. Thus, let $w > 0$. We have to show that M belongs to $\mathcal{Q}_\alpha$ for any rational number α with $0 < \alpha < w$. Take such an α and assume that M does not belong to $\mathcal{Q}_\alpha$. Since $(\mathcal{C}_\alpha, \mathcal{Q}_\alpha)$ is a split torsion pair and M is indecomposable, we conclude that M belongs to $\mathcal{C}_\alpha$. Thus $\operatorname{Hom}(\mathbf{q}_\alpha, M) = 0$. But by the definition of w this implies that $w \leq \alpha$, a contradiction.

We add a further property of $\mathcal{M}(w)$ which is quite useful to know:

Proposition 9. *The subcategories $\mathcal{C}_w, \mathcal{B}_w$ and $\mathcal{M}(w)$ are closed under products and direct limits.*

We only have to consider the first two subcategories. Now $\mathcal{C}_w$ is the torsionfree class of a torsion pair, thus closed under products. Also, since $\mathcal{C}_w = r(\mathbf{q}_w)$, and $\mathbf{q}_w$ consists of finitely generated modules, we see that $\mathcal{C}_w$ is closed under direct limits.

Consider now $\mathcal{B}_w$. All the subcategories $\mathcal{Q}_w = g(\mathbf{q}_w)$ are closed under direct limits, thus the same is true for $\mathcal{B}_w$. It remains to be seen that $\mathcal{B}_w$ is closed under products. Assume that there are given modules $M_i \in \mathcal{B}_w$ and let $M = \prod_{i\in I} M_i$.

Consider first the case $w > 0$. Choose some $0 < \beta < w$ in $\mathbb{Q}$. Then all the modules M_i are generated by G_β, thus there exist epimorphisms $\bigoplus_{I(i)} G_\beta \to M_i$ for some index set $I(i)$. But $\operatorname{Add} G_\beta$ is closed under products, thus $\prod_{I(i)} G_\beta$ maps onto $\bigoplus_{I(i)} G_\beta$ and therefore $\prod_i \prod_{I(i)} G_\beta$ maps onto $\prod_i M(i)$. Again using that $\operatorname{Add} G_\beta$ is closed under products, we see that M is generated by G_β.

We proceed quite similarly for $w = 0$. Write ${}_\Lambda\Lambda = P \oplus P'$, where P belongs to $\mathbf{p}_0$ and P' to $\mathbf{t}_0$ (thus $P = {}_\Lambda\Lambda_0$). Since M_i does not split off any indecomposable module from $\mathbf{p}_0$, any map $P \to M_i$ can be factored through a module in $\operatorname{add} \tau^{-t} P$, for any $t \in \mathbb{N}_0$. It follows that for any $t \in \mathbb{N}_0$, all the modules M_i are generated by $P_t = \tau^{-t} P \oplus P'$. This module P_t is a finite dimensional module, thus the products of copies of P_t are direct sums of copies of P_t. This shows that M itself is generated by P_t. Now consider an indecomposable module N from $\mathbf{p}_0$. There is $t \in \mathbb{N}_0$ with $\operatorname{Hom}(P_t, N) = 0$ and this implies that $\operatorname{Hom}(M, N) = 0$. This shows that M cannot split off a copy of N, as we had to show.

Remark. Note that in contrast to $\mathcal{B}_w$, the subcategories $\mathcal{Q}_w$ are **not** closed under products.

Examples. For $\alpha \in \mathbb{Q}_0^\infty$, examples of modules in $\mathcal{M}(\alpha)$ have been mentioned above. Let us now consider the case of an arbitrary $w \in \mathbb{R}^+$. In case w is not rational, $\mathcal{M}(w)$ cannot contain any non-zero module of finite length. We are going to provide two recipes for constructing non-zero modules in $\mathcal{M}(w)$.

The first construction: *Let $\alpha_1 > \alpha_2 > \dots$ be a sequence of rational numbers converging to w and choose modules $M_i \in \operatorname{add} \mathbf{t}_{\alpha_i}$. Then $\prod_i M_i / \bigoplus_i M_i$ belongs to $\mathcal{M}(w)$.* Proof: Let $M = \prod_i M_i$, and $M' = \bigoplus_i M_i$. Let us show that the maximal submodule of M which belongs to $\mathcal{Q}_w$ is M'. On the one hand, M' is generated by $\mathbf{q}_w$. On the other hand, given an indecomposable module $N \in \mathbf{q}_w$ and a non-zero map $f\colon N \to M$, then N belongs to some $\mathbf{t}_\beta$ with $\beta > w$. Since the sequence $(\alpha_i)_i$ converges to w, there is some natural number n with $\alpha_i < \beta$ for all $i > n$. Thus the image of f is contained in $\prod_{i \le n} M_i \subseteq M'$. Since $(\mathcal{C}_w, \mathcal{Q}_w)$ is a torsion pair, it follows that M/M' belongs to $\mathcal{C}_w$. In addition, we have to show that M/M' belongs to $\mathcal{B}_w$. Since $\mathcal{B}_w$ is closed under products, M belongs to $\mathcal{B}_w$. But $\mathcal{B}_w = l(\mathbf{p}_w)$ is closed under factor modules, thus with M also M/M' belongs to $\mathcal{B}_w$.

The second construction: *Let $\alpha_1 < \alpha_2 < \dots$ be a sequence of rational numbers converging to w and choose modules $M_i \in \operatorname{add} \mathbf{t}_{\alpha_i}$ with inclusions $M_1 \subseteq M_2 \subseteq \dots$. Then the direct limit $\varinjlim M_i$ belongs to $\mathcal{M}(w)$.* Proof: All the modules M_i belong to $\mathcal{C}_w$ and $\mathcal{C}_w$ is closed under direct limits, therefore $M = \varinjlim M_i$ belongs to $\mathcal{C}_w$. Consider a rational number α where $0 < \alpha < w$. There exists i with $\alpha < \alpha_i$. Then M_j belongs to $\operatorname{add} \mathbf{q}_\alpha$ for all $j \ge i$. This shows that M is generated by $\mathbf{q}_\alpha$, and therefore M belongs to $\mathcal{Q}_\alpha$. As a consequence, M belongs to $\mathcal{B}_w$.

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