MODULES OF FINITE PROJECTIVE DIMENSION FOR STANDARDLY STRATIFIED ALGEBRAS

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Introduction

Contravariantly finite resolving subcategories of the category $\operatorname{mod}\Lambda$ of finitely generated (left) Λ-modules have played an important role in the representation theory of artin algebras. There are interesting connections with tilting theory and filtrations of modules (see [4]).

A central subcategory of $\operatorname{mod}\Lambda$ for an artin algebra Λ is the category $\mathcal{P}^{<\infty}(\Lambda)$ of modules of finite projective dimension. This is always a resolving subcategory, but it is in general not contravariantly finite [14]. When it is, the finitistic dimension of Λ is known to be finite [4], and hence sufficient conditions for $\mathcal{P}^{<\infty}(\Lambda)$ to be contravariantly finite are of interest. This is known to be true for algebras stably equivalent to hereditary algebras [4], for split left serial algebras over a field [7] and for a class of weakly triangular algebras [9].

Let $P_1, \dots, P_n$ be an ordered sequence of the nonisomorphic indecomposable projective modules over an artin algebra Λ, and denote by Δ_i the largest factor module of P_i with composition factors only amongst $S_1, \dots, S_i$, where S_j is the simple top of P_j. Denote by $\mathcal{F}(\Delta)$, or $\mathcal{F}_\Lambda(\Delta)$, the full subcategory of $\operatorname{mod}\Lambda$ consisting of the Λ-modules having a Δ-filtration, that is, a filtration with the Δ_i as "composition factors." The subcategory $\mathcal{F}(\Delta)$ is known to be contravariantly finite [15]. We want to get sufficient conditions for $\mathcal{P}^{<\infty}(\Lambda)$ to be contravariantly finite by finding conditions for $\mathcal{P}^{<\infty}(\Lambda)$ to coincide with $\mathcal{F}(\Delta)$. This leads us naturally to the standardly stratified algebras [8] (see also [10], [1]), which by definition are those with $\Lambda \in \mathcal{F}(\Delta)$, for some ordering of the indecomposable projective modules. This condition is clearly necessary in order to have $\mathcal{P}^{<\infty}(\Lambda) = \mathcal{F}(\Delta)$. This class of algebras is a natural generalization of the quasihereditary algebras introduced in [8], where it in addition is assumed that each $\operatorname{End}_\Lambda(\Delta_i)$ is a division algebra.

Our work was motivated by the work on weakly triangular algebras in [9], where Λ is said to be weakly triangular if there is an ordering $P_1, \dots, P_n$ of the nonisomorphic indecomposable projective Λ-modules such that $\operatorname{Hom}_\Lambda(P_i, P_j) = 0$ for $i < j$. Under an additional assumption, which by [13] is equivalent to, in our terminology, that each Δ_i has finite projective dimension, it is shown in [9] that $\mathcal{F}(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

For an artin algebra Λ it is known from the work of Butler and Auslander that the relations for the Grothendieck group are generated by almost split sequences if and only if Λ is of finite representation type. This suggests the question of whether there is an analogous result for any subcategory having almost split sequences, for example for functorially finite and extension closed subcategories [6]. When Λ is standardly stratified, we prove such a result for $\mathcal{F}(\Delta)$, generalizing work from [13]. In particular, in view of the previous results in this paper, we get sufficient conditions for the analogue of the Butler–Auslander Theorem to hold for $\mathcal{P}^{<\infty}(\Lambda)$.

The paper is organized as follows. In Section 1 we give some preliminary results on standardly stratified algebras, including known results from the literature. We observe that $\mathcal{F}(\Delta) \subset \mathcal{P}^{<\infty}(\Lambda)$ for a standardly stratified algebra Λ, and deduce from this, like for quasihereditary algebras [15], that there is always an associated tilting module. In Section 2 we start with a standardly stratified algebra Λ with ordering $P_1, \dots, P_n$ of the indecomposable projective Λ-modules. Then $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$, where $P = P_2 \coprod \cdots \coprod P_n$, is again standardly stratified. When $\mathcal{F}_\Gamma(\Delta) = \mathcal{P}^{<\infty}(\Gamma)$ we give necessary and sufficient conditions for having $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$. On the basis of this we get sufficient conditions, in terms of quivers with relations, for $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$ when Λ is standardly stratified. We give examples to illustrate, and we see in particular that the main theorem of [9] can be obtained as a consequence. The result on Grothendieck groups is given in Section 3. For this we need to investigate the Δ-socle for the indecomposable summands of T, where the results of [15] extend to the more general setting.

Some results have been obtained independently in [2], in particular Theorem 1.4.

1 Preliminaries

In this section we recall some relevant definitions and results from the literature, and give the connection with tilting modules for standardly stratified algebras.

Let Λ be an artin algebra. Recall that a full subcategory $\mathcal{X}$ of the finitely generated Λ-modules $\operatorname{mod}\Lambda$ closed under isomorphisms and summands is contravariantly finite in $\operatorname{mod}\Lambda$ if for each C in $\operatorname{mod}\Lambda$ there is a map $f\colon X_C \to C$ with X_C in $\mathcal{X}$, such that for any map $g\colon X \to C$ with X in $\mathcal{X}$ there is some $h\colon X \to X_C$ with $fh = g$ [6]. There is the dual concept of a covariantly finite subcategory, and a subcategory is said to be functorially finite if it is both contravariantly and covariantly finite [6]. The subcategory $\mathcal{X}$ is said to be resolving if it contains the projectives and is closed under extensions and kernels of epimorphisms (and there is the dual concept of coresolving subcategory).

We start by recalling the following [15], [11].

Proposition 1.1. *Let Λ be an artin algebra with an ordering $P_1, \dots, P_n$ of the nonisomorphic indecomposable projective Λ-modules, and let $\mathcal{F}(\Delta)$ be the Λ-modules with Δ-filtration.*

Then $\operatorname{Ext}^1_\Lambda(\Delta_j, \Delta_i) = 0$ for $j \geq i$, and $\mathcal{F}(\Delta)$ is a functorially finite subcategory closed under extensions and kernels of epimorphisms.

Note that in view of 1.1, Λ is standardly stratified if and only if $\mathcal{F}(\Delta)$ is resolving.

We shall also need the following essentially known result (see [15], [11]).

Proposition 1.2. *Let Λ be a standardly stratified algebra. Then we have the following.*

(a) For M in $\mathcal{F}(\Delta)$, any Δ-composition series can be rearranged, with the same Δ-composition factors, to get $0 = M_{t+1} \subset M_t \subset \cdots \subset M_2 \subset M_1 =$

M such that $M_i/M_{i+1} \simeq \Delta_{k_i}$, with $k_1 \leq \cdots \leq k_t$.

(b) *The number of times Δ_1 occurs as a Δ-composition factor in a Δ-composition series for M is the multiplicity of S_1 in M/rM.*

(c) *The multiplicity of Δ_i as a Δ-composition factor in a Δ-composition series for $M \in \mathcal{F}(\Delta)$ is independent of the choice of the series.*

(d) *Δ_i occurs exactly once as a Δ-composition factor of P_i, and for all other Δ_j we have $j > i$.*

(e) *$\mathrm{Ext}^2_\Lambda(\Delta_j, \Delta_i) = 0$ for $j \geq i$.*

Proof. (a) This follows by using $\mathrm{Ext}^1_\Lambda(\Delta_j, \Delta_i) = 0$ for $j \geq i$ (see [11]).

(b) By (a) we have $N \subset M$ with $M/N \simeq \Delta_1^s$ for some $s \geq 0$, and Δ_1 not a Δ-composition factor of N. Then the multiplicity of P_1 in the projective cover of M is clearly s, and we are done.

(c) This follows using (a) and (b).

(d) This is obvious from $\mathrm{Ext}^1_\Lambda(\Delta_i, \Delta_j) = 0$ for $j \geq i$ and the above.

(e) This follows from (d) and 1.1.

□

One of the aims in this paper is to find sufficient conditions for $\mathcal{F}(\Delta)$ to coincide with $\mathcal{P}^{<\infty}(\Lambda)$. For this it is useful to note that we always have the following.

Proposition 1.3. *Let Λ be a standardly stratified algebra. Then we have $\mathcal{F}(\Delta) \subset \mathcal{P}^{<\infty}(\Lambda)$.*

Proof. Let $P_1, \ldots, P_n$ be the ordering of the nonisomorphic indecomposable projective modules. Since P_n has a Δ-filtration, we must have $P_n = \Delta_n$. For $i < n$ consider the exact sequence $0 \to K_i \to P_i \to \Delta_i \to 0$. Assume by induction that $\mathrm{pd}_\Lambda \Delta_j < \infty$ for $j > i$. Then $\mathrm{pd}_\Lambda K_i < \infty$ by 1.2(d), and hence $\mathrm{pd}_\Lambda \Delta_i < \infty$. It follows that each module with a Δ-filtration has finite projective dimension.

□

An important property of quasihereditary algebras established in [15] is that there is a natural tilting module associated with $\mathcal{F}(\Delta)$. This generalizes to standardly stratified algebras. But first we must introduce some more terminology, notation and results.

For a full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, denote by $\mathcal{C}^{\perp}$ the full subcategory with objects $\{\, C; \operatorname{Ext}^i_\Lambda(\mathcal{C}, C) = 0 \text{ for } i > 0 \,\}$, and by ${}^{\perp}\mathcal{C}$ the one with objects $\{\, C; \operatorname{Ext}^i_\Lambda(C, \mathcal{C}) = 0 \text{ for } i > 0 \,\}$. Further, $\hat{\mathcal{C}}$ has as objects the B for which there is an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to B \to 0$ with $C_i \in \mathcal{C}$, and $\check{\mathcal{C}}$ is defined dually. When $\mathcal{X}$ is contravariantly finite resolving, then $\mathcal{Y} = \mathcal{X}^{\perp}$ is covariantly finite coresolving and ${}^{\perp}\mathcal{Y} = \mathcal{X}$, and $\mathcal{X} \cap \mathcal{Y}$ clearly gives the Ext-injective objects in $\mathcal{X}$ and the Ext-projective objects in $\mathcal{Y}$ [4]. A Λ-module T is a tilting module if $\operatorname{pd}_\Lambda T < \infty$, $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for $i > 0$ and there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with each T_i in the category $\operatorname{add} T$ whose objects are the summands of finite direct sums of copies of T.

The assignment $\mathcal{X} \mapsto \mathcal{X} \cap \mathcal{X}^{\perp} = \{\text{Ext-injectives in } \mathcal{X}\}$ gives a one-one correspondence between contravariantly finite resolving subcategories $\mathcal{X}$ of $\operatorname{mod}\Lambda$, with $\mathcal{X} \subset \mathcal{P}^{<\infty}(\Lambda)$ and (basic) tilting modules T: Given such an $\mathcal{X}$, then $\mathcal{X} \cap \mathcal{X}^{\perp} = \operatorname{add} T$ for a tilting module T. If conversely T is a tilting module, we associate $\mathcal{X} = \check{\operatorname{add}}\, T$ with T, and then $\mathcal{X}^{\perp} = T^{\perp}$ [4]. Combining these facts with 1.3, we obtain the following.

Theorem 1.4. *Let Λ be a standardly stratified algebra and $\mathcal{F}(\Delta)$ the modules with Δ-filtration.*

There is a (basic) tilting module T associated with $\mathcal{F}(\Delta)$ such that $\operatorname{add} T = \mathcal{F}(\Delta) \cap \mathcal{F}(\Delta)^{\perp}$ *is the* Ext*-injective objects in $\mathcal{F}(\Delta)$ (and the* Ext*-projective objects in $\mathcal{F}(\Delta)^{\perp}$), and we have $\mathcal{F}(\Delta) = \check{\operatorname{add}}\, T (\subset {}^{\perp} T)$ and $\mathcal{F}(\Delta)^{\perp} = T^{\perp}$.*

We can also give another description of $\mathcal{F}(\Delta)$, showing the relationship to ${}^{\perp}T$.

Proposition 1.5. *Let Λ be a standardly stratified algebra and $\mathcal{F}(\Delta)$ the modules with Δ-filtration. Then we have*

$$\mathcal{F}(\Delta) = {}^{\perp} T \cap \mathcal{P}^{<\infty}(\Lambda) = {}^{\perp} T \cap \{\, C; \operatorname{pd}_\Lambda C \leq \operatorname{pd}_\Lambda T \,\}.$$

Proof. In [12, 2.1] it is proved that if $\mathcal{C}$ is a resolving subcategory of $\mathcal{P}^{\leq t}(\Lambda)$ ($= \{C; \operatorname{pd}_\Lambda C \leq t\}$) for some t, and $\mathcal{C}$ contains a tilting module which is Ext-injective in $\mathcal{C}$, then $\mathcal{C}$ is contravariantly finite in $\operatorname{mod}\Lambda$. Using this, we see that the subcategories ${}^{\perp}T \cap \mathcal{P}^{\leq t}(\Lambda)$ for each $t \geq \operatorname{pd}_\Lambda T$ are contravariantly

finite resolving subcategories of $\operatorname{mod}\Lambda$ contained in $\mathcal{P}^{<\infty}(\Lambda)$, with $\operatorname{add}T$ the Ext-injective objects. By the unique correspondence cited above, using that $\mathcal{F}(\Delta)$ is also a contravariantly finite resolving subcategory of $\operatorname{mod}\Lambda$ contained in $\mathcal{P}^{<\infty}(\Lambda)$, we have $\mathcal{F}(\Delta) =$
${}^{\perp}T \cap \mathcal{P}^{\leq t}(\Lambda)$ for any $t \geq \operatorname{pd}_\Lambda T$. It follows that $\mathcal{F}(\Delta) = {}^{\perp}T \cap \mathcal{P}^{<\infty}(\Lambda)$.

□

Note that we do not in general have $\mathcal{F}(\Delta) = {}^{\perp}T$ for a standardly stratified algebra Λ. For example, if T is a cotilting module (which by definition is the dual of a tilting module) and $\operatorname{gl.dim}\Lambda = \infty$, then $\widehat{{}^{\perp}T} = \operatorname{mod}\Lambda$ [4, 5.5], whereas $\widehat{\mathcal{F}(\Delta)} \subset \mathcal{P}^{<\infty}(\Lambda) \subsetneq \operatorname{mod}\Lambda$. Hence we have $\mathcal{F}(\Delta) \neq {}^{\perp}T$. A selfinjective nonsimple local algebra Λ with $T = \Lambda$ gives a concrete example. If $\operatorname{gl.dim}\Lambda < \infty$, the tilting module T is a cotilting module, and ${}^{\perp}T$ is contravariantly finite resolving, and must hence coincide with $\mathcal{F}(\Delta)$ [4].

2 Relationship between $\mathcal{F}_\Lambda(\Delta)$ and $\mathcal{P}^{<\infty}(\Lambda)$

Let again Λ be an artin algebra and $P_1, \dots, P_n$ a fixed ordering of the nonisomorphic indecomposable projective Λ-modules. Let $P = P_2 \coprod \cdots \coprod P_n$ and $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$. We first observe that if Λ is standardly stratified, then so is Γ, and in this case we compare the categories $\mathcal{F}_\Lambda(\Delta)$ and $\mathcal{F}_\Gamma(\Delta)$. We use this to find conditions for $\mathcal{F}_\Lambda(\Delta)$ to coincide with $\mathcal{P}^{<\infty}(\Lambda)$, under the assumption $\mathcal{F}_\Gamma(\Delta) = \mathcal{P}^{<\infty}(\Gamma)$. Such a result is useful in view of the fact that when Λ has only one simple module up to isomorphism, then we have $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda) = \operatorname{add}\Lambda$. We use these results of a theoretical nature to formulate sufficient conditions, in terms of quivers with relations, for a standardly stratified algebra Λ to have the property that $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

In the above notation, we start with the comparison between modules with Δ-filtration for Λ and Γ. We denote by $\operatorname{mod}P$ the full subcategory of $\operatorname{mod}\Lambda$ whose objects C are such that there is a projective presentation $Q_1 \to Q_0 \to C \to 0$ with Q_0 and Q_1 in $\operatorname{add}P$. Then the restriction functor $(P, \ )|_{\operatorname{mod}P}\colon \operatorname{mod}P \to \operatorname{mod}\Lambda$ is known to be an equivalence of categories.

Proposition 2.1. *Let Λ be a standardly stratified algebra, and let $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ be as above. Then we have the following.*

(a) Γ is standardly stratified, with $\overline{\Delta_i} = (P, \Delta_i)$ for $i \geq 2$ as the associated Δ's.

(b) If M is in $\mathcal{F}_\Lambda(\Delta)$, then (P,M) is in $\mathcal{F}_\Gamma(\Delta)$.

(c) If M is in $\operatorname{mod} P$, then (P,M) is in $\mathcal{F}_\Gamma(\Delta)$ if and only if M is in $\mathcal{F}_\Lambda(\Delta)$.

Proof. (a) Consider for $i>1$ the exact sequences $0\to K\to P_i\to\Delta_i\to 0$ and $0\to(P,K)\to(P,P_i)\to(P,\Delta_i)\to 0$. Since the Δ-composition factors of K are Δ_j for $j>i$, the indecomposable summands of the projective cover of (P,K) are of the form (P,P_j) for $j>i$. This shows that $\overline{\Delta_i}$ for Γ is (P,Δ_i), for each $i>1$. Then clearly Γ is standardly stratified when Λ is.

(b) This is a direct consequence of (a).

(c) Assume that M is in $\operatorname{mod} P$, and assume first that $(P,M)\simeq(P,\Delta_i)$ for some $i\geq 2$. It follows from 1.2 (d) that Δ_i is in $\operatorname{mod} P$, and since also M is in $\operatorname{mod} P$, we get $M\simeq\Delta_i$, so that $M\in\mathcal{F}(\Delta)$.

Assume now that (P,M) has a Δ-filtration in $\operatorname{mod}\Gamma$ of Δ-length greater than one. Then we have an exact sequence $0\to(P,N)\to(P,M)\to(P,\Delta_i)\to 0$, with $i\geq 2$, where (P,N) has a Δ-filtration and we can assume that N is in $\operatorname{mod} P$. By induction on the Δ-length, we conclude that N has a Δ-filtration.

Consider the sequence of maps $N\xrightarrow{f}M\xrightarrow{g}\Delta_i$, which gives rise to the exact sequence $0\to(P,N)\to(P,M)\to(P,\Delta_i)\to 0$. Since N is in $\operatorname{mod} P$, we clearly have $gf=0$ since $(P,g)(P,f)=0$. Now $K=\operatorname{Ker} f$ and $\operatorname{Coker} g$ both must have only S_1's as composition factor, and hence $\operatorname{Coker} g=0$ since $\operatorname{Coker} g$ is a factor of Δ_i with $i>1$. Also $H=\operatorname{Ker} g/\operatorname{Im} f$ can have only S_1's as composition factor. Then the exact sequence $0\to H\to M/\operatorname{Im} f\to\Delta_i\to 0$ shows that since M is in $\operatorname{mod} P$, then $M/\operatorname{Im} f$ is a factor of P_i. Then H must be 0 by the definition of Δ_i.

We now have an exact sequence $0\to K\to N\to M\to\Delta_i\to 0$. From the exact sequence $0\to K\to N\to N/K\to 0$ we obtain the exact sequence $\operatorname{Ext}^1_\Lambda(\Delta_i,N)\xrightarrow{\alpha}\operatorname{Ext}^1_\Lambda(\Delta_i,N/K)\to\operatorname{Ext}^2_\Lambda(\Delta_i,K)$. Since the only composition factors of K are S_1's, it follows easily from 1.2 (e) that $\operatorname{Ext}^2_\Lambda(\Delta_i,K)$ is 0. Since then α is an epimorphism, there is some commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & \Delta_i \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & N/K \arrow[r] & M \arrow[r] & \Delta_i \arrow[r] & 0
\end{tikzcd}$$

for some M' in $\operatorname{mod}\Lambda$. This gives rise to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (P,N) \arrow[r] \arrow[d] & (P,M') \arrow[r] \arrow[d] & (P,\Delta_i) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & (P,N/K) \arrow[r] & (P,M) \arrow[r] & (P,\Delta_i) \arrow[r] & 0
\end{tikzcd}$$

where $(P,M') \to (P,M)$ must be an isomorphism since $(P,N) \to (P,N/K)$ is an isomorphism. Since N is in $\operatorname{mod}P$ by assumption, and Δ_i is in $\operatorname{mod}P$, then M' is in $\operatorname{mod}P$. It then follows that M is isomorphic to M'. Since N has a Δ-filtration, so does M, and the proof is finished.

□

Note that if Q is an arbitrary projective module for a standardly stratified algebra Λ, it is not necessarily true that $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ is again standardly stratified. For example, let the indecomposable projective Λ-modules have the following structure

$$P_1 = \begin{tikzcd}[row sep=small] S_1 \arrow[d, dash] \\ S_2 \arrow[d, dash] \\ S_1 \end{tikzcd} \qquad P_2 = \begin{tikzcd}[row sep=small, column sep=small] & S_2 \arrow[dl, dash] \arrow[dr, dash] & \\ S_1 \arrow[dr, dash] & & S_3 \arrow[dl, dash] \\ & S_2 & \end{tikzcd} \qquad P_3 = \begin{tikzcd}[row sep=small] S_3 \arrow[d, dash] \\ S_2 \end{tikzcd}$$

Then we have $\Delta_1 = S_1$, $\Delta_2 = \begin{matrix} S_2 \\ S_1 \end{matrix}$ and $\Delta_3 = P_3$, and $\operatorname{End}_\Lambda(P_1 \bigoplus P_2)^{\mathrm{op}}$ is easily seen not to be standardly stratified.

We next want to compare $\mathcal{F}_\Gamma(\Delta) = \mathcal{P}^{<\infty}(\Gamma)$ and $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$ with Λ and Γ as above. For this the following preliminary results will be useful.

Lemma 2.2. *Let Λ be a standardly stratified algebra and $\mathcal{F}_\Lambda(\Delta)$ as before. If X is in $\mathcal{F}_\Lambda(\Delta)$ and $\Delta_1 \subseteq rX$, where r is the radical of Λ, then $X/\Delta_1 \notin \mathcal{F}_\Lambda(\Delta)$.*

Proof. If $\Delta_1 \subseteq rX$,then X and X/Δ_1 have the same projective cover. If $X/\Delta_1 \in \mathcal{F}_\Lambda(\Delta)$, then by 1.2 (b) the multiplicity of Δ_1 as a Δ-composition factor of X would be the same as for X/Δ_1, which gives a contradiction. Hence we conclude that $X/\Delta_1 \notin \mathcal{F}_\Lambda(\Delta)$. □

Lemma 2.3. *Let Λ be a standardly stratified algebra. Write as before $P = P_2 \oplus \cdots \oplus P_n$, $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ and let $0 \to K \to Q \xrightarrow{f} M \to 0$ be an exact sequence in $\operatorname{mod}\Lambda$, with Q projective. If S_1, but not Δ_1, is a factor of M, then S_1 is a factor of K.*

Proof. We have the exact sequence $K/\tau_P(K) \to Q/\tau_P(Q) \xrightarrow{\overline{f}} M/\tau_P(M) \to 0$ using that $\tau_P(X) = \tau_P(\Lambda)X$ (see [3]), and $Q/\tau_P(Q) \simeq \Delta_1^t$ for some $t \geq 0$. Since Δ_1 is not a factor of M, $\overline{f}$ is not an isomorphism. Since then $K/\tau_P(K) \neq 0$, and S_1 is the only possible composition factor, we are done. $\square$

We are now ready to give a criterion for $\mathcal{P}^{<\infty}(\Lambda) = \mathcal{F}_\Lambda(\Delta)$.

Theorem 2.4. *Let Λ be a standardly stratified algebra.*

(a) If $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$, then $\Delta_1 \not\subseteq rX$ for all X in $\mathcal{P}^{<\infty}(\Lambda)$.

(b) If $\mathcal{F}_\Gamma(\Delta) = \mathcal{P}^{<\infty}(\Gamma)$ and $\Delta_1 \not\subseteq \underline{r}X$ for all X in $\mathcal{P}^{<\infty}(\Lambda)$, then $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

Proof. (a) Assume that $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$ and $\Delta_1 \subseteq rX$ for some X in $\mathcal{P}^{<\infty}(\Lambda)$. Then $X \in \mathcal{F}_\Lambda(\Delta)$, and hence $X/\Delta_1 \notin \mathcal{F}_\Lambda(\Delta)$ by 2.2. Since clearly $\mathrm{pd}_\Lambda(X/\Delta_1) < \infty$, we have a contradiction, showing that $\Delta_1 \not\subseteq rX$ for all $X \in \mathcal{P}^{<\infty}(\Lambda)$.

(b) Assume that $\mathcal{F}_\Gamma(\Delta) = \mathcal{P}^{<\infty}(\Gamma)$ and that $\Delta_1 \not\subseteq rX$ for all $X \in \mathcal{P}^{<\infty}(\Lambda)$. Then assume that there is some M in $\mathrm{mod}\,\Lambda$ with $M \in \mathcal{P}^{<\infty}(\Lambda) \setminus \mathcal{F}_\Lambda(\Delta)$. We choose M of minimal length with this property. Then no Δ_i, in particular Δ_1, is a factor of M. Let $0 \to Q_r \to \cdots \to Q_1 \to Q_0 \to M \to 0$ be a minimal projective resolution of M.

Assume first that each Q_i is in $\mathrm{add}\,P$, where $P = P_2 \coprod \cdots \coprod P_n$. Then we have $\mathrm{pd}_\Gamma(P, M) < \infty$, and hence by our hypothesis $(P, M) \in \mathcal{F}_\Gamma(\Delta)$. Since M is in $\mathrm{mod}\,P$, it follows from 2.1 that M is in $\mathcal{F}_\Lambda(\Delta)$, which is a contradiction.

Assume now that P_1 is a summand of some Q_i with $i \geq 0$. We know that Δ_i is not a factor of M. Also Δ_1 is not a factor of $\Omega^j M$ for $j > 0$. For if it was, we would have $\Delta_1 \simeq \Omega^j M/J$ for some $J \subset \Omega^j M$, and hence $\Delta_1 \subset r(Q_{j-1}/J)$. This is impossible under our assumption, since clearly $\mathrm{pd}_\Lambda(Q_{j-1}/J) < \infty$. We have that S_1 is a factor of $\Omega^i M$, but since Δ_1 is not a factor of $\Omega^i M$, it follows from 2.3 that S_1 is a factor of $\Omega^{i+1} M$. Continuing we get that S_1 is a factor of $Q_r = \Omega^r M$, which is a contradiction since Δ_1 is not a factor of Q_r. $\square$

Assume again that $\mathcal{P}^{<\infty}(\Gamma) = \mathcal{F}_\Gamma(\Delta)$, with the previous notation. When Λ is given by a quiver with relations over a field k, we can use 2.4 to formulate sufficient conditions on the quiver with relations in order to have $\mathcal{P}^{<\infty}(\Lambda) = \mathcal{F}_\Lambda(\Delta)$. The condition $\Delta_1 \not\subseteq rX$ for all X in $\mathrm{mod}\,\Lambda$ is clearly sufficient. In particular, it is sufficient that there is a nonzero element in Δ_1 which is sent to zero for any map $f\colon \Delta_1 \to rX$. For a map $f\colon \Delta_1 \to rX$ we can choose a map $\tilde{f}\colon P_1 \to P_X$ inducing f, where P_X is the projective cover of X. Then $\operatorname{Im}\tilde{f}$ is contained in rP_X. Any map $P_1 \to rP_i$ is given by a k-linear combination of nontrivial paths from i to 1. We then have the following sufficient condition for $\mathcal{P}^{<\infty}(\Lambda)$ to coincide with $\mathcal{F}_\Lambda(\Delta)$ in terms of quivers with relations, given that $\mathcal{P}^{<\infty}(\Gamma)$ is equal to $\mathcal{F}_\Gamma(\Delta)$: "There is a path (or a k-linear combination of paths) δ starting at the vertex 1 such that the image $\overline{\delta}$ in Δ_1 of the element $\delta \in P_1$ is nonzero, and such that for each nontrivial path μ ending at the vertex 1 we have $\delta\mu = 0$."

Note that saying $\bar{\delta} \neq 0$ is equivalent to saying that δ can not be written as a k-linear combination of paths which all go through some vertex different from 1 (in the quiver with relations).

Putting things together we get the following.

Theorem 2.5. *Let Λ be a standardly stratified algebra given by a quiver with relations over a field k, with vertices $1, \dots, n$ corresponding to the indecomposable projective modules $P_1, \dots, P_n$.*

Assume that for each i there is a path (or a k-linear combination of paths) δ_i starting and ending at i such that $\overline{\delta_i} \neq 0$ in Δ_i, and such that $\delta_i\mu = 0$ for all nontrivial paths μ starting at some $j \geq i$. Then we have $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

Proof. We write $\Gamma_i = \operatorname{End}(P_i \oplus P_{i+1} \oplus \cdots \oplus P_n)^{op}$. Clearly $\mathcal{P}^{<\infty}(\Gamma_n)$ and $\mathcal{F}_{\Gamma_n}(\Delta)$ both consist of the projective Γ_n-modules. Let $i < n$, and assume we have proved that $\mathcal{P}^{<\infty}(\Gamma_{i+1}) = \mathcal{F}_{\Gamma_{i+1}}(\Delta)$. By assumption there is a path, or a k-linear combination of paths $\delta_i \in P_i$ starting and ending at i, such that the image $\overline{\delta}_i$ in Δ_i is nonzero, and such that $\delta_i\mu = 0$ for all nontrivial paths μ starting at some $j \geq i$. Considering the induced elements in the quiver for Γ_i, we can apply the above remarks to conclude $\mathcal{P}^{<\infty}(\Gamma_i) = \mathcal{F}_{\Gamma_i}(\Delta)$. Hence we are done by induction. □

Note that the main theorem of [9] can be obtained as a consequence, when Λ is given by a quiver with relations. Here $\tau_Q(\Lambda)$ denotes the sum of the images of maps from Q to Λ.

Corollary 2.6. *Let Λ be a weakly triangular algebra given by a quiver with relations over a field k, with $\tau_{Q_i}(\Lambda)$ of finite projective dimension for each i, where $P_1, \dots, P_n$ are the indecomposable projective modules ordered such that $\operatorname{Hom}_\Lambda(P_i, P_j) = 0$ for $i < j$ and $Q_i = P_i \coprod \cdots \coprod P_n$.*

Then Λ is standardly stratified, and $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

Proof. In view of the criterion given by 2.5, we only need to prove that Λ is standardly stratified. For each indecomposable projective Λ-module P_j, consider $Z = \tau_{Q_i}(P_j)/\tau_{Q_{i+1}}(P_j)$. Here Δ_i is itself a ring, which is local. We clearly have $\operatorname{pd}_\Lambda Z < \infty$, and Z is a Δ_i-module, where also $\operatorname{pd}_{\Delta_i} Z < \infty$ since $\tau_{Q_{i+1}}(P_j)$ is a strong idempotent ideal because it has a projective resolution in $\operatorname{add} Q_{i+1}$ (see [3]). It then follows that Z is a free Δ_i-module, and we are done. $\square$

We now give some examples to illustrate the results. First note that if Λ is quasihereditary and not all Δ_i are simple, then $\mathcal{P}^{<\infty}(\Lambda) = \operatorname{mod}\Lambda$ properly contains $\mathcal{F}_\Lambda(\Delta)$.

Example. Let Λ be given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta_1"] & 2 \arrow[l, shift left, "\beta_2"] \arrow[r, "\gamma_1"] & 3 \end{tikzcd}$$

with the relations $\{\alpha^2,\ \gamma_1\beta_1,\ \beta_1\beta_2,\ \alpha\beta_2\}$. Then the indecomposable projective modules have the following structure

$$P_1: \begin{array}{cc} \multicolumn{2}{c}{S_1} \\ S_1 & S_2 \\ S_2 & S_1 \\ S_1 & \end{array} \qquad P_2: \begin{array}{cc} \multicolumn{2}{c}{S_2} \\ S_1 & S_3 \end{array} \qquad P_3: \begin{array}{c} S_3 \\ S_2 \end{array}$$

For the vertex 1, choose α, for vertex 2 the trivial path e_2 and for vertex 3 the trivial path e_3. Then we have $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$ by 2.5, since Λ is clearly standardly stratified.

Example. Let Λ be given the following quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta_1"] & 2 \arrow[l, shift left, "\beta_2"] \arrow[loop above, "\gamma"] \arrow[r, shift left, "\mu_1"] & 3 \arrow[l, shift left, "\mu_2"] \end{tikzcd}$$

where the relations are such that the indecomposable projective modules have the following structure

$$P_1: \begin{tikzcd}[column sep=small, row sep=small] & S_1 \arrow[dl, dash] \arrow[dr, dash] & & \\ S_1 & & S_2 \arrow[dl, dash] \arrow[dr, dash] & \\ & S_2 \arrow[dl, dash] & & S_1 \\ S_1 & & & \end{tikzcd} \qquad P_2: \begin{tikzcd}[column sep=small, row sep=small] & S_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ S_2 \arrow[d, dash] & S_1 \arrow[dr, dash] & S_3 \arrow[d, dash] \\ S_1 & & S_2 \end{tikzcd} \qquad P_3: \begin{tikzcd}[row sep=small] S_3 \arrow[d, dash] \\ S_2 \end{tikzcd}$$

For 1 choose α, for 2 choose γ, and for 3 choose the trivial path e_3. Then we have $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

3 Relations for the Grothendieck group $K_0(\mathcal{F}(\Delta))$

Let Λ be an artin algebra and denote by $K_0(\operatorname{mod}\Lambda, 0)$ the group whose elements are the M in $\operatorname{mod}\Lambda$, modulo the relations given by split exact sequences. For M in $\operatorname{mod}\Lambda$ denote by $[M]$ the corresponding element in $K_0(\operatorname{mod}\Lambda, 0)$. If $E(\Lambda)$ denotes the subgroup generated by the $[A] + [C] - [B]$ for any exact sequence, we have by definition that the Grothendieck group $K_0(\operatorname{mod}\Lambda)$ is $K_0(\operatorname{mod}\Lambda, 0)/E(\Lambda)$. Let $A(\Lambda)$ be the subgroup of $E(\Lambda)$ generated by the relations given by the almost split sequences. Then a result of Butler-Auslander says that $E(\Lambda) = A(\Lambda)$ if and only if Λ is of finite representation type (see [5]).

The corresponding question is interesting for any functorially finite extension closed subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$, since in this case we have existence of almost split sequences [6]. And, indeed, the corresponding result has been proved in [13] for $\mathcal{F}_\Lambda(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$ when Λ is standardly stratified and $\operatorname{Hom}_\Lambda(P_i, P_j) = 0$ for $i < j$. Adapting the proof in [13] we prove more generally this result for $\mathcal{F}_\Lambda(\Delta)$ when Λ is standardly stratified, which by Section 2 also gives a more general result for $\mathcal{P}^{<\infty}(\Lambda)$.

We start with the following direct consequence of 1.2 (c).

Lemma 3.1. *Let Λ be a standardly stratified algebra. Then $K_0(\mathcal{F}(\Lambda))$ is a free abelian group of rank n, with basis $[\Delta_1], \dots, [\Delta_n]$.*

Denote by $\langle [M], [N] \rangle$ the length of $\operatorname{Hom}_\Lambda(M, N)$ as an R-module, when Λ is an artin R-algebra. If $0 \to A \to B \to C \to 0$ is almost split in $\mathcal{F}_\Lambda(\Delta)$ we denote the element $[A] + [C] - [B]$ in $K_0(\mathcal{F}_\Lambda(\Delta), 0)$ by r_C and by l_A. The following result will be useful.

Proposition 3.2. *Let Λ be a standardly stratified algebra. Let $x = \sum a_M[M]$ and $y = \sum b_M[M]$ be elements in $K_0(\mathcal{F}_\Delta, 0)$, where M runs through the nonisomorphic indecomposable modules in $\mathcal{F}_\Lambda(\Delta)$. Then the following are equivalent.*

(a) $x{=}y$

(b) $\langle x, [N] \rangle = \langle y, [N] \rangle$ *for all (indecomposable) N in $\mathcal{F}_\Lambda(\Delta)$*

Proof. Clearly (a) implies (b). So assume (b) holds. Then we have $\langle x, r_C \rangle = \langle y, r_C \rangle$ for each indecomposable nonprojective module C in $\mathcal{F}(\Lambda)$. Since $\langle [M], r_C \rangle = 0$ when the indecomposable module M is not isomorphic to C, we get $a_C = b_C$ when C is indecomposable nonprojective. Hence we can assume that $x = \sum_{i=1}^n a_i[P_i]$ and $y = \sum_{i=1}^n b_i[P_i]$ where $P_1, \dots, P_n$ are the nonisomorphic indecomposable projective Λ-modules.

By hypothesis we have $\langle x, [\Delta_1]\rangle = \langle y, [\Delta_1]\rangle$. Since $\langle [P_i], [\Delta_1]\rangle = 0$ for $i > 1$, it follows that $a_1 = b_1$. Further we have $\langle x - a_1[P_1], [\Delta_2]\rangle = \langle y - a_1[P_1], [\Delta_2]\rangle$, and we conclude similarly that $a_2 = b_2$. Continuing this way, we get $x = y$, and the proof is done. $\square$

We now investigate more closely elements x in $K_0(\mathcal{F}(\Delta), 0)$, which can be written as a linear combination of the l_A (or of the r_C).

Lemma 3.3. *Let $x = \sum_A c_A l_A$ with $c_A \in \mathbb{Z}$. Then we have $c_A = \langle x, [A]\rangle / \langle l_A, [A]\rangle$. (And if $x = \sum_C d_C r_C$, then $d_C = \langle [C], x\rangle / \langle [C], r_C\rangle$.)*

Proof. Since $\langle l_A, [D]\rangle = 0$ for any indecomposable module D in $\mathcal{F}(\Delta)$ which is not isomorphic to A, we get $\langle x, [D]\rangle = c_D\langle l_D, [D]\rangle$, and we have $\langle l_D, [D]\rangle \neq 0$. This proves the first claim, and the second is analogous. $\square$

We now associate to each M in $\mathcal{F}(\Delta)$ the module $\hat{M} = \bigoplus_{i=1}^n \Delta_i^{m_i}$, where m_i is the multiplicity of Δ_i as a Δ-composition factor of M. Then we have $[M] = [\hat{M}]$ in $K_0(\mathcal{F}(\Delta))$, and moreover the following.

Lemma 3.4. *The kernel $E(\Lambda)$ of the natural map $K_0(\mathcal{F}(\Delta), 0) \to K_0(\mathcal{F}(\Delta))$ is generated by the elements $[M] - [\hat{M}]$, with $[M]$ in $\mathcal{F}(\Delta)$.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{F}(\Delta)$. Then we clearly have $[\hat{B}] = [\hat{A}] + [\hat{C}]$ in $K_0(\mathcal{F}(\Delta)), 0)$. Hence we have $[A] + [C] - [B] = ([A] + [C] - [B]) - ([\hat{A}] + [\hat{C}] - [\hat{B}]) = (([A] - [\hat{A}]) + ([C] - [\hat{C}]) - ([B] - [\hat{B}])$ in $K_0(\mathcal{F}(\Delta), 0)$, which finishes the proof. $\square$

Before we prove the main result of this section, we need the following direct consequence of the proof of [15, 5.2].

Lemma 3.5. *Let Λ be a standardly stratified algebra with associated (basic) tilting module T, and n nonisomorphic simple modules. Then we can write $T = T_1 \coprod \cdots \coprod T_n$, where the T_i are indecomposable, such that $T_i \supset \Delta_i$ and all Δ-composition factors of T_i/Δ_i are of the form Δ_j for $j < i$.*

Proof. There is by [15, 2.4] for each Δ_i an exact sequence $0 \to \Delta_i \to Y \to X \to 0$ where $X \in \mathcal{F}(\Delta)$ has a Δ-filtration using $\{\Delta_1, \dots, \Delta_{i-1}\}$, and Y is in $\mathcal{F}(\Delta)^\perp$. Since $\mathcal{F}(\Delta)$ is closed under extensions, it follows that Y is in $\mathcal{F}(\Delta) \cap \mathcal{F}(\Delta)^\perp = \operatorname{add} T$. Since Δ_i occurs exactly once in a Δ-composition series for Y, there is a unique, up to isomorphism, indecomposable summand T_i of T in $\operatorname{add} Y$ with Δ_i in a Δ-composition series. Since the other Δ-composition factors of T_i are amongst $\{\Delta_1, \cdots, \Delta_{i-1}\}$, it follows using Proposition 1.2 that Δ_i is contained in T_i and that T_i/Δ_i has Δ-composition factors amongst $\{\Delta_1, \cdots, \Delta_{i-1}\}$. Clearly $T_1, \cdots, T_n$ are nonisomorphic, and hence they are exactly the indecomposable summands of T up to isomorphism. It follows that $T = T_1 \coprod \cdots \coprod T_n$. $\square$

Theorem 3.6. *We have $A(\Lambda) = E(\Lambda)$ if and only if $\mathcal{F}(\Delta)$ is of finite representation type.*

Proof. Assume first that $\mathcal{F}(\Delta)$ is of finite representation type. To prove $A(\Lambda) = E(\Lambda)$, it is by 3.4 sufficient to prove that $[M] - [\hat{M}] \in A(\Lambda)$ for each M in $\mathcal{F}(\Delta)$. Let $x = \sum_A \langle [M] - [\hat{M}], [A] \rangle / \langle l_A, [A] \rangle l_A$, where A runs through the indecomposable noninjective objects in $\mathcal{F}(\Delta)$. It is then sufficient to show $x = [M] - [\hat{M}]$.

To see this, let N be an indecomposable noninjective object in $\mathcal{F}(\Delta)$. Then we have $\langle x, [N] \rangle = \langle [M] - [\hat{M}], [N] \rangle$, since $\langle l_A, [N] \rangle = 0$ if $A \not\simeq N$. If $N = I$ is injective in $\mathcal{F}(\Delta)$, then $\langle l_A, [N] \rangle = 0$, so that also $\langle x, [N] \rangle = 0$. Since $(\ , I)$ is exact on exact sequences in $\mathcal{F}(\Delta)$, it is easy to see that $\langle [M], [I] \rangle = \langle [\hat{M}], [I] \rangle$. Hence we have $\langle [M] - [\hat{M}], [I] \rangle = 0 = \langle x, [I] \rangle$. We can then get our desired conclusion $x = [M] - [\hat{M}]$ by applying 3.2.

Assume conversely that $A(\Lambda) = E(\Lambda)$. Then each $[M] - [\hat{M}]$ is a finite linear combination $\sum u_C r_C$ of the r_C. By 3.3, $u_C \neq 0$ if and only if $\langle [C], [M] - [\hat{M}] \rangle \neq 0$. In particular, the set $\mathcal{C}_M = \{\, C$ indecomposable in $\mathcal{F}(\Delta); \langle [C], [M] - [\hat{M}] \rangle \neq 0 \,\}$ is finite. Then also $\mathcal{U} = \bigcup_{i=1}^n \mathcal{C}_{T_i} \cup (\bigcup_{i=1}^n \mathcal{C}_{T_i/\Delta_i})$ is finite, where $T = T_1 \coprod \cdots \coprod T_n$ is the tilting module associated with $\mathcal{F}(\Delta)$.

Let now C be indecomposable not in $\mathcal{U}$. We want to show that C is projective. For each $i = 1, \ldots, n$, apply $(C,\ )$ to the exact sequence $0 \to \Delta_i \to T_i \to T_i/\Delta_i \to 0$. We have $[\hat{T_i}] = [\Delta_i] + [\widehat{T_i/\Delta_i}]$. Since $C \notin \mathcal{U}$, we have $\langle [C], [\hat{T_i}] \rangle = \langle [C], [T_i] \rangle$ and $\langle [C], [\widehat{T_i/\Delta_i}] \rangle = \langle [C], [T_i/\Delta_i] \rangle$ by the definition of $\mathcal{U}$. Hence we have $\langle [C], [T_i] \rangle = \langle [C], [\Delta_i] \rangle + \langle [C], [T_i/\Delta_i] \rangle$, and consequently $0 \to (C, \Delta_i) \to (C, T_i) \to (C, T_i/\Delta_i) \to 0$ is exact. Since $\operatorname{Ext}^1_\Lambda(C, T_i) = 0$, we obtain $\operatorname{Ext}^1_\Lambda(C, \Delta_i) = 0$ for $i = 1, \ldots, n$. Then $\operatorname{Ext}^1_\Lambda(C, M) = 0$ for all M in $\mathcal{F}(\Delta)$, so that C is projective in $\mathcal{F}(\Delta)$, and is hence a projective Λ-module. It follows that $\mathcal{F}(\Delta)$ has only a finite number of indecomposable modules. $\square$

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Notes

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