Dualities on generalized Koszul algebras

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Abstract

Koszul rings are graded rings which have played an important role in algebraic topology, algebraic geometry, noncommutative algebraic geometry, and in the theory of quantum groups. One aspect of the theory is to compare the module theory for a Koszul ring and its Koszul dual. There are dualities between subcategories of graded modules; the Koszul modules. When Λ is an artin algebra and T is a cotilting Λ-module, the functor $\mathrm{Hom}_\Lambda(\ ,T)$ induces a duality between certain subcategories of the finitely generated modules over Λ and $\mathrm{End}_\Lambda(T)$.

The purpose of this paper is to develop a unified approach to both the Koszul duality and the duality for cotilting modules. This theory specializes to these two cases as well as containing new interesting examples. The starting point for the theory is a positively $\mathbb{Z}$-graded ring $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \cdots$ and a (Wakamatsu) cotilting Λ_0-module T, satisfying additional assumptions. The theory gives a duality between certain subcategories of the finitely generated graded modules generated in degree zero over Λ and the extension algebra $\oplus_{i\geq 0} \mathrm{Ext}^i_\Lambda(T,T)$.

1991 *Mathematics Subject Classification.* Primary 16W50, 16S37, 16D90; Secondary 18E05, 16E05.

Introduction

Let Λ be a ring and M a (left) Λ-module. Let $\Gamma = \mathrm{End}_\Lambda(M)$. Then M is a Λ-Γ-bimodule, and we have a natural ring homomorphism $\Lambda \xrightarrow{\alpha} \mathrm{End}_\Gamma(M)$. The case when α is an isomorphism is of special interest, and in this case M is said to be faithfully balanced. Denote by $\mathrm{Mod}\,\Lambda$ the category of left Λ-modules. For X in $\mathrm{Mod}\,\Lambda$, the homomorphism set $\mathrm{Hom}_\Lambda(M,X)$ is a Γ^{op}-module. Hence we have a natural functor $(M,\ )\colon \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\Gamma^{\mathrm{op}}$. Here we use the notation (M,X) for $\mathrm{Hom}_\Lambda(M,X)$, since no confusion can arise. Then M is taken to Γ and Λ is taken to the Γ^{op}-module (M,Λ). So the pair (Λ, M) is taken to the pair $(\Gamma,(M,\Lambda))$. This correspondence with a pair consisting of a ring and a module over the ring was investigated by M. Auslander for finitely generated modules over artin algebras, and was called Wedderburn correspondence in [**1**]. It has been of interest to find appropriate subcategories of $\mathrm{Mod}\,\Lambda$ and of $\mathrm{Mod}\,\Gamma^{\mathrm{op}}$ such that $(M,\ )$ induces an equivalence of categories, as generalization of the Morita equivalence theorem.

We also have the functor $(\ ,M)$ from $\mathrm{Mod}\,\Lambda$ to $\mathrm{Mod}\,\Gamma$. In this case M is taken to Γ and Λ is taken to M, so that the pair (Λ, M) is taken to the pair (Γ, M) via the natural correspondence of pairs. Hence from the point of view of correspondence of such pairs, the contravariant functor $(\ ,M)$ is more natural than the covariant functor $(M,\ )$. It is now of interest to find natural subcategories of $\mathrm{Mod}\,\Lambda$ and of $\mathrm{Mod}\,\Gamma$ such that $(\ ,M)$ induces a duality.

Examples in the representation theory of artin algebras come from tilting and cotilting modules, and there are also generalizations to other rings. A module T is a tilting module over Λ if $\mathrm{pd}_\Lambda T < \infty$, $\mathrm{Ext}^i_\Lambda(T,T) = (0)$ for $i > 0$, and there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to T_2 \to \cdots \to T_n \to 0$ with T_i in $\mathrm{add}\,T$. Then T is also a tilting Γ-module, and $\mathrm{End}_\Gamma(T) \simeq \Lambda$. A module T is a cotilting Λ-module if $\mathrm{id}_\Lambda T < \infty$, $\mathrm{Ext}^i_\Lambda(T,T) = (0)$ for $i > 0$, and there is an exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to D(\Lambda^{\mathrm{op}}) \to 0$ with T_i in $\mathrm{add}\,T$. Then T is a cotilting Γ-module, and the contravariant functor $(\ ,T)$ induces a duality between ${}^{\perp}_{\Lambda}T$ and ${}^{\perp}_{\Gamma}T$, where ${}^{\perp}X = \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(M,X) = (0) \text{ for all } i > 0\}$.

Instead of associating with the pair (Λ, M) the endomorphism ring $\mathrm{End}_\Lambda(M)$, we can associate the Ext-algebra $\Gamma = \oplus\,\mathrm{Ext}^i_\Lambda(M,M)$. Since Γ is in a natural way a graded ring, it is natural to also start with a graded ring Λ and a graded Λ-module M. Then M is again a Γ-module, as before. So assume we have the following setting: Λ is a graded ring $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_i + \cdots$, where Λ_0 is an artin algebra and the Λ_i are finitely generated as left and as right Λ_0-modules, and let $M = \Lambda_0$ and let Γ be the Ext-algebra $E(\Lambda) = \oplus\,\mathrm{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)$. Here $\mathrm{Ext}^i_\Lambda(\ ,\ )$ denotes the ordinary i-th ungraded extension group. Then we can consider the condition that $\Lambda \to E(E(\Lambda))$ is an isomorphism. Let $\mathcal{E}$ denote the functor $\oplus\,\mathrm{Ext}^i_\Lambda(\ ,M)$. This functor has been investigated in the following setting: Λ_0 is semisimple (not necessarily an artin algebra) and $M = \Lambda_0$. Then Λ is said to be *Koszul* if there is an isomorphism $\Lambda \to \oplus \operatorname{Ext}^i_\Gamma(\Gamma_0, \Gamma_0)$. Let $\operatorname{Gr}\Lambda$ denote the subcategory of $\operatorname{Mod}\Lambda$ consisting of graded modules with degree 0 morphisms. Denote by $\operatorname{gr}\Lambda$ the full subcategory of $\operatorname{Gr}\Lambda$ consisting of the finitely generated modules. A natural functor from $\operatorname{Gr}\Lambda$ to $\operatorname{Gr}\Gamma$ is in this case $\mathcal{E} = \oplus \operatorname{Ext}^i_\Lambda(\ ,M)$, and when Λ is Koszul, then there are natural subcategories of $\operatorname{Gr}\Lambda$ and of $\operatorname{Gr}\Gamma$, which are called the Koszul modules, where $\mathcal{E}$ induces a duality.

The two cases discussed above are analogous, since for a cotilting module the endomorphism algebra and the Ext-algebra coincide. Actually, one of the motivations for this paper was to provide a general framework covering both the cotilting situation and the case of classical Koszul algebras, with respect to duality of subcategories.

We also point out that a more general framework is natural. To explain this we need to recall some constructions concerning relative homology from [**6**].

Given an additive subfunctor F of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ an exact sequence $\eta\colon 0 \to A \to B \to C \to 0$ is *F-exact* if η is in $F(C,A)$. A module P is called *F-projective* or *relative projective* if $0 \to (P,A) \to (P,B) \to (P,C) \to 0$ is exact for all for F-exact sequences $0 \to A \to B \to C \to 0$. We denote the full subcategory of relative projective modules by $\mathcal{P}(F)$. An additive subfunctor F of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ is said to have *enough projectives* if for every module C in $\operatorname{Mod}\Lambda$ (or $\operatorname{mod}\Lambda$, depending on the context we work in) there exists an F-exact sequence $0 \to K \to P \to C \to 0$ with P relative projective.

For a subcategory $\mathcal{X}$ in $\operatorname{Mod}\Lambda$ let $F_\mathcal{X}$ be the additive subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ given by

$$F_\mathcal{X}(C,A) = \{0 \to A \to B \to C \to 0 \mid \operatorname{Hom}_\Lambda(\ ,B)|_\mathcal{X} \to \operatorname{Hom}_\Lambda(\ ,C)|_\mathcal{X} \to 0 \text{ is exact}\}$$

for all pairs of modules A and C in $\operatorname{Mod}\Lambda$. Then $F_\mathcal{X}$ defines a family of exact sequences which is closed under pushouts, pullbacks and Baer sums. Dually we have that $F^\mathcal{X}$ given by

$$F^\mathcal{X}(C,A) = \{0 \to A \to B \to C \to 0 \mid \operatorname{Hom}_\Lambda(B,\ )|_\mathcal{X} \to \operatorname{Hom}_\Lambda(A,\ )|_\mathcal{X} \to 0 \text{ is exact}\}$$

for all pairs of modules A and C in $\operatorname{Mod}\Lambda$ is an additive subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$.

If $\mathcal{X}$ is a contravariantly finite subcategory of $\operatorname{Mod}\Lambda$ closed under summands and containing the projective Λ-modules, it is straightforward to see that $\mathcal{P}(F_\mathcal{X}) = \mathcal{X}$ and that $F_\mathcal{X}$ has enough projectives. In this case we can construct the relative derived functors of $\operatorname{Hom}_\Lambda(\ ,\ )$ using relative projective resolutions, and we denote them by $\operatorname{Ext}^i_{F_\mathcal{X}}(\ ,\ )$.

The general framework is described by quadruples $(\Lambda, T, \mathcal{C}, F_\mathcal{X})$, where Λ is a ring, T a Λ-module, $\mathcal{C}$ a subcategory of $\mathcal{M}$ (which is $\operatorname{mod}\Lambda$ or $\operatorname{Mod}\Lambda$) containing T and $F_\mathcal{X}$ a subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ given by a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{M}$ closed under summands and containing $\mathcal{P}(\Lambda)$, where $\mathcal{P}(\Lambda)$ denotes the subcategory of projective Λ-modules. Then we define $E_T(\Lambda)$ to be the Ext-algebra $\oplus_{i\geq 0}\operatorname{Ext}^i_{F_\mathcal{X}}(T,T)$ and $\mathcal{E}$ the functor $\oplus_{i\geq 0}\operatorname{Ext}^i_{F_\mathcal{X}}(\ ,T)\colon \operatorname{Gr}\Lambda \to \operatorname{Gr}\mathcal{E}_T(\Lambda)$. From the quadruple $(\Lambda, T, \mathcal{C}, F_\mathcal{X})$ we construct the quadruple $(E_T(\Lambda), \mathcal{E}(\Lambda), \mathcal{E}(\mathcal{X}), F_{\mathcal{E}(\mathcal{C})})$, where it is clear from the assumptions that $\mathcal{E}(\Lambda)$ is in $\mathcal{E}(\mathcal{X})$ and that $\mathcal{P}(E_T(\Lambda))$ is in $\mathcal{E}(\mathcal{C})$. However it is not necessarily true in general that $\mathcal{E}(\mathcal{C})$ is a contravariantly finite subcategory in the subcategory we would be interested in.

Now we explain how the above constructions can be viewed in this framework. We start by considering the construction of an endomorphism algebra.

Consider the pair (Λ, M) where Λ is an artin algebra, and M is a finitely generated Λ-module. Then let $\mathcal{C} = \operatorname{add}(\Lambda \oplus M)$ and $\mathcal{X} = \mathcal{C}$. As we pointed out above, $\mathcal{P}(F_{\mathcal{X}}) = \mathcal{X}$, and therefore $\operatorname{Ext}^i_{F_{\mathcal{X}}}(M, M) = (0)$ for $i \geq 1$. Hence the quadruple corresponding to $(\Lambda, M, \mathcal{C}, F_{\mathcal{X}})$ is given by $(\Gamma = \operatorname{End}_\Lambda(M), {}_\Gamma M, \operatorname{add}(\Gamma \oplus {}_\Gamma M), F_{\operatorname{add}(\Gamma \oplus_\Gamma M)})$. We see that we can do the construction again, and we get back to the first quadruple if M is a faithful and balanced module.

Also associated with the pair (Λ, T) when T is a cotilting module, there is a natural quadruple $(\Lambda, T, \operatorname{add} T, F_{\mathcal{P}(\Lambda)})$, where $F_{\mathcal{P}(\Lambda)} = \operatorname{Ext}^1_\Lambda(\ ,\ )$. This induces the quadruple $(\Gamma = \operatorname{End}_\Lambda(T), {}_\Gamma T, \operatorname{add} {}_\Gamma T, F_{\mathcal{P}(\Gamma)})$. Repeating the construction, it is well known that we get back to the quadruple we started with. We also point out that the construction of the endomorphism algebra of a relative cotilting module can be viewed in this framework.

Now we discuss the classical Koszul situation from this point of view. Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a positively $\mathbb{Z}$-graded ring generated in degrees 0 and 1, where Λ_0 is semisimple artin algebra and Λ_i is a Λ_0-Λ_0-bimodule of finite length over Λ_0 for all $i \geq 1$. Then consider the quadruple $(\Lambda, \Lambda_0, \operatorname{add} \Lambda_0, F_{\mathcal{P}(\Lambda)})$, which induces the quadruple $(\Gamma = \oplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0), \Gamma_0, \operatorname{add} \Gamma_0, F_{\mathcal{P}(\Gamma)})$. Clearly we can do the construction again, and we return to the original quadruple if and only if Λ is a Koszul algebra.

The main object of study in this paper is to give a natural framework for a duality theory in a setting containing both the duality given by cotilting modules for artin algebras and the duality for Koszul modules over Koszul rings $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$ when the semisimple ring Λ_0 is an artin algebra. So we want to identify appropriate pairs (Λ, T) for this purpose.

We start with a positively $\mathbb{Z}$-graded ring $\Lambda = \oplus_{i \geq 0} \Lambda_i$ generated in degrees 0 and 1, where Λ_0 is an artin algebra and Λ_i is a Λ_0-Λ_0-bimodule of finite length over Λ_0 for all $i \geq 1$. For a Λ-module T we want to study the functors

$$\mathcal{E} = \oplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(\ , T)\colon \operatorname{Gr} \Lambda \to \operatorname{Gr} E_T(\Lambda)$$

and

$$\mathcal{E}' = \oplus_{i \geq 0} \operatorname{Ext}^i_{E_T(\Lambda)}(\ , T')\colon \operatorname{Gr} E_T(\Lambda) \to \operatorname{Gr} E_{T'}(E_T(\Lambda)),$$

where $T' = \mathcal{E}(\Lambda)$.

In generalizing the classical cases one needs to decide which of the characteristic properties one wants to keep. We have already emphasized the duality property. In addition we would like to have as many essential properties of the category of Koszul modules as possible. We consider operations under which this category is closed, and compare with properties of the subcategory ${}^\perp T$ associated with a cotilting module. Actually, we consider a generalization of a cotilting module, investigated by Wakamatsu, which we call a Wakamatsu cotilting module.

We first give an approach to this problem in Chapters I and II, which follows closely the classical approach. To start with we assume some conditions on Λ, which we call the $\otimes$-Tor-conditions. They are trivially satisfied in the two special cases, and may look artificial. However, we show later that they are almost necessary in order to have a duality theory. We prove a duality theorem in this setting. For the benefit of the reader we collect in Chapter I definitions and properties which are essential from our point of view, from each of the two classical cases.

A second approach, which turns out to be more general, is treated in Chapter III. Here we emphasize which categorical properties we would like to have for the category of Koszul modules, by analyzing the two motivating cases. At the same time we have in mind that we would like to have a good duality theory.

In Chapter IV we discuss some further results, along with some open questions.

There is another generalization of a graded ring $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$ being Koszul, due to Woodcock [**18**]. He assumes that Λ is projective as a left and as a right Λ_0-module, i.e. Λ_i is projective as a left and as a right Λ_0-module. This is not necessarily the case in our setting. Moreover he has no restrictions on Λ_0. His point of view is considering the Koszul complex as the defining property he wants to generalize. However, assuming that Λ is a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra, and choosing $T = D(\Lambda_0)$, one can show that Λ is a T-Koszul algebra in the sense of this paper.

Acknowledgment

The first author wants to thank NSF, NSA and the American Scandinavian Foundation for support and his coauthors for their kind hospitality and support during his visits in Trondheim.

The last two authors would like to thank their coauthor and Virginia Tech for kind hospitality and support during their visits in Blacksburg. They also thank the Norwegian Research Council and NTNU for support during the preparation of this paper.

Notation and general preliminary results

In this chapter we try to make this paper as selfcontained as possible, as we attempt to include all the different notation, definitions and preliminary results that we need in the succeeding chapters.

We start by recalling some notions, definitions and results from the representation theory of artin algebras. For a general reference on this we refer the reader to [**4**].

A *quiver* is an oriented graph Γ, where Γ_0 denotes the set of vertices and Γ_1 denotes the set of arrows. A *path* in the quiver Γ is an ordered sequence of arrows $a_n a_{n-1} \cdots a_1$ or v for some v in Γ_0, where the arrow a_i ends where the arrow a_{i+1} starts for $i = 1, 2, \ldots, n-1$. The paths v for v in Γ_0 is called the *trivial paths*. Now, let k be any field. Let $k\Gamma$ denote the vector space over k with the paths of Γ as a k-basis. The *path algebra* of Γ over k is $k\Gamma$, where the multiplication is defined as follows. It is enough to define the multiplication of two paths p and q. We define

$$p \cdot q = \begin{cases} pq & \text{if } q \text{ ends where } p \text{ starts and } p \text{ and } q \text{ are both nontrivial paths,} \\ p & \text{if } q = v \text{ for the vertex } v \text{ where } p \text{ starts in } \Gamma, \\ q & \text{if } p = v \text{ for the vertex } v \text{ where } q \text{ ends in } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

A finite dimensional algebra Λ over a field k is said to be *basic* if $\Lambda \simeq \oplus_{i=1}^{n} P_i$ for some indecomposable projective Λ-modules P_i, then $P_i \not\simeq P_j$ for $i \neq j$.

It is known that for a finite dimensional basic algebra Λ over an algebraically closed field k, there exists a finite quiver Γ and an ideal I in $k\Gamma$, such that $\Lambda \simeq k\Gamma/I$. Moreover, the ideal I contains some power of the ideal J in $k\Gamma$ generated by the arrows in Γ and I is contained in J^2. An ideal I satisfying these conditions is called an *admissible* ideal. The radical of $k\Gamma/I$ is J/I and $k\Gamma/J \simeq k^n$, where n is the number of vertices in Γ.

The following is a list of definitions related to subcategories of the category $\operatorname{mod}\Sigma$ of finitely generated Σ-modules for an artin algebra Σ.

Definition 0.1. Let Σ be an artin algebra, and let $\mathcal{X}$ be a full subcategory of $\operatorname{mod}\Sigma$.

(a) The subcategory $\mathcal{X}$ is called *extension closed* if for all exact sequence $0 \to A \to B \to C \to 0$ with A and C in $\mathcal{X}$, then B is in $\mathcal{X}$.
(b) The subcategory $\mathcal{X}$ is called *resolving* if $\mathcal{X}$ is closed under kernel of epimorphisms and extensions and $\mathcal{X}$ contains the finitely generated projective Σ-modules.
(c) The full subcategory $\widehat{\mathcal{X}}$ consists of all modules C in $\operatorname{mod}\Sigma$ such that there exists an exact sequence $0 \to X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to X_0 \to C \to 0$ for some n and with X_i in $\mathcal{X}$ for all $i \geq 0$.

(d) A morphism $f\colon X_C \to C$ with X_C in $\mathcal{X}$ is called a *left $\mathcal{X}$-approximation* if for any morphism $g\colon X \to C$ for X in $\mathcal{X}$, there exists a morphism $h\colon X \to X_C$ such that $fh = g$. A left $\mathcal{X}$-approximation $f\colon X_C \to C$ is called *minimal* if for any morphism $g\colon X_C \to X_C$ with $fg = f$ implies that g is an isomorphism.

(d) A morphism $f\colon A \to X^A$ with X^A in $\mathcal{X}$ is called a *right $\mathcal{X}$-approximation* if for any morphism $g\colon A \to X$ for X in $\mathcal{X}$, there exists a morphism $h\colon X^A \to X$ such that $hf = g$. A right $\mathcal{X}$-approximation $f\colon A \to X^A$ is called *minimal* if for any morphism $g\colon X^A \to X^A$ with $gf = f$ implies that g is an isomorphism.

(e) The subcategory $\mathcal{X}$ is called *contravariantly finite* in $\operatorname{mod}\Sigma$ if each module C in $\operatorname{mod}\Sigma$ has a left $\mathcal{X}$-approximation. The subcategory $\mathcal{X}$ is called *covariantly finite* in $\operatorname{mod}\Sigma$ if each module C in $\operatorname{mod}\Sigma$ has a right $\mathcal{X}$-approximation.

(f) The subcategory $\mathcal{X}$ is called *funtorically finite* in $\operatorname{mod}\Sigma$ if $\mathcal{X}$ is both covariantly and contravariantly finite in $\operatorname{mod}\Sigma$.

Next we recall some notions concerning tilting and cotilting theory for artin algebras.

Definition 0.2. Let Σ be an artin algebra, and let $\mathcal{X}$ be a full subcategory of $\operatorname{mod}\Sigma$.

(a) A Σ-module T is called *selforthogonal* if $\operatorname{Ext}^i_\Sigma(T,T) = (0)$ for all $i > 0$.

(b) Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Sigma$. Then ${}^\perp\mathcal{C}$ denotes the full subcategory $\{X \in \operatorname{mod}\Sigma \mid \operatorname{Ext}^i_\Sigma(X,C) = (0) \text{ for all } C \in \mathcal{C}\}$.

(c) Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Sigma$. Then $\mathcal{C}^\perp$ denotes the full subcategory $\{X \in \operatorname{mod}\Sigma \mid \operatorname{Ext}^i_\Sigma(C,X) = (0) \text{ for all } C \in \mathcal{C}\}$.

(d) For a Σ-module M the full subcategory $\operatorname{Sub}(M)$ of $\operatorname{mod}\Sigma$ consists of all submodules of a finite direct sum of direct summands of M.

(e) For a Σ-module M the full subcategory $\operatorname{add} M$ of $\operatorname{mod}\Sigma$ consists of all direct summands of finite direct sums of copies of M.

For a ring Σ and a Σ-module M we denote by M^* the Σ^{op}-module $\operatorname{Hom}_\Sigma(M,\Sigma)$.

Recall that an artin algebra Σ is said to satisfy the Generalized Nakayama conjecture if $\operatorname{Ext}^i_\Sigma(M, M \oplus \Sigma) = (0)$ for all $i > 0$ for a Σ-module M implies that M is projective.

We need to recall the following notion from [**6**].

Let X be a finitely generated Λ_0-module. A Λ_0-module U is said to be *dualized* by X if the natural homomorphism

$$U \to \operatorname{Hom}_\Sigma(\operatorname{Hom}_{\Lambda_0}(U,X),X)$$

is an isomorphism, where $\Sigma = \operatorname{End}_{\Lambda_0}(X)$. It is shown in [**6**] that U is dualized by X if and only if there exists an exact sequence

$$0 \to U \xrightarrow{f} X_0 \to X_1$$

with X_0 and X_1 in $\operatorname{add} X$, where f is a (minimal) left $\operatorname{add} X$-approximation.

Recall that we introduced the subfunctor $F^{\mathcal{X}}$ of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ for a ring Λ and a subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ in the introduction. When the subcategory $\mathcal{X}$ is equal to $\operatorname{add} T$ for some module T, then we write F^T for $F^{\operatorname{add} T}$.

We now recall some notions and definitions for graded rings.

DEFINITION 0.3. Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a positively $\mathbb{Z}$-graded ring.

(a) Λ is called a *left finite* graded ring if Λ_i is a finitely generated left Λ_0-module for all $i \geq 1$.
(b) Λ is said to be *generated in degrees* 0 *and* 1 if $\Lambda_i = \Lambda_1\Lambda_{i-1}$ for all $i \geq 1$.
(c) Let M be a graded Λ-module, then
   (i) $M[n]$ is the graded Λ-module such that $M[n]_i = M_{-n+i}$ for all i in $\mathbb{Z}$.
   (ii) $M_{\geq n}$ is the graded submodule of M such that
$$(M_{\geq n})_i = \begin{cases} (0) & \text{for } i < n, \\ M_i & \text{otherwise.} \end{cases}$$
   (iii) $\langle M_n\rangle$ denotes the graded submodule of M generated by M_n.
(d) Let M be a graded Λ-module. If U is a Λ_0-module of M, then $\langle U\rangle$ denotes the (graded) Λ-submodule of M generated by U.

Let Λ be a positively $\mathbb{Z}$-graded ring, with Λ_0 an artin algebra over some commutative artin ring R. Assume that Λ_i is finitely generated as a left Λ_0-module. Then Λ_i is finitely generated as a module over R. This implies that Λ_i is finitely generated as a right Λ_0-module. Hence being a left finite graded and a right finite graded ring is equivalent when Λ_0 is an artin algebra.

Let Λ be a positively $\mathbb{Z}$-graded ring. Denote by $\operatorname{Gr}\Lambda$ the category of graded Λ-modules with degree zero morphisms. The full subcategory of $\operatorname{Gr}\Lambda$ consisting of the finitely generated modules is denoted by $\operatorname{gr}\Lambda$. We denote by $\operatorname{gr}_0\Lambda$ the full subcategory of $\operatorname{gr}\Lambda$ consisting of the modules generated in degree 0. Denote by $\operatorname{fgsyz}\Lambda$ the full subcategory of $\operatorname{gr}\Lambda$ consisting of all modules X such that $\Omega^i_\Lambda(X)$ is finitely generated for all $i \geq 1$.

For any ring Λ we sometimes abbreviate $\operatorname{Hom}_\Lambda(A, B)$ as ${}_\Lambda(A, B)$ or just (A, B). For a positively $\mathbb{Z}$-graded ring Λ we denote by $\operatorname{Ext}^i_\Lambda(C, A)$ the ordinary i-th ungraded extension group and by $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(C, A)$ the i-th graded extension group.

We end this chapter with an elementary result on graded modules, where we leave the proof to the reader.

LEMMA 0.4. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a positively $\mathbb{Z}$-graded ring. Let X be a graded Λ-module generated in positive degrees, and let T be a graded Λ-module concentrated in degree* 0*. Then the exact sequence $0 \to \langle X_0\rangle X \to X/\langle X_0\rangle \to 0$ induces a split exact sequence*
$$0 \to \operatorname{Hom}_\Lambda(X/\langle X_0\rangle, T) \to \operatorname{Hom}_\Lambda(X, T) \to \operatorname{Hom}_{\Lambda_0}(X_0, T) \to 0.$$

CHAPTER I

Main results and examples

In order to get an understanding for the background for our theory of T-Koszul algebras, we first give an account of some of the main results in the two classical cases. We then state some of the central results of this paper. We want to quickly get to a variety of examples, so we postpone the proofs of these results to the next chapter.

1. Two classical cases

In this section we recall relevant properties for cotilting modules and Koszul algebras, in particular with respect to existence of dualities for subcategories.

Tilting theory has played a central role in the representation theory of artin algebras [**3, 12, 13, 17**]. One of the aspects of the theory deals with equivalence or duality between appropriate subcategories of module categories. We deal here with artin algebras, even though there are analogous results in more general contexts.

Let Λ be an artin algebra and denote by $\operatorname{mod}\Lambda$ the category of finitely generated (left) Λ-modules. Let T in $\operatorname{mod}\Lambda$ be selforthogonal. We say T is a *tilting* module if $\operatorname{pd}_\Lambda T < \infty$ and there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with the T_i in $\operatorname{add} T$. Dually, T is a *cotilting* module if $\operatorname{id}_\Lambda T < \infty$ and there is an exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to D\Lambda \to 0$ with the T_i in $\operatorname{add} T$, where $D(\Lambda)$ is an injective generator.

Associated with a cotilting module T is the full subcategory ${}^\perp T$ of $\operatorname{mod}\Lambda$, whose objects are modules C with $\operatorname{Ext}^i_\Lambda(C,T) = (0)$ for $i > 0$. We have the following important properties for ${}^\perp T$.

THEOREM 1.1. *If T is a cotilting module, ${}^\perp T$ has the following properties.*
- (a) *${}^\perp T$ is resolving; that is, contains Λ and is closed under extensions and kernels of epimorphisms.*
- (b) *${}^\perp T$ is functorially finite.*
- (c) *${}^\perp T$ has almost split sequences.*

Let $\Gamma = \operatorname{End}_\Lambda(T)$ for a cotilting Λ-module T. Then T is a cotilting module over Γ, the algebra Λ is isomorphic to $\operatorname{End}_\Lambda(T)$, and the module theories for $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ are closely connected in the following way.

THEOREM 1.2. *If T is a cotilting module, the functor* $\operatorname{Hom}_\Lambda(\ ,T)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ *induces a duality* $\operatorname{Hom}_\Gamma(\ ,T)\colon {}^\perp_\Lambda T \to {}^\perp_\Gamma T$.

Note that $D(\Lambda)$ is clearly a cotilting module. In this case ${}^\perp_\Lambda D(\Lambda) = \operatorname{mod}\Lambda$, the algebra Γ is isomorphic to Λ^{op}, the subcategory ${}^\perp_\Gamma D(\Lambda)$ is $\operatorname{mod}\Lambda^{\mathrm{op}}$, and we recover the duality $D = \operatorname{Hom}_\Lambda(\ ,D(\Lambda))\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$.

In the general case T is an injective object in ${}^\perp T$ in the sense that $\operatorname{Ext}^i_\Lambda(C,T) = (0)$ for $i > 0$ and C in ${}^\perp T$. In addition we have the following.

Proposition 1.3. *Let T be a cotilting Λ-module. Then for each C in ${}^{\perp}T$ there is an exact sequence $0 \to C \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \to \cdots \to T_n \to \cdots$ with T_i in $\operatorname{add} T$ and $\operatorname{Ext}^1_\Lambda(\operatorname{Ker} f_i, T) = (0)$ for $i > 0$.*

We shall mainly deal with the following common generalization of a tilting and cotilting module considered by Wakamatsu [**17**]. Let T be selforthogonal module, and denote by $\mathcal{X}_{{}_\Lambda T}$ or, simply by $\mathcal{X}_T$, the full subcategory of ${}^{\perp}T$ whose objects are the modules C for which there is an exact sequence $0 \to C \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \to \cdots$ with T_i in $\operatorname{add} T$ and $\operatorname{Ext}^1_\Lambda(\operatorname{Ker} f_i, T) = (0)$ for all $i > 0$. We have the following property for $\mathcal{X}_T$.

Proposition 1.4. *$\mathcal{X}_T$ is closed under extensions and kernels of epimorphisms.*

We say that T is a *Wakamatsu cotilting* module if Λ is in $\mathcal{X}_T$ or, equivalently, if $\mathcal{X}_T$ is resolving. Replacing ${}^{\perp}T$ by $\mathcal{X}_T$ in the more general context we have some similar results. The next result shows that this more general concept is preserved by duality, while this is not the case for cotilting (or tilting) modules.

Theorem 1.5. *Let T be a Wakamatsu cotilting Λ-module and $\Gamma = \operatorname{End}_\Lambda(T)$.*

(a) *T is a Wakamatsu cotilting Γ-module.*
(b) *$D(T)$ is a Wakamatsu cotilting Λ^{op}-module.*
(c) *$\operatorname{End}_\Gamma(T) \simeq \Lambda$.*
(d) *There is a duality $\operatorname{Hom}_\Lambda(\ ,T)\colon \mathcal{X}_{{}_\Lambda T} \to \mathcal{X}_{{}_\Gamma T}$.*

We shall need the following, which is a direct consequence of the definitions.

Proposition 1.6. *Let T be a Wakamatsu cotilting Λ-module.*

(a) *If $0 \to A \to B \to C \to 0$ is an exact sequence in $\operatorname{mod} \Lambda$ such that $0 \to (C,T) \to (B,T) \to (A,T) \to 0$ is exact, and A and B are in $\mathcal{X}_T$, then C is in $\mathcal{X}_T$.*
(b) *$\mathcal{X}_T \subset \operatorname{Sub} T$.*

There are examples of Wakamatsu cotilting modules T where both the projective and injective dimension is infinite. For example, this is the case for the path algebra of the quiver

$$1 \rightleftarrows 2 \longleftarrow 3 \longleftarrow 4 \rightleftarrows 5$$

over a field k, modulo the relations which are compositions of two arrows [W]. Let $T = I_2 \oplus P_2 \oplus P_4 \oplus P_5 \oplus S_3$, where I_j, P_j and S_j denote the indecomposable injective, the indecomposable projective and the simple module over the path algebra associated to vertex j. In this example we have $\mathcal{X}_T = {}^{\perp}T$, even though T is not a cotilting module.

In addition to the definition there are the following criteria for a selforthogonal (in particular, Wakamatsu cotilting) module to be a cotilting module.

Theorem 1.7. *Let T be a selforthogonal Λ-module.*

(a) *T is a cotilting module if and only if $\widehat{\mathcal{X}_T} = \operatorname{mod} \Lambda$.*
(b) *T is a cotilting module if and only if ${}^{\perp}T$ is contravariantly finite and $\widehat{{}^{\perp}T} = \operatorname{mod} \Lambda$.*

Koszul algebras and Koszul duality have been of interest in algebraic topology, algebraic geometry, noncommutative algebraic geometry, and in the theory of quantum groups. We now recall definitions and relevant results for Koszul algebras (see [**9**], [**11**] and [**10**]).

Let k be a field and $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_i + \cdots$ a left finite graded k-algebra, where we assume $\Lambda_0 \simeq k^n$ for some n. Consider the Ext-algebra $E(\Lambda) = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$. Note that $E(\Lambda)$ is a graded k-algebra with $E(\Lambda)_i = \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$. We say that Λ is *generated in degrees* 0 *and* 1 if for each $i \geq 1$ we have $\Lambda_i = \Lambda_1\Lambda_{i-1}$. The algebra Λ is *Koszul* if Λ and $E(\Lambda)$ are generated in degrees 0 and 1.

A graded projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to C \to 0$$

of C in $\operatorname{gr}\Lambda$ is said to be *linear* if each P_i is generated in degree i. We denote the full subcategory of $\operatorname{gr}\Lambda$ whose objects have linear projective resolutions by $\mathcal{K}(\Lambda)$, and call these modules the *Koszul modules*.

We recall some characterizations of Koszul algebras and Koszul modules. Denote by $\mathcal{E} = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\ ,\Lambda_0)\colon \operatorname{Gr}\Lambda \to \operatorname{Gr} E(\Lambda)$ the natural functor.

THEOREM 1.8. *Let k be a field and $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_i + \cdots$ a left finite graded k-algebra, where we assume $\Lambda_0 \simeq k^n$ for some n. Then the following are equivalent when Λ is generated in degrees* 0 *and* 1.

(a) *Λ is a Koszul algebra.*
(b) *Λ_0 has a linear projective resolution.*
(c) *$E(E(\Lambda)) \simeq \Lambda$ as graded algebras.*

THEOREM 1.9. *Let Λ be a Koszul algebra, and let C be in $\operatorname{gr}\Lambda$. Then C is a Koszul module if and only if C and $\mathcal{E}(C)$ are generated in degree* 0.

Of special interest for this paper is the fact that there is an associated duality.

THEOREM 1.10. *Let Λ be a Koszul algebra.*

(a) *The functor $\mathcal{E}\colon \operatorname{Gr}\Lambda \to \operatorname{Gr} E(\Lambda)$ induces a duality $\mathcal{E}\colon \mathcal{K}(\Lambda) \to \mathcal{K}(E(\Lambda))$ between the Koszul modules.*
(b) *The category $\mathcal{K}(\Lambda)$ contains all the finitely generated projective and semisimple modules generated in degree* 0.
(c) *The module C is a projective module if and only if the $E(\Lambda)$-module $\mathcal{E}(C)$ is semisimple, and C is semisimple if and only if $\mathcal{E}(C)$ is projective.*
(d) *If C is in $\mathcal{K}(\Lambda)$, then $\Omega_\Lambda(C)[-1]$ and $C_{\geq 1}[-1]$ are in $\mathcal{K}(\Lambda)$.*

We have the following basic properties for the category of Koszul modules.

PROPOSITION 1.11. *Let Λ be a Koszul algebra, and $0 \to A \to B \to C \to 0$ an exact sequence of graded Λ-modules with A, B and C generated in degree* 0.

(a) *If A and C are in $\mathcal{K}(\Lambda)$, then B is in $\mathcal{K}(\Lambda)$.*
(b) *If A and B are in $\mathcal{K}(\Lambda)$, then C is in $\mathcal{K}(\Lambda)$.*

Another important class of algebras are the quadratic algebras. Again let $\Lambda_0 = k^n$ and let V be a Λ_0-bimodule which is a Λ_0-module of finite length. Let K in $V \otimes_{\Lambda_0} V$ be a Λ_0-sub-bimodule. Then the factor algebra $\Lambda = T_{\Lambda_0}(V)/\langle K\rangle$, where $T_{\Lambda_0}(V)$ denotes the tensor algebra, is said to be a *quadratic algebra*. Associated with Λ is the algebra $\Lambda^! = T_{\Lambda_0}(V^*)/\langle K^\perp\rangle$, where $V^* = \operatorname{Hom}_{\Lambda_0}(V, \Lambda_0)$ and $K^\perp \subset V^* \otimes_{\Lambda_0} V^* = (V \otimes_{\Lambda_0} V)^*$ is perpendicular to K in $V \otimes_{\Lambda_0} V$. We recall the following.

THEOREM 1.12. *Let Λ be a Koszul algebra.*

(a) Λ *is quadratic.*

(b) $E(\Lambda) \simeq \Lambda^!$.

2. Statements of main results

In this section we state some of the main results of this paper, and give the definitions necessary for this.

Throughout this section, $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$ will be a positively $\mathbb{Z}$-graded algebra generated in degrees 0 and 1. Let $\mathrm{gr}_0 \Lambda$ denote the full subcategory of $\mathrm{gr}(\Lambda)$ whose objects consist of graded modules generated in degree 0. We will freely view the category of Λ_0-modules and its various subcategories as embedded in $\mathrm{gr}_0 \Lambda$ by viewing a Λ_0-module M_0 as the graded module $\sum_i N_i$ with $N_i = (0)$ if $i \neq 0$ and $N_0 = M_0$.

We make the following assumptions throughout the section unless otherwise stated. First we will always assume that Λ_0 is an artin algebra and that Λ is left finite. By T we will denote a Wakamatsu cotilting Λ_0-module.

Let $\Gamma = \oplus_{n \geq 0} \operatorname{Ext}^n_\Lambda(T, T)$. We define $\mathcal{E}\colon \operatorname{gr} \Lambda \to \operatorname{gr} \Gamma$ by

$$\mathcal{E}(M) = \oplus_{n \geq 0} \operatorname{Ext}^n_\Lambda(M, T),$$

where $\mathcal{E}(M)$ has a natural graded Γ-module structure via the Yoneda product. We note that if $\Lambda = \Lambda_0$, then $\mathcal{E}$, when restricted to $\mathcal{X}_T$, is the Wakamatsu cotilting duality described in the previous section. Next we note that if Λ_0 is a product of n copies of a field k, then the only Wakamatsu Λ_0-cotilting module is $T = \Lambda_0$ up to multiplicity of direct summands. If Λ is a Koszul algebra, then the functor $\mathcal{E}$ restricts to the Koszul duality on the category of Koszul modules $\mathcal{K}(\Lambda)$. In this section, we define a general setting where Wakamatsu cotilting duality and Koszul duality fit in as special cases.

We shall define and introduce two conditions on Λ which will be important for our definition of an algebra which is Koszul with respect to T.

DEFINITION 2.1. Let $\mathcal{Y}$ be a subcategory of $\operatorname{mod} \Lambda_0$.

(a) We say that a Λ_0-bimodule B satisfies the $\otimes$-*condition* with respect to $\mathcal{Y}$ if $B \otimes_{\Lambda_0} \mathcal{Y} \subseteq \mathcal{Y}$.

(b) A Λ_0-bimodule B satisfies the Tor-*condition* with respect to $\mathcal{Y}$, if $\operatorname{Tor}_i^{\Lambda_0}(B, \mathcal{Y}) = (0)$ for each $i \geq 1$.

(c) When T is a Wakamatsu cotilting Λ_0-module, we say that the graded ring Λ satisfies the $\otimes$-condition and/or the Tor-condition with respect to T if each Λ_i satisfies the corresponding condition with respect to $\mathcal{X}_T$.

When both the $\otimes$-condition and the Tor-condition hold, we say that the $\otimes$-Tor-conditions are satisfied.

These conditions may seem artificial, but in Section 4 of Chapter III, they will be motivated by showing that conditions like $\otimes$-Tor-conditions are a necessary consequence of the existence of a duality theory.

Note that in both Wakamatsu cotilting theory and Koszul algebra theory, the $\otimes$-Tor-conditions are trivially satisfied, and these conditions will be part of our definition of an algebra being Koszul with respect to T. In Chapter III we give a more general setting for a duality theory to hold, but we postpone the general rather technical conditions for clarity of exposition.

Next we define a class of modules which will play the role of projective modules in the duality theory.

DEFINITION 2.2. Assume that Λ satisfies the $\otimes$-Tor-conditions with respect to T. We say a graded Λ-module M is *fat* (with respect to T) if there is Λ_0-module X_0 in $\mathcal{X}_T$ such that M is isomorphic to $\Lambda \otimes_{\Lambda_0} X_0$ as graded modules, where $(\Lambda \otimes_{\Lambda_0} X_0)_n = \Lambda_n \otimes_\Lambda X_0$. In particular, if M is a fat module, then M is in $\operatorname{gr}_0 \Lambda$ and $M_i = \Lambda_i \otimes_{\Lambda_0} M_0$.

We note that if $\Lambda = \Lambda_0$, then the fat modules are just the modules in $\mathcal{X}_T$. If Λ is a Koszul algebra, then the fat modules are just the projective modules since the modules in $\mathcal{X}_T$ are just the finitely generated projective Λ_0-modules.

Next we define a type of resolution that will play the role that linear projective resolutions play in Koszul theory.

DEFINITION 2.3. Let M be in $\operatorname{gr} \Lambda$. We say that M has a *linear fat resolution* if there is an exact sequence of graded Λ-modules

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where for each $n \geq 0$ the module $F_n[-n]$ is fat.

Note that if Λ is a Koszul algebra, then the concept of linear fat resolutions coincides with the concept of linear projective resolutions.

We now define a Koszul algebra with respect to T.

DEFINITION 2.4. Let $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting Λ_0-module and assume that Λ satisfies the $\otimes$-Tor-conditions with respect to T. Then Λ is a *Koszul algebra with respect to* T if Λ_0 has a linear fat resolution. For simplicity we say that Λ is a T-Koszul algebra.

Under this definition and because of our earlier remarks, it is clear that if $\Lambda = \Lambda_0$ and T is a Wakamatsu cotilting module, then Λ is T-Koszul. If Λ is a left finite graded algebra generated in degrees 0 and 1 with Λ_0 a finite product of a field, then $T = \Lambda_0$ is the only Wakamatsu cotilting module up to multiplicity of direct summands. This implies that Λ is a Koszul algebra with respect to T if and only if it is a Koszul algebra (in the classical case).

The full subcategory of $\operatorname{gr} \Lambda$ whose objects are the graded modules M with linear fat resolutions, and where M_i is in $\mathcal{X}_T$ for all $i \geq 0$, is denoted by $\mathcal{K}_T(\Lambda)$. If Λ is a Koszul algebra with respect to T, then the objects of the category $\mathcal{K}_T(\Lambda)$ will be called *T-Koszul modules.*

We now state some of the main results of the paper. The proofs of these results can be found in the next chapter.

THEOREM 2.5. *Let Λ be a Koszul algebra with respect to T. Then $E_T(\Lambda)$ is a Koszul algebra with respect to $T = \mathcal{E}(\Lambda)$.*

Note that $\mathcal{E}(\Lambda) = \oplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(\Lambda, T) \simeq \operatorname{Hom}_\Lambda(\Lambda, T) \simeq T$. Thus T has a natural Γ-module structure. It will be clear from the context whether we view T as a Λ- or Γ-module.

THEOREM 2.6. *Let Λ be a Koszul algebra with respect to T. Then, as graded algebras, Λ is isomorphic to $\oplus_{n \geq 0} \operatorname{Ext}^n_\Gamma(T, T)$.*

We view this isomorphism as an identification when Λ is a T-Koszul algebra, and in this case we let $\mathcal{E}'\colon \operatorname{gr}\Gamma \to \operatorname{gr}\Lambda$ denote the functor given by $\mathcal{E}'(M) = \oplus_{n\geq 0}\operatorname{Ext}^n_\Gamma(M,T)$. The next theorem is the main duality result. If M is in $\mathcal{K}_T(\Lambda)$ and $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ is a linear fat resolution of M, then we denote the kernel of $F_0 \to M$ by M^1.

THEOREM 2.7. *Let Λ be a Koszul algebra with respect to T.*

(a) *The functor $\mathcal{E}\colon \operatorname{gr}\Lambda \to \operatorname{gr}\Gamma$ specializes to a duality $\mathcal{E}\colon \mathcal{K}_T(\Lambda) \to \mathcal{K}_T(\Gamma)$, with inverse given by $\mathcal{E}'\colon \mathcal{K}_T(\Gamma) \to \mathcal{K}_T(\Lambda)$.*
(b) *If M_0 is in $\mathcal{X}_T$, then $\Lambda\otimes_{\Lambda_0} M_0$ and M_0 are T-Koszul modules.*
(c) *If M is a T-Koszul module, then $M_{\geq 1}[-1]$ and $M^1[-1]$ are also T-Koszul modules.*
(d) (i) $\mathcal{E}(M_{\geq 1}[-1]) = (\mathcal{E}(M))^1[-1]$.
  (ii) $\mathcal{E}(M^1[-1]) = (\mathcal{E}(M))_{\geq 1}[-1]$.
  (iii) *If $F = \Lambda\otimes_{\Lambda_0} M_0$ is a fat module, then $\mathcal{E}(F) = \mathcal{E}(F)_0$.*
  (iv) $\mathcal{E}(M_0) = \Gamma\otimes_{\Gamma_0}(\mathcal{E}(M))_0$.

To see that this duality specializes to the two situations discussed in the last section, we note that if $\Lambda = \Lambda_0$, it is immediate that $\mathcal{K}_T(\Lambda) = \mathcal{X}_T$ and $\mathcal{E}(M) = \operatorname{Hom}_\Lambda(M,T)$ if M is in $\mathcal{X}_T$. Thus, Wakamatsu cotilting duality is a special case of Theorem 2.7. When Λ_0 is a finite product of copies of a field, then fat modules are projective modules generated in degree 0, the category $\mathcal{X}_T$ is $\operatorname{mod}\Lambda_0$, and linear fat resolutions are just linear projective resolutions. Hence, in this case, the category $\mathcal{K}_T(\Lambda)$ is the category of Koszul modules, and the duality described in Theorem 2.7 is Koszul duality.

The statements that fat modules and Λ_0-modules are T-Koszul modules correspond to the fact that for Koszul algebras, projective modules and semisimple modules are Koszul modules. The statement about $M_{\geq 1}$ and M^1 correspond to the statement that if M is a Koszul module, then so are the negative -1-shift of the graded radical of M and the first syzygy of M. Finally, using these correspondences, we see that the T-Koszul duality extends the usual Koszul duality.

Our duality result implies that if Λ is a T-Koszul algebra and M is in $\mathcal{K}_T(\Lambda)$ then, since $\mathcal{E}(M)$ is in $\mathcal{K}_T(\Gamma)$, we see that $\mathcal{E}(M)$ is generated in degree 0. Unlike what happens in the Koszul algebra case, $\mathcal{E}(M)$ being generated in degree 0 is not sufficient for M to be in $\mathcal{K}_T(\Lambda)$. Example 4.1 below provides an example of this. These observations lead to the following definitions.

DEFINITION 2.8. Let Λ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting Λ_0-module. We denote by $\mathcal{G}_T(\Lambda)$ the full subcategory of $\operatorname{gr}_0\Lambda$ consisting of modules M such that $\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Ext}^i_\Lambda(M,T) = \operatorname{Ext}^{i+1}_\Lambda(M,T)$ for all $i\geq 0$.

If Γ is generated in degrees 0 and 1, for example if Λ is T-Koszul, then $\mathcal{G}_T(\Lambda)$ consists of the modules M in $\operatorname{gr}_0\Lambda$ such that $\mathcal{E}(M)$ is generated in degree 0 as a Γ-module. We shall see that for a T-Koszul algebra Λ we always have that $\mathcal{K}_T(\Lambda)$ is contained in $\mathcal{G}_T(\Lambda)$. The case when we have equality is especially interesting, so it is convenient to give the following definition.

DEFINITION 2.9. A T-Koszul algebra Λ is said to be *strongly* T-Koszul if $\mathcal{K}_T(\Lambda) = \mathcal{G}_T(\Lambda)$.

Many of the examples at the end of this section are strongly T-Koszul. In Example 4.1 in section 4 we give a T-Koszul algebra which is not a strongly T-Koszul algebra.

Next we note that the category of T-Koszul modules has some nice properties.

PROPOSITION 2.10. *Let Λ be a Koszul algebra with respect to T. Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of graded Λ-modules generated in degree 0. Suppose that the sequence is F^T-exact. Then we have the following.*

(a) *If A and C are T-Koszul modules, then B is a T-Koszul module.*
(b) *If A and B are T-Koszul modules, then C is a T-Koszul module.*

The next result continues the strong analogy between Koszul algebras and T-Koszul algebras. It is immediate that Λ is generated in degrees 0 and 1 if and only if Λ is a quotient of the tensor algebra $T_{\Lambda_0}(\Lambda_1) = \Lambda_0 + \Lambda_1 + (\Lambda_1 \otimes_{\Lambda_0} \Lambda_1) + (\Lambda_1^{\otimes^3_{\Lambda_0}}) + \cdots$ by an ideal contained in $(\Lambda_1 \otimes_{\Lambda_0} \Lambda_1) + (\Lambda_1^{\otimes^3_{\Lambda_0}}) + \cdots$. We have the following result which generalizes the fact that Koszul algebras are quadratic algebras and that the Ext-algebra Γ has orthogonal quadratic relations.

THEOREM 2.11. *Let Λ be a Koszul algebra with respect to T, and let $\Gamma = E_T(\Lambda)$. Let X_2^2 be the kernel of the multiplication map $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1 \to \Lambda_2$. Then*

(a) *Λ is isomorphic to $T_{\Lambda_0}(\Lambda_1)/I$ where I is the ideal generated by X_2^2.*
(b) *Γ_1 is isomorphic to $\mathrm{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$.*
(c) *Identify Γ_0 and Γ_1 with $\mathrm{End}_{\Lambda_0}(T)$ and $\mathrm{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$, respectively. Then Γ is isomorphic to $T_{\Gamma_0}(\Gamma_1)/J$ where J is the ideal generated by $\{\sum \beta_i \otimes \alpha_i \in \Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \mid \sum \beta_i(\mathrm{id}_{\Lambda_1} \otimes \alpha_i)|_{X_2^2 \otimes_{\Lambda_0} T} = 0\}$.*

3. T-Koszul algebras concentrated in degrees 0 and 1

We now turn our attention to the question of the existence of T-Koszul algebras beyond the classical cases discussed in Section 1. In general, most graded algebras Λ are not T-Koszul. Theorem 2.11 shows that they must be quadratic in a natural sense. But, as with the classical Koszul algebras, it is not clear which quadratic algebras are T-Koszul. This section deals with the "easiest" case where $\Lambda = \Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$. The following gives a criterion for such an algebra to be T-Koszul and gives a description of the T-Koszul modules.

THEOREM 3.1. *Let $\Lambda = \Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$ be a left finite graded algebra with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting Λ_0-module. Then*

(a) *Λ is a Koszul algebra with respect to T if and only if the $\otimes$-Tor-conditions with respect to T are satisfied.*
(b) *If Λ is a Koszul algebra with respect to T, then $\Gamma = \oplus_{n\geq 0} \mathrm{Ext}^n_\Lambda(T,T)$ is isomorphic to the tensor algebra $T_{\mathrm{End}_{\Lambda_0}(T)}(\mathrm{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T))$.*
(c) *If Λ is a Koszul algebra with respect to T, then the category of T-Koszul modules is the full subcategory of $\mathrm{gr}_0 \Lambda$ consisting of modules $M_0 + M_1 + 0 + \cdots$ such that M_0 and M_1 are in $\mathcal{X}_T$.*
(d) *If T is a cotilting Λ_0-module and Λ is a Koszul algebra with respect to T, then Λ is a strongly T-Koszul algebra.*

PROOF. (a) From the definition of a T-Koszul algebra, we need to show that if Λ satisfies the $\otimes$-Tor-conditions then Λ_0 is a T-Koszul module. For this we note that if M_0 is a Λ_0-module with M_0 in $\mathcal{X}_T$ then, viewing M_0 as a Λ-module,

$(M_0)^1 = 0 + (\Lambda_1 \otimes_{\Lambda_0} M_0) + 0 + \cdots$. By the $\otimes$-condition, $(\Lambda_1 \otimes_{\Lambda_0} M_0)$ is in $\mathcal{X}_T$. It follows that $M^1[-1]$ is again a Λ_0-module in $\mathcal{X}_T$ in degree 0 and (0) in degree 1. Hence, applying this observation repeatedly, we see that M_0 has a linear fat resolution of the form:

$$\cdots \to \Lambda \otimes_{\Lambda_0} \Lambda_1^{\otimes^2_{\Lambda_0}} \otimes_{\Lambda_0} M_0 \to \Lambda \otimes_{\Lambda_0} \Lambda_1 \otimes_{\Lambda_0} M_0 \to \Lambda \otimes_{\Lambda_0} M_0 \to M_0 \to 0$$

Setting $M_0 = \Lambda_0$, we see that Λ_0 has a linear fat resolution and hence Λ is a Koszul algebra with respect to T.

(b) Assume that Λ is a T-Koszul algebra. Since $\Lambda_2 = (0)$, the kernel of $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1 \to \Lambda_2$ is $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1$. By Theorem 2.11, the statement in (b) immediately follows.

(c) We begin by assuming that $M = M_0 + M_1 + 0 + \cdots$ is a graded module generated in degree 0 with M_i in $\mathcal{X}_T$. Consider the short exact sequence $0 \to M^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$. Then M^1 has support in degree 1. At degree 1, we obtain a short exact sequence of Λ_0-modules $0 \to M_1^1 \to \Lambda_1 \otimes_{\Lambda_0} M_0 \to M_1 \to 0$. By the $\otimes$-condition, $\Lambda_1 \otimes_{\Lambda_0} M_0$ is in $\mathcal{X}_T$. Since $\mathcal{X}_T$ is closed under kernels of epimorphisms by Proposition 1.4, we conclude that M_1^1 is in $\mathcal{X}_T$. By our remarks earlier in the proof, $M_1^1[-1]$ has a linear fat resolution. It now follows that M does. We have shown that M is a T-Koszul module.

Conversely, assume that M is a T-Koszul module. Then by definition M is generated in degree 0 with M_i in $\mathcal{X}_T$ for all i. This completes the proof of (c).

The proof of (d) will be postponed (see Theorem 2.3 in Chapter IV). □

The following example provides an interesting class of Koszul algebras with respect to T, where Λ_0 is an arbitrary finite dimensional algebra of the form kQ/I for a field k and the quiver of $\Lambda = \Lambda_0 + \Lambda_1$ has quiver Q together arrows from the "top" of Λ_1. These extra arrows can be arbitrary as shown in the next example. Thus the quiver of Λ as a finite dimensional algebra can be viewed as a directed graph with arrows of degree 0 (from Q) and arrows of degree 1 from Λ_1.

EXAMPLE 3.2. Let Λ_0 be a k-algebra of the form kQ/I where kQ is a path algebra and I is an admissible ideal. Let X be a finite dimensional semisimple Λ_0-bimodule. Let $\mathbf{r}$ denote the Jacobson radical of Λ_0. Set $\Lambda_1 = \Lambda_0 \otimes_{\Lambda_0/\mathbf{r}} X \otimes_{\Lambda_0/\mathbf{r}} \Lambda_0$. By Theorem 3.1, we see that $\Lambda = \Lambda_0 + \Lambda_1$ is a Koszul algebra with respect to any Wakamatsu cotilting Λ_0-module T. One can see this since $(\Lambda_i)_{\Lambda_0}$ is projective and $\Lambda_1 \otimes_{\Lambda_0} Y$ is a left projective Λ_0-module for any left Λ_0-module Y. Hence the $\otimes$-Tor-conditions hold. Note that X is arbitrary and the bimodule decomposition of X as a semisimple $\Lambda/\mathbf{r}$-module gives the "degree 1" arrows in the quiver of Λ.

We obtain two immediate corollaries to Theorem 3.1 that will allow constructing other classes of examples.

COROLLARY 3.3. *Let $\Lambda = \Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$ and $T = \Lambda_0$. Assume also that Λ_0 has finite global dimension. If Λ_1 is projective as a left Λ_0-module, then Λ is a Koszul algebra with respect to T.*

PROOF. Since Λ_0 has finite global dimension, Λ_0 is a cotilting module and $\mathcal{X}_T$ is the category of finitely generated projective modules. By Theorem 3.1, to show that Λ is a T-Koszul algebra, we need only show that the $\otimes$-Tor-conditions hold. Let M be in $\mathcal{X}_T$. Then by projectivity of Λ_1 and M, we see that $\Lambda_1 \otimes_{\Lambda_0} M$ is left projective and hence the $\otimes$-condition follows. Since T is left projective Λ_0-module, the Tor-condition is clear. □

Examples of this can be obtained if we have $\Lambda_0 = \Lambda_0' \oplus R_0$ as left modules, Λ_0 of finite global dimension, $\operatorname{Hom}_{\Lambda_0}(\Lambda_0', R_0) = (0)$ and choose $\Lambda_1 = R_0$. It is easy to see that R_0 in this case is a Λ_0-bimodule which is left projective. We provide two concrete examples of these types of T-Koszul algebras.

EXAMPLE 3.4. Let Λ_0 be the path algebra kQ where k is a field and Q is the quiver $v_0 \xrightarrow{a} v_2 \xleftarrow{b} v_1$. We write $kQ = (kQ)(v_0 + v_1) \oplus (kQ)v_2$. Let $R_0 = (kQ)v_2$. Note that $\Lambda_1 = R_0$ is the Λ_0-bimodule ${}_2k_2$; that is, k as a vector space and the simple module corresponding to the vertex 2 on both the left and right. We note that Λ_1 is a left projective but not a right projective Λ_0-module. It is easy to see the conditions of Corollary 3.3 are satisfied.

It is immediate that the quiver Q' of Λ is given by $\begin{tikzcd} v_0 \arrow[r, "a"] & v_2 \arrow[loop below, "c"] & v_1 \arrow[l, "b"'] \end{tikzcd}$ and that $\Lambda = KQ'/J^2$ where J is the ideal generated by the arrows. Note that the grading is obtained by giving the arrows a and b degree 0 and c degree 1.

The indecomposable T-Koszul modules are $\Lambda_0 v_i + 0$ for $i = 0, 1, 2$ and $\Lambda_0 v_2 + \Lambda_0 v_2$.

EXAMPLE 3.5. Let Q be a quiver with a set of relations $\rho = \{r_1, \ldots, r_n\}$ such that we can partition the vertices $Q_0 = V_1 \cup V_2$ to satisfy the following conditions:

(i) There are no arrows from a vertex in V_2 to a vertex in V_1.
(ii) $kQ/\langle\rho\rangle$ is of finite global dimension.

Let I be the ideal in kQ generated by the set of relations ρ. Set $\Lambda_0 = kQ/I$, and let v_i be the sum of vertices in V_i, for $i = 1, 2$. Then $\Lambda_0 = \Lambda_0 v_1 + \Lambda_0 v_2$ and $\operatorname{Hom}_{\Lambda_0}(\Lambda_0 v_2, \Lambda_0 v_1) = (0)$ by the condition above. Taking $\Lambda_1 = R_0 = \Lambda_0 v_1$ and $T = \Lambda_0$, we obtain T-Koszul algebras of the form $\Lambda_0 + \Lambda_1 + 0 + \cdots$.

Another class of algebras with $\Lambda = \Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$ can be obtained by using the next corollary of Theorem 3.1.

COROLLARY 3.6. *Let Λ_0 be a finite dimensional k-algebra, where k is a field, and let T be a Wakamatsu cotilting module. Suppose, for $i = 1, \ldots, n$, we choose left Λ_0-modules U_i in $\mathcal{X}_T$ and Q_i in $\mathcal{X}_T^{\perp}$. Let $\Lambda_1 = \oplus_{i=1}^n U_i \otimes_k D(Q_i)$, where D is the usual artin algebra duality. Then $\Lambda = \Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$ is a Koszul algebra with respect to T.*

PROOF. By Theorem 3.1, we must show that the $\otimes$-Tor-conditions hold. Let X be in $\mathcal{X}_T$. For each i, the module $U_i \otimes_k D(Q_i) \otimes_{\Lambda_0} X$ is a direct sum of copies of U_i and hence is in $\mathcal{X}_T$. Hence the $\otimes$-condition follows. Next consider $\operatorname{Tor}_1^{\Lambda_0}(U_i \otimes_k D(Q_i), X)$. As a right Λ_0-module, $U_i \otimes_k D(Q_i)$ is a direct sum of copies of $D(Q_i)$. Since Q_i is in $\mathcal{X}_T^{\perp}$, we have that $\operatorname{Ext}^1_{\Lambda_0}(X, D(U_i \otimes_k D(Q_i))) = (0)$. But

$$\operatorname{Tor}_1^{\Lambda_0}(U_i \otimes_k D(Q_i), X) = D\operatorname{Ext}^1_{\Lambda_0}(X, D(U_i \otimes_k D(Q_i)))$$

by [**16**, Theorem 9.51]. Hence the Tor-condition holds and we are done. □

Note that if Q_i is a left injective Λ_0-module, Q_i is in $\mathcal{X}_T^{\perp}$, and if U_i is a left projective Λ_0-module, then U_i is in $\mathcal{X}_T$. This allows an easy construction of T-Koszul algebras where Λ_0 need not be of finite global dimension and T can be an arbitrary Wakamatsu cotilting module. We also note that U_i need only to be in $\mathcal{X}_T$ and not necessarily projective. In such cases, the Λ_1 that we obtain is not left projective.

There are examples of T-Koszul algebras of the form $\Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$ that do not satisfy the hypothesis of the above two corollaries. One such class is the following. If Λ_0 is a selfinjective artin algebra, we may take $\Lambda_1 = \Lambda_0$ together with $T = \Lambda_0$. The $\otimes$-Tor-conditions are clearly satisfied under these hypotheses. We describe the T-Koszul modules and the duality in a special case.

Since T is an injective cogenerator, we see that $\mathcal{X}_T$ is $\operatorname{mod}\Lambda_0$. It follows that $\mathcal{K}_T(\Lambda)$ is $\operatorname{gr}_0 \Lambda$ in this case by Theorem 3.1. Furthermore, since $\Gamma_1 = \operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$, we see that Γ_1 can be identified with Λ_0^{op}. By Theorem 3.1 it follows that Γ is isomorphic to the tensor algebra $T_{\Lambda_0^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}})$.

Lest one begins to believe that most graded algebras of the form $\Lambda_0 + \Lambda_1 + 0 + 0 + \cdots$ is a T-Koszul algebra we give some 'nonexamples'. This is different from the Koszul algebra case where all algebras of the form $\Lambda_0 + \Lambda_1 + 0 + \cdots$ with Λ_0 a product of a field are Koszul algebras. We omit the details from the following examples and just state the results.

Example 3.7. a) Let Λ_0 be the path algebra kQ associated to the quiver $Q = v_1 \xrightarrow{a} v_2$. Let $\Lambda_1 = D(\Lambda_0)$ and $T = \Lambda_0$. Then $\Lambda_0 + D(\Lambda_0) + 0 + 0 + \cdots$ is not a T-Koszul algebra. This shows that trivial extensions $\Lambda_0 \ltimes D(\Lambda_0)$ of an artin algebra Λ_0 need not be a Koszul algebra with respect to $T = \Lambda_0$.

b) Let $\Lambda_0 = k[x]/(x^2)$, $T = \Lambda_0$ and let Λ_1 be the trivial Λ_0-bimodule k for a field k. Then $k[x]/(x^2) + k + 0 + \cdots$ is not a T-Koszul algebra.

c) Let Λ_0 be the path algebra as in part a) above. Let $T = \Lambda_0$, and let $\Lambda_1 = {}_1k_1$; that is, Λ_1 is the bimodule that is the simple corresponding to the vertex v_1 on the left and on the right. Then $\Lambda_0 + \Lambda_1 + 0 + \cdots$ is not a T-Koszul algebra. Note that by taking Λ_1 to be ${}_2k_2$, the algebra $\Lambda_0 + \Lambda_1$ is a T-Koszul algebra by Corollary 3.3.

d) Let Λ be the radical square zero algebra with quiver $v_1 \underset{b}{\overset{a}{\rightleftarrows}} v_2$. We view Λ as $\Lambda_0 + \Lambda_1 + 0 + \cdots$ by putting the arrow a in degree 0 and the arrow b in degree 1. Then Λ is not a T-Koszul algebra for $T = \Lambda_0$. We note that Λ is a preprojective algebra (see [**15**] for definition of a preprojective algebra).

4. Further examples and classes of T-Koszul algebras

In this section we give various examples of T-Koszul algebras which are not concentrated in degrees 0 and 1. We give an example of a T-Koszul algebra that is not strongly T-Koszul, and generalize results on quadratic monomial algebras from classical Koszul theory.

We start with giving an example of a T-Koszul algebra which is not strongly T-Koszul.

Example 4.1. Let k be a field and Λ_0 the path algebra associated to the quiver

$$v_1 \qquad v_2 \qquad v_{2'} \qquad v_3 \xrightarrow{a} v_4$$

We let P_i denote the left projective Λ_0-module associated to vertex i, and let P_i^* denote the right projective Λ_0-module associated to vertex i. Next let $\Lambda_1 = (P_2 \otimes_k P_1^*) \oplus (P_{2'} \otimes_k P_1^*) \oplus (P_3 \otimes_k P_2^*) \oplus (P_4 \otimes_k P_{2'}^*)$. It is now easy to see that the quiver Q of the tensor algebra $T_{\Lambda_0}(\Lambda_1)$ is

$$\begin{tikzcd} v_1 \arrow[r, "x"] \arrow[dr, "z"'] & v_2 \arrow[r, "y"] & v_3 \arrow[d, "a"] \\ & v_{2'} \arrow[r, "w"] & v_4 \end{tikzcd}$$

The tensor algebra is isomorphic to the path algebra kQ. The grading is given by the arrow a being in degree 0, and the arrows x, y, z, and w are in degree 1. Let I be the ideal in kQ generated by $ayx - wz$. Note that this is a homogeneous element of degree 2. We let $\Lambda = kQ/I = \Lambda_0 + \Lambda_1 + \Lambda_2 + 0 + 0 + \cdots$. Next, we note that Λ_2 is isomorphic to $P_3 \otimes_k P_1^*$.

First we claim that Λ is a Koszul algebra with respect to $T = \Lambda_0$. Since Λ_0 is an hereditary algebra, we see that T is a cotilting module. Furthermore, in this case, $\mathcal{X}_T$ is the category of finitely generated projective Λ_0-modules. To show that the $\otimes$-Tor-conditions hold, we note that both Λ_1 and Λ_2 are direct sums of Λ_0-bimodules of the form $P \otimes_K P'$ where P is a left projective Λ_0-module and P' is a right projective Λ_0-module. The $\otimes$-Tor-conditions are then immediate. It remains to show that Λ_0 has a linear fat resolution. For this we describe the graded structure of the indecomposable projective Λ-modules, $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_{2'}$, $\widehat{P}_3$, and $\widehat{P}_4$, where $\widehat{P}_i$ is the projective module associated to vertex i in the quiver Q. The module $\widehat{P}_1$ has basis v_1 in degree 0, basis x and z in degree 1, and basis yx and $ayx = wz$ in degree 2. The module $\widehat{P}_2$ has basis v_2 in degree 0, y and ay in degree 1. The module $\widehat{P}_{2'}$ has basis $v_{2'}$ in degree 0, and w in degree 1. The module $\widehat{P}_3$ has basis v_3 and a in degree 0. Finally, $\widehat{P}_4$ has basis v_4 in degree 0.

With these preliminaries we give a linear fat resolution of Λ_0, projective by projective (all tensors are over Λ_0).

$$0 \to \Lambda \otimes P_3[2] \xrightarrow{\binom{ay}{w}} ((\Lambda \otimes P_2) \oplus (\Lambda \otimes P_{2'})[1] \xrightarrow{(x,z)} \Lambda \otimes P_1 \to P_1 \to 0$$

$$0 \to \Lambda \otimes P_3[1] \xrightarrow{y} \Lambda \otimes P_2 \to P_2 \to 0$$

$$0 \to (\Lambda \otimes P_4)[1] \xrightarrow{(w)} \Lambda \otimes P_{2'} \to P_{2'} \to 0$$

$$0 \to \Lambda \otimes P_3 \to P_3 \to 0$$

$$0 \to \Lambda \otimes P_4 \to P_4 \to 0$$

This completes the proof that Λ is a T-Koszul algebra.

Next we show that there is a graded Λ-module M such that M is in $\mathrm{gr}_0\,\Lambda$, $M = M_1 + M_2 + 0 + \cdots$ with M_i in $\mathcal{X}_T$, $\mathcal{E}(M)$ is a Γ-module generated in degree 0, but M is not in $\mathcal{K}_T(\Lambda)$. Thus, although $\mathcal{E}(M)$ is generated in degree 0, M is not a T-Koszul module.

We let $M = \widehat{P}_1/L$ where L is the submodule of $\widehat{P}_1$ with basis z, yx, and $ayx = wz$. Thus M has a basis with two elements, $v_1 + L$ in degree 0 and $x + L$ in degree 1. First we show that M in not in $\mathcal{K}_T(\Lambda)$. We begin by noting that both M_0 and M_1 are in $\mathcal{X}_T$, since they are respectively isomorphic to the simple projective Λ_0-modules P_1 and P_2. We show that M does not have a linear fat resolution. Consider the kernel N of $\Lambda \otimes P_1 \to M$. We identify $\Lambda \otimes P_1$ with Q_1. If M were to be in $\mathcal{K}_T(\Lambda)$, the module N would need to be generated in degree 1. But N has two generators, z and yx, and yx is in degree 2. Then we conclude that M is not a T-Koszul module.

To complete the example, we need to compute $\mathcal{E}(M)$ and show that it is generated in degree 0 as a Γ-module. First we describe Γ. We have $\Gamma_0 = \mathrm{End}_{\Lambda_0}(\Lambda_0)$ since $T = \Lambda_0$. The algebra Γ_0 is the path algebra associated to the quiver

$$v_1 \qquad v_2 \qquad v_{2'} \qquad v_3 \xleftarrow{a^o} v_4$$

where the vertices correspond the obvious simple modules. Next, using Theorem 2.11 or by directly computing the minimal Λ-projective resolution of Λ_0 given above, and then computing the Ext-algebra, we see that Γ is the quotient of the path algebra kQ^o/J where Q^o is the quiver

$$\begin{tikzcd} v_1 \arrow[r, "y^o"] \arrow[dr, "x^o"] & v_{2'} \arrow[r, "w^o"] & v_4 \arrow[d, "a^o"] \\ & v_2 \arrow[r, "z^o"] & v_3 \end{tikzcd}$$

and where J is the ideal generated by $a^o w^o y^o + z^o x^o$. Here a^o is in degree 0 and x^o, y^o, z^o, and w^o are in degree 1.

To continue our computation of $\mathcal{E}(M)$, we provide the minimal graded Λ-projective resolution of M below.

$$0 \to Q_4[2] \xrightarrow{\binom{a}{w}} (Q_3[2]) \oplus (Q_{2'}) \xrightarrow{(zx,y)} Q_1 \to M \to 0.$$

Now it is easy to compute that $\mathcal{E}(M) = \mathcal{E}(M)_0 + \mathcal{E}(M)_1 + \mathcal{E}(M)_2 + 0 + \cdots = \Gamma v_1/U$ where U is the left submodule of Γv_1 generated by x^0. Thus, $\mathcal{E}(M)$ is generated in degree 0 even though M does not have a linear fat resolution.

We do remark though that the duality we have described on $\mathcal{K}_T(\Lambda)$ does extend to include M. We also note that although $\mathcal{E}(M)$ has a linear fat resolution, $\mathcal{E}(M)_2$ is isomorphic to the simple Γ_0-module associated to vertex v_4 and is not in $\mathcal{X}_T$. Thus, $\mathcal{E}(M)$, as must be, is not in $\mathcal{K}_T(\Gamma)$.

Some of the motivation for this paper was to provide a common framework for Koszul and cotilting theory. The next class of examples, pointed out to us by Lutz Hille, is obtained from the two classical cases using tensor product.

PROPOSITION 4.2. *Let $\Sigma = \Sigma_0 + \Sigma_1 + \cdots$ be a Koszul k-algebra, where k is a field. Suppose that A is a finite dimensional k-algebra, and choose any Wakamatsu cotilting A-module T'. Then the left finite graded ring $\Lambda = \Sigma \otimes_k A$ is a Koszul algebra with respect to the $\Sigma_0 \otimes_k A$-module $T = \Sigma_0 \otimes_k T'$.*

Moreover, $E_T(\Lambda) \simeq E(\Sigma) \otimes_k \mathrm{End}_A(T')$ as graded algebras, where $E(\Sigma)$ is the Koszul dual of Σ.

PROOF. It is easy to see that

$$\mathrm{Hom}_{\Sigma_0 \otimes_k A}(P \otimes_k X, Q \otimes_k Y) \simeq \mathrm{Hom}_{\Sigma_0}(P, Q) \otimes_k \mathrm{Hom}_A(X, Y),$$

where P and Q are Σ_0-modules and X and Y are in $\mathrm{mod}\, A$. This implies that

$$\mathrm{Ext}^i_{\Sigma_0 \otimes_k A}(\Sigma_0 \otimes_k X, \Sigma_0 \otimes_k Y) \simeq \Sigma_0 \otimes_k \mathrm{Ext}^i_A(X, Y)$$

for all modules X and Y in $\mathrm{mod}\, A$. From this we obtain that $T = \Sigma_0 \otimes_k T'$ is a selforthogonal module and that $\mathcal{X}_T = \mathrm{add}(\Sigma_0 \otimes_k \mathcal{X}_{T'})$. Hence T is a Wakamatsu cotilting Λ_0-module.

Since $\Lambda_i = \Sigma_i \otimes_k A$ is a projective right Λ_0-module, Λ satisfies the Tor-condition with respect to T.

By [**8**, page 165] we have that

$$(\Sigma_i \otimes_k X) \otimes_{\Sigma_0 \otimes_k A} (\Sigma_0 \otimes_k Y) \simeq \Sigma_i \otimes_{\Sigma_0} (X \otimes_A Y)$$

for all right A-modules X and left A-modules Y. Hence we infer that Λ satisfies the $\otimes$-condition.

Since Σ is a Koszul algebra, Σ_0 has a linear projective resolution. Tensoring this resolution with A over k yields a linear fat resolution of $\Lambda_0 = \Sigma_0 \otimes_k A$, and by definition Λ is a T-Koszul algebra with respect to T.

Similar as above one can show that

$$\operatorname{Hom}_{\Sigma \otimes_k A}(P \otimes_k X, Q \otimes_k Y) \simeq \operatorname{Hom}_\Sigma(P, Q) \otimes_k \operatorname{Hom}_A(X, Y),$$

where P is a projective Σ-module, Q is a Σ-module and X and Y are in $\operatorname{mod} A$. Using this and the description of $\Gamma = E_T(\Lambda)$ in Theorem 2.11 (c), it follows that $\Gamma_0 \simeq \Sigma_0^{\mathrm{op}} \otimes_k \operatorname{End}_A(T')$, the degree one part $\Gamma_1 \simeq \operatorname{Hom}_{\Sigma_0}(\Sigma_1, \Sigma_0) \otimes_k \operatorname{End}_A(T')$, and $E_T(\Lambda) \simeq T_{\Gamma_0}(\Gamma_1)/\langle A_2 \rangle$, where $A_2 = A_2' \otimes_k \operatorname{End}_A(T')$ with

$$E(\Sigma) = T_{\Sigma_0^{\mathrm{op}}}(\operatorname{Hom}_{\Sigma_0}(\Sigma_1, \Sigma_0))/\langle A_2' \rangle.$$

Since tensoring over the field k is an exact functor and

$$T_{\Sigma_0^{\mathrm{op}} \otimes_k \operatorname{End}_A(T')}(\operatorname{Hom}_{\Sigma_0}(\Sigma_1, \Sigma_0) \otimes_k \operatorname{End}_A(T')) \simeq T_{\Sigma_0^{\mathrm{op}}}(\operatorname{Hom}_{\Sigma_0}(\Sigma_1, \Sigma_0)) \otimes_k \operatorname{End}_A(T'),$$

we obtain that $E_T(\Lambda) \simeq E(\Sigma) \otimes_k \operatorname{End}_A(T')$ as graded algebras. □

The following result is a useful result for constructing further examples.

PROPOSITION 4.3. *Suppose Λ is a graded algebra generated in degrees* 0 *and* 1. *Let $T = D(\Lambda_0)$ (where Λ_0 is viewed as a right projective Λ_0-module). If Λ_i is a right projective Λ_0-module for each $i \geq 0$, then Λ is T-Koszul if and only if Λ_0 is in $\mathcal{K}_T(\Lambda)$. Moreover, if Λ is a T-Koszul algebra, then Λ is a strongly T-Koszul algebra.*

PROOF. By definition, if Λ is T-Koszul, then Λ_0 is in $\mathcal{K}_T(\Lambda)$. Assuming that Λ_0 is in $\mathcal{K}_T(\Lambda)$, we need to show that the $\otimes$-Tor-conditions hold. Since each Λ_i is a right projective Λ_0-module the Tor-condition holds. Since T is an injective cogenerator of $\operatorname{mod} \Lambda_0$, it follows that $\mathcal{X}_T = \operatorname{mod}(\Lambda_0)$. Hence, the $\otimes$-condition holds. We postpone the last part of the proof until Proposition 3.3 of Chapter IV. □

An easy example of a T-Koszul algebra obtained by applying the previous proposition is the following.

EXAMPLE 4.4. Let Λ_0 be a self-injective artin algebra, $T = \Lambda_0$, and $\Lambda = \Lambda_0 + \Lambda_0 + \Lambda_0 + \cdots$. Then Λ_0 is in $\mathcal{K}_T(\Lambda)$ since

$$0 \to (\Lambda \otimes_{\Lambda_0} \Lambda_0)[1] \to \Lambda \otimes_{\Lambda_0} \Lambda_0 \to \Lambda_0 \to 0$$

is a linear fat resolution of Λ_0. By Proposition 4.3 the algebra Λ is a T-Koszul algebra. It follows from Theorem 2.11 that $\Gamma = \Lambda_0^{\mathrm{op}} + \Lambda_0^{\mathrm{op}} + 0 + \cdots$.

A second example is the following.

EXAMPLE 4.5. Let Q the quiver

$$a \circlearrowleft v_1 \qquad v_2 \qquad v_3 \qquad v_4$$

and $\Lambda_0 = kQ/\langle a^2\rangle$. Thus Λ_0 is isomorphic to $k[x]/(x^2)\times k\times k\times k$ and is selfinjective. Let $\Lambda_1 = {}_1k_2 + {}_3k_2 + {}_4k_2 + {}_3k_3 + {}_4k_4$. Let $\Lambda_2 = (\Lambda_1 \otimes_{\Lambda_0} \Lambda_1)/({}_3k_3 \otimes {}_3k_2 + {}_3k_3 \otimes {}_3k_3)$. Thus, as a Λ_0-bimodule, $\Lambda_2 = {}_4k_2 + {}_4k_4$. Setting $\Lambda_i = \Lambda_1 \otimes_{\Lambda_0} \Lambda_{i-1}$ for $i \geq 3$, we see that $\Lambda_i = \Lambda_2$ as Λ_0-bimodules for $i \geq 3$. If we set $T = \Lambda_0$, noting that each Λ_i is a right projective Λ_0-module, we see that the conditions of Proposition 4.3 are met and Λ is T-Koszul if and only if Λ_0 is in $\mathcal{K}_T(\Lambda)$.

We must show that Λ_0 is in $\mathcal{K}_T(\Lambda)$. For this, we provide a linear fat resolution for each indecomposable projective summand of Λ_0. We let Q_i denote the left projective Λ_0-module $\Lambda_0 v_i$ and S_i denote the associated simple module. Then we have

$$0 \to \Lambda \otimes_{\Lambda_0} Q_1 \to Q_1 \to 0$$

$$\Lambda \otimes_{\Lambda_0} (S_1 \oplus Q_3 \oplus Q_4) \to \Lambda \otimes_{\Lambda_0} Q_2 \to Q_2 \to 0$$

$$\cdots \to \Lambda \otimes_{\Lambda_0} (S_1 \oplus Q_3)[2] \to \Lambda \otimes_{\Lambda_0} (S_1 \oplus Q_3)[1] \to$$

$$\cdots \to \Lambda \otimes_{\Lambda_0} Q_3[2] \to \Lambda \otimes_{\Lambda_0} Q_3[1] \to\to \Lambda \otimes_{\Lambda_0} Q_3 \to Q_3 \to 0$$

$$0 \to \Lambda \otimes_{\Lambda_0} Q_4[1] \to \Lambda \otimes_{\Lambda_0} Q_4 \to Q_4 \to 0$$

We remark that Λ_1 is not a left projective Λ_0-module.

We now provide an interesting class of Koszul algebras with respect to T whose construction is related to the construction given in Example 3.2. As in that example, we give a method for constructing quotients of path algebras where the arrows occur in degree 0 or 1. This method is quite general, and given any finite dimensional k-algebra Λ_0 of the form kQ, we provide a method of 'adding' an arbitrary finite number of arrows in degree 1 with no restrictions. Thus, the 'quiver' of Λ will be the quiver of Λ_0 plus the degree one arrows.

EXAMPLE 4.6. Let Λ_0 be a k-algebra of the form kQ/I where kQ is a path algebra and I is an admissible ideal. Let $\{v_1, \ldots, v_n\}$ be the vertices of Q. Let X be a finite dimensional semisimple Λ_0-bimodule. Thus X is a direct sum of modules of the form ${}_ik_j$ where ${}_ik_j$ is Λ_0-bimodule which is isomorphic to the simple left Λ_0-module associated to the vertex v_i and is isomorphic to the simple right Λ_0-module associated to the vertex v_j. Let $\mathbf{r}$ denote the Jacobson radical of Λ_0. Since I is an admissible ideal, we may view Λ_0 as a left or right $\Lambda_0/\mathbf{r}$-module.

Set $\Lambda_1 = \Lambda_0 \otimes_{\Lambda_0/\mathbf{r}} X \otimes_{\Lambda_0/\mathbf{r}} \Lambda_0$. We now take Λ to be the tensor algebra $T_{\Lambda_0}(\Lambda_1)$. First we show that the $\otimes$-Tor-conditions are satisfied. Let $n \geq 1$ and consider Λ_n. As a right Λ_0-module it is of the form $Y \otimes_{\Lambda_0/\mathbf{r}} \Lambda_0$ for some Λ_0-module Y. But such a module is a right projective Λ_0-module, and hence the Tor-condition holds. Next, if X is in $\mathcal{X}_T$, the module $\Lambda_n \otimes_{\Lambda_0} X$ is of the form $\Lambda_0 \otimes_{\Lambda_0/\mathbf{r}} Z$ for some $\Lambda_0/\mathbf{r}$-module Z. Hence $\Lambda_n \otimes_{\Lambda_0} X$ is a left projective Λ_0-module. Since projective Λ_0-modules are in $\mathcal{X}_T$, we see that the $\otimes$-condition is satisfied.

It remains to show that Λ_0 has a linear fat resolution. We see that the kernel of $\Lambda \otimes_{\Lambda_0} \Lambda_0 \to \Lambda_0$ is just $\Lambda_{\geq 1}$. But by the construction of Λ we see that $\Lambda_{\geq 1} = \Lambda \otimes_{\Lambda_0} \Lambda_1[1]$, which is a fat module. Thus, the linear fat resolution is

$$0 \to \Lambda \otimes_{\Lambda_0} \Lambda_1[1] \to \Lambda \otimes_{\Lambda_0} \Lambda_0 \to \Lambda_0 \to 0.$$

Hence Λ is a T-Koszul algebra for any Wakamatsu cotilting module T.

EXAMPLE 4.7. Let Λ_0 and Λ_1 be as above. If we set $\Lambda = \Lambda_0 + \Lambda_1 + 0 + \cdots$, then by some of the above arguments we have that Λ satisfies the $\otimes$-Tor-conditions for any Wakamatsu cotilting module T. Hence, by Theorem 3.1, it follows that Λ is a T-Koszul algebra.

Our next class of examples generalizes the fact that quadratic monomial algebras are Koszul algebras. We begin with an arbitrary artin algebra Λ_0 and T a Wakamatsu cotilting Λ_0-module. Let Λ_1 be a Λ_0-bimodule which satisfies the $\otimes$-Tor-conditions. Note, that as a left Λ_0-module, Λ_1 is in $\mathcal{X}_T$. Let $\Lambda_1 = \oplus_{i=1}^n B_i$ be a decomposition of Λ_1 into indecomposable Λ_0-bimodules. We introduce some notational conventions. Let $S(q)$ denote the set of length q sequences of integers taken from the set of integers between 1 and n. If α is in $S(q)$ with $\alpha = (a_1, \ldots, a_q)$, then we define B_α to be the Λ_0-bimodule $B_{a_1} \otimes_{\Lambda_0} B_{a_2} \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} B_{a_q}$. Thus, for example, $\Lambda_1^{\otimes^q_{\Lambda_0}} = \oplus_{\alpha \in S(q)} B_\alpha$.

Our assumption that Λ_1 satisfies the $\otimes$-Tor-conditions implies that for each $q \geq 1$ and each α in S_q, as a left module, B_α is in $\mathcal{X}_T$ and B_α satisfies the same conditions. We now fix a subset G of $S(2)$. We let $G(m) = \{\alpha = (a_1, \ldots, a_m) \in S(m) \mid \text{for some } i, 1 \leq i < m, (a_i, a_{i+1}) \in G\}$ for $m \geq 2$. Thus, $G(2) = G$. Let $L(m) = S(m) \setminus G(m)$ for $m \geq 2$. Thus, $L(m)$ are the sequences $(a_1, \ldots, a_m)$ of length m such that for each i where $1 \leq i < m$, the pair (a_i, a_{i+1}) is not in G. Also of interest is the set $K(m) = \{(a_1, \ldots, a_m) \in S(m) \mid \text{for each } i, 1 \leq i < m, (a_i, a_{i+1}) \in G(2)\}$.

Let I_G denote the ideal in the tensor algebra $T_{\Lambda_0}(\Lambda_1)$ generated by $\{B_g \mid g \in G\}$. Note that I_G is graded ideal in $T_{\Lambda_0}(\Lambda_1)$ generated in degree 2 where $(I_G)_m = \oplus_{\alpha \in G(m)} B_\alpha$. Let $\Lambda = T_{\Lambda_0}(\Lambda_1)/I_G = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3 + \cdots$ where $\Lambda_m = \Lambda_1^{\otimes^m_{\Lambda_0}}/(I_G)_m$. It is easy to see that Λ_m is isomorphic to $\oplus_{\alpha \in L(m)} B_\alpha$ and that we have exact sequences

$$0 \to \oplus_{\alpha \in G(m)} B_\alpha \to \oplus_{\alpha \in S(m)} B_\alpha \to \Lambda_m \to 0.$$

We note that the $\otimes$-condition implies that if α is in $S(m)$, then B_α is in $\mathcal{X}_T$. We will show that Λ is a T-Koszul algebra.

We begin by noting that our earlier remarks imply that Λ satisfies the $\otimes$-Tor-conditions. Thus, we must show that Λ_0 has a linear fat resolution. For ease of notation we let $\otimes$ mean $\otimes_{\Lambda_0}$ for the remainder of this class of examples. We first consider the fat Λ-module $F_m = \Lambda \otimes \oplus_{\alpha \in K(m)} B_\alpha[m]$. In degree $p \geq m$, $(F_m)_p = \Lambda_{p-m} \otimes \oplus_{\alpha \in K(m)} B_\alpha$. Using the observation that $\Lambda_k = \oplus_{\alpha \in L(k)} B_\alpha$, we see that

$$(F_m)_p = \oplus_{\substack{\alpha=(a_1,\ldots,a_p) \\ (a_1,\ldots,a_{p-m}) \in L(p-m) \\ (a_{p-m+1},\ldots,a_p) \in K(m)}} B_\alpha.$$

We now give a linear fat resolution of Λ_0. Consider

$$\cdots \xrightarrow{\mu_{m+1}} F_m \xrightarrow{\mu_m} \cdots \to F_3 \xrightarrow{\mu_3} F_2 \xrightarrow{\mu_2} \Lambda \otimes \Lambda_1[1] \xrightarrow{\mu_1} \Lambda \otimes \Lambda_0 \xrightarrow{\mu_0} \Lambda_0 \to 0, \quad (**)$$

where μ_0 and μ_1 are the obvious maps and μ_m for $m \geq 2$ is described as follows. By the description of $(F_m)_p$ for $p \geq m$, we need to define μ_m in degree p from the summands of $(F_m)_p$ to the $(F_{m-1})_p$. A summand of $(F_m)_p$ is B_α where $\alpha = (a_1, \ldots, a_d)$ where $(a_1, \ldots, a_{p-m})$ is in $L(p-m)$ and $(a_{p-m+1}, \ldots, a_p)$ is in $K(m)$. We have that B_α is a term in the direct sum for $(F_{m-1})_p$ if and only if (a_{p-m}, a_{p-m+1}) is not in $G(2)$. We define μ_m in degree p on B_α to be the identity if B_α is a summand in $(F_{m-1})_p$ and to be 0 on B_α otherwise. It is now a straightforward computation that $(**)$ is a linear fat resolution.

We now turn our attention to $\Gamma = \oplus_{n\geq 0}\operatorname{Ext}^n_\Lambda(T,T)$. Let $(**)$ be the linear fat resolution of Λ_0 and let $X^n = \operatorname{Im}(\mu_n\colon F^n \to F^{n-1})$. It follows from the definition of a linear fat resolution that $F^n = \Lambda\otimes_{\Lambda_0} X^n_n$. We will show that $\Gamma_n = \operatorname{Hom}_{\Lambda_0}(X^n_n\otimes T, T)$ for $n\geq 1$. Thus, $\Gamma_1 = \oplus_{i=1}^n \operatorname{Hom}_{\Lambda_0}(B_i\otimes T,T)$. Setting $C_i = \operatorname{Hom}_{\Lambda_0}(B_i\otimes T,T)$, we see that Γ_1 decomposes into a direct sum of bimodules $\oplus_{i=1}^n C_i$. By Theorem 2.5 the algebra Γ is a T-Koszul algebra with respect to $T=\mathcal{E}(\Lambda)$ and hence is a quadratic algebra by Theorem 2.11. Since Γ is generated in degrees 0 and 1, we see that Γ is isomorphic to a quotient of the tensor algebra $T_{\Gamma_0}(\Gamma_1)/J$ by an ideal J generated by elements of degree 2. The ideal J is generated by all $C_i\otimes_{\Gamma_0} C_j$ where (i,j) is in $L(2) = S(2)\setminus G(2)$. This corresponds to the 'orthogonal' relations to the generators $B_i\otimes B_j$ with (i,j) in $G(2)$ of I_G.

We summarize the above discussion.

Example 4.8. Let Λ_0 be an artin algebra and T a Wakamatsu cotilting Λ_0-module. Let Λ_1 be a Λ_0-bimodule such that $\Lambda_1 = \oplus_{i=1}^n B_i$ with each B_i an indecomposable Λ_0-bimodule. Assume that Λ_1 satisfies the $\otimes$-Tor-conditions and let I be an ideal in the tensor algebra $T_{\Lambda_0}(\Lambda_1)$ generated by some of the summands of $\Lambda_1\otimes_{\Lambda_0}\Lambda_1$ of the form $B_i\otimes_{\Lambda_0} B_j$. Then $T_{\Lambda_0}(\Lambda_1)/I$ is a T-Koszul algebra. Furthermore, Γ is isomorphic to the tensor algebra $T_{\Gamma_0}(\Gamma_1)/J$, the bimodule $\Gamma_1 = \oplus_{i=1}^n C_i$ where the C_i are defined above, and J is generated by the summands $C_i\otimes_{\Gamma_0} C_j$ for the pairs (i,j) such that $B_i\otimes B_j$ is not a generator of I.

We end this section with an example that does not fit into the definitions of this section. The following algebra is T-Koszul in the more general sense we define later, but does not satisfy the Tor-condition.

Example 4.9. Let $\Lambda_0 = k[x]/(x^2)\times k[y]/(y^2)$ for a field k. Let $e_1 = (1,0)$ and $e_2=(0,1)$. Let $\mathbf{r}$ denote the Jacobson radical of Λ_0. Let $S = (\Lambda_0/\mathbf{r})e_1$ and $S' = e_2(\Lambda_0/\mathbf{r})$. We let $\Lambda_1 = S\otimes_k S'$, so Λ_1 is isomorphic to S as a left Λ_0-module and to S' as a right Λ_0-module. Let $\Lambda = \Lambda_0+\Lambda_1+0+\cdots$. Let $T=\Lambda_0$. Then $\mathcal{X}_T = \operatorname{mod}(\Lambda_0)$. But $\operatorname{Tor}_1^{\Lambda_0}(\Lambda_1,T)\neq 0$. Thus the Tor-condition fails. We will see that there is a more general theory in which Λ is a Koszul algebra with respect to T.

CHAPTER II

Main results with proofs

The aim of this chapter is to prove a duality theorem for the Koszul modules of a T-Koszul algebra and its Ext-algebra. We have collected various preliminary results in the first section, and in the last section we deal with quadratic algebras. These results were stated without proof in Chapter I.

1. Preliminaries

In this section we give some preliminary results needed in the proof of the duality theorem. It is natural to study subcategories of the category $\mathcal{G}_T(\Lambda)$ of modules which are generated in degree zero and such that the modules in the image of $\mathcal{E}$ are also generated in degree zero as a module over the subalgebra of $E_T(\Lambda)$ generated by $E_T(\Lambda)_0$ and $E_T(\Lambda)_1$. We show that for a T-Koszul algebra Λ the category $\mathcal{K}_T(\Lambda)$ of Koszul modules is a subcategory of $\mathcal{G}_T(\Lambda)$. But first we investigate when bimodules satisfy the $\otimes$-Tor-conditions with respect to a resolving subcategory. We show that such bimodules are closed under tensor products and kernels of epimorphisms of bimodules.

LEMMA 1.1. *Let Σ be an artin algebra, and let $\mathcal{Y}$ be a resolving subcategory of* $\operatorname{mod}\Sigma$. *Let X and Y be two Σ-bimodules satisfying the* $\otimes$-Tor-*conditions with respect to $\mathcal{Y}$.*

(a) *$X \otimes_\Sigma Y$ satisfies the* $\otimes$-Tor-*conditions with respect to $\mathcal{Y}$.*
(b) *If $0 \to K \to Y \to X \to 0$ is an exact sequence of Σ-bimodules with X and Y as above, then K also satisfies the* $\otimes$-Tor-*conditions with respect to $\mathcal{Y}$.*

PROOF. (a) Let Z be in $\mathcal{Y}$. Since $\operatorname{Tor}_i^\Sigma(X,Y) = (0)$ and $\operatorname{Tor}_i^\Sigma(Y,Z) = (0)$ for all $i \geq 1$, it follows essentially from [**8**, Theorem 2.8] that $\operatorname{Tor}_i^\Sigma(X \otimes_\Sigma Y, Z) \simeq \operatorname{Tor}_i^\Sigma(X, Y \otimes_\Sigma Z)$, which is zero by assumption. Hence $X \otimes_\Sigma Y$ satisfies the Tor-condition with respect to $\mathcal{Y}$. Clearly $X \otimes_\Sigma Y$ satisfies the $\otimes$-condition with respect to $\mathcal{Y}$.

(b) The long exact sequence in Tor-functors immediately implies that K satisfies the Tor-condition with respect to $\mathcal{Y}$.

Let Z be in $\mathcal{Y}$. Since X satisfies the Tor-condition, the sequence $0 \to K \otimes_\Sigma Z \to Y \otimes_\Sigma Z \to X \otimes_\Sigma Z \to 0$ is exact. By assumption $X \otimes_\Sigma Z$ and $Y \otimes_\Sigma Z$ are in $\mathcal{Y}$, hence $K \otimes_\Sigma Z$ is in $\mathcal{Y}$, since $\mathcal{Y}$ is resolving. Therefore K satisfies the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$. $\square$

Now we formalize and generalize the $\otimes$-Tor-conditions for graded rings introduced in the previous chapter (see Definition 2.1 of Chapter I).

DEFINITION 1.2. Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, and let $\mathcal{Y}$ be a subcategory of $\operatorname{mod}\Lambda_0$. Let X be a Λ-Λ_0-bimodule generated as a left Λ-module in positive degrees. Then X is said to satisfy the $\otimes$-Tor-*conditions* with respect to $\mathcal{Y}$ if X_i satisfies the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$ for all $i \geq 0$.

The next result is used to show that for a T-Koszul algebra Λ the kernels X^i in the fat resolution of Λ_0 satisfy the $\otimes$-Tor-conditions with respect to $\mathcal{X}_T$.

PROPOSITION 1.3. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, and let $\mathcal{Y}$ be a resolving subcategory of $\operatorname{mod}\Lambda_0$. Suppose Λ satisfies the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$. Let X be in $\operatorname{gr}_0\Lambda$ with X a Λ-Λ_0-bimodule satisfying the $\otimes$-Tor-condition with respect to $\mathcal{Y}$.*

Then X^1 is a Λ-Λ_0-bimodule satisfying the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$, where we recall that X^1 is given by the exact sequence $0 \to X^1 \to \Lambda \otimes_{\Lambda_0} X_0 \to X \to 0$.

PROOF. This follows directly from the definitions involved and Lemma 1.1. $\square$

COROLLARY 1.4. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, and let $\mathcal{Y}$ be a resolving subcategory of $\operatorname{mod}\Lambda_0$. Suppose that Λ satisfies the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$ and that*

$$\cdots \to \Lambda \otimes_{\Lambda_0} X_2^2[2] \xrightarrow{d_2} \Lambda \otimes_{\Lambda_0} X_1^1[1] \xrightarrow{d_1} \Lambda \otimes_{\Lambda_0} \Lambda_0 \xrightarrow{d_0} \Lambda_0 \to 0$$

is exact, with $X^i = \operatorname{Ker} d_{i-1}$.

Then the Λ-Λ_0-bimodules X^i satisfy the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$ for all $i \geq 1$.

PROOF. This is a direct consequence of Proposition 1.3 and the assumptions on Λ, since Λ_0 is a Λ-Λ_0-bimodule that satisfies the $\otimes$-Tor-conditions with respect to $\mathcal{Y}$. $\square$

In order to show that $\mathcal{K}_T(\Lambda)$ is a subcategory of $\mathcal{G}_T(\Lambda)$, the following criterion for computation of Ext-groups is essential.

LEMMA 1.5. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting module over Λ_0. Let M be a module in $\operatorname{gr}_0\Lambda$ with M_0 in ${}^{\perp}_{\Lambda_0}T$.*

(a)
$$\operatorname{Ext}^1_\Lambda(M,T) \simeq \operatorname{Hom}_\Lambda(M^1,T)$$
as $\Gamma_0 = \operatorname{End}_{\Lambda_0}(T)$-modules, where M^1 as before is given by the exact sequence $0 \to M^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$.

(b) *If M^1 is generated in degree 1, then $\operatorname{Ext}^1_\Lambda(M,T) \simeq \operatorname{Hom}_{\Lambda_0}(M^1_1,T)$ and the isomorphism is given by sending a map $f\colon M^1_1 \to T$ to the extension given by the pushout*

$$\begin{tikzcd} 0 \arrow[r] & M^1 \arrow[r] \arrow[d, "fp"] & \Lambda \otimes_{\Lambda_0} M_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0, \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where $p\colon M^1 \to M^1_1$ is the natural epimorphism.

PROOF. (a) Let $\eta\colon 0 \to \Omega_\Lambda(M) \xrightarrow{g} \Lambda \otimes_{\Lambda_0} P_{\Lambda_0}(M_0) \to M \to 0$ be the projective cover of M. From Lemma 0.4 we have that

$$\operatorname{Hom}_\Lambda(\Omega_\Lambda(M), T) \simeq \operatorname{Hom}_{\Lambda_0}(\Omega_{\Lambda_0}(M_0), T) \oplus \operatorname{Hom}_\Lambda(M^1, T),$$

since $\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle = M^1$. Applying the functor $\operatorname{Hom}_\Lambda(\ ,T)$ to η then gives rise to the following exact sequence, using in addition that $\operatorname{Hom}_\Lambda(X,T) \simeq \operatorname{Hom}_{\Lambda_0}(X_0, T)$ for all X in $\operatorname{gr}_0 \Lambda$.

$$0 \to \operatorname{Hom}_{\Lambda_0}(M_0, T) \to \operatorname{Hom}_{\Lambda_0}(P_{\Lambda_0}(M_0), T) \to \\ \operatorname{Hom}_{\Lambda_0}(\Omega_{\Lambda_0}(M_0), T) \oplus \operatorname{Hom}_\Lambda(M^1, T) \to \operatorname{Ext}^1_\Lambda(M, T) \to 0.$$

Since the image of the map $g^*\colon \operatorname{Hom}_\Lambda(\Lambda \otimes_{\Lambda_0} P_{\Lambda_0}(M_0), T) \to \operatorname{Hom}_\Lambda(\Omega_\Lambda(M), T)$ is contained in $\operatorname{Hom}_{\Lambda_0}(\Omega_{\Lambda_0}(M_0), T)$ and M_0 is in ${}^{\perp}_{\Lambda_0}T$, we infer that $\operatorname{Hom}_\Lambda(M^1, T) \simeq \operatorname{Ext}^1_\Lambda(M, T)$ as Γ_0-modules.

(b) This is a straightforward consequence of (a) and the definition of the connecting homomorphism for the long exact sequence. □

Let Λ and T be as above, and let $\Gamma = E_T(\Lambda)$. A central ingredient for proving that $\mathcal{K}_T(\Lambda)$ and $\mathcal{K}_T(\Gamma)$ are dual categories is the following result, here formulated in a more general setting.

LEMMA 1.6. *Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a left finite graded ring generated in degrees 0 and 1 with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting module over Λ_0. If M is in $\mathcal{G}_T(\Lambda)$, then the sequence*

$$0 \to \mathcal{E}(M_{\geq 1})[1] \to \mathcal{E}(M_0) \to \mathcal{E}(M) \to 0$$

is exact.

PROOF. The sequence $\eta\colon 0 \to M_{\geq 1} \to M \xrightarrow{p} M_0 \to 0$ is clearly an exact sequence of graded Λ-modules. Hence we obtain an induced homomorphism $\mathcal{E}(p)\colon \mathcal{E}(M_0) \to \mathcal{E}(M)$. Since $\mathcal{E}(p)$ is onto in degree 0 and M is in $\mathcal{G}_T(\Lambda)$, we infer that $\mathcal{E}(p)$ is an epimorphism. In addition $\operatorname{Hom}_\Lambda(M_0, T) \simeq \operatorname{Hom}_\Lambda(M, T)$, so that the long exact sequence induced from η implies that the sequence

$$0 \to \mathcal{E}(M_{\geq 1})[1] \to \mathcal{E}(M_0) \to \mathcal{E}(M) \to 0$$

is exact. □

The proof of the duality involves identifying the groups $\operatorname{Ext}^i_\Lambda(M, T)$ with homomorphism groups over Λ_0 for a module M in $\mathcal{K}_T(\Lambda)$ over a T-Koszul algebra Λ. For this it is useful to give a orthogonality property of the Λ_0-module $\operatorname{Hom}_{\Lambda_0}(X, T)$ for a Λ_0-bimodule X satisfying the $\otimes$-Tor-conditions with respect to $\mathcal{X}_T$. This is done in the next result. Furthermore, given the Tor-condition this result reduces the verification of the $\otimes$-condition for a bimodule to checking a property of only one module.

LEMMA 1.7. *Let Σ be an artin algebra, and let T be a Wakamatsu cotilting module over Σ. Let X be a Σ-bimodule with ${}_\Sigma X$ in $\mathcal{X}_T$, and let $\mathcal{Y}$ be a resolving subcategory of $\mathcal{X}_T$. Assume that X satisfies the* Tor*-condition with respect to $\mathcal{Y}$. Then $X \otimes_\Sigma Y$ is in ${}^\perp T$ for all Y in $\mathcal{Y}$ if and only if* $\operatorname{Hom}_\Sigma(X, T)$ *is in $\mathcal{Y}^\perp$.*

PROOF. Let Y be in $\mathcal{Y}$, and let $\cdots \to P^1 \to P^0 \to Y \to 0$ be a projective resolution of Y over Σ. By the Tor-condition with respect to $\mathcal{Y}$ the sequence

$$\cdots \to X \otimes P^1 \to X \otimes P^0 \to X \otimes Y \to 0$$

is exact. Since ${}_\Sigma X$ is in $\mathcal{X}_T$, the module $X \otimes P^j$ is in $\operatorname{add}{}_\Sigma X$ for all j and hence in $\mathcal{X}_T$. Using these observations it follows that the homology of the upper complex in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (X \otimes P^0, T) \arrow[r] \arrow[d, dash, "\wr"] & (X \otimes P^1, T) \arrow[r] \arrow[d, dash, "\wr"] & (X \otimes P^2, T) \arrow[r] \arrow[d, dash, "\wr"] & \cdots \\
0 \arrow[r] & (P^0, (X,T)) \arrow[r] & (P^1, (X,T)) \arrow[r] & (P^2, (X,T)) \arrow[r] & \cdots
\end{tikzcd}$$

is $\operatorname{Ext}^j_\Sigma(X \otimes Y, T)$ and the homology of the lower complex is $\operatorname{Ext}^j_\Sigma(Y, (X,T))$. Hence $X \otimes Y$ is in ${}^\perp T$ for all Y in $\mathcal{Y}$ if and only if $\operatorname{Hom}_\Sigma(X,T)$ is in $\mathcal{Y}^\perp$. □

We have seen that T being a cotilting module gives nice properties for T-Koszul algebras. That this is also the case for the $\otimes$-Tor-conditions is an easy consequence of Lemma 1.7.

Corollary 1.8. *Let Σ be an artin algebra, and let T be a cotilting module over Σ. Let X be a Σ-bimodule with ${}_\Sigma X$ in $\mathcal{X}_T$. Assume that X satisfies the* Tor-*condition with respect to $\mathcal{X}_T$. Then X satisfies the $\otimes$-*Tor-*conditions with respect to $\mathcal{X}_T$ if and only if* $\operatorname{Hom}_\Sigma(X,T)$ *is in $\mathcal{X}_T^\perp$.*

In making further identifications involving $\operatorname{Ext}^1_\Lambda(M,T)$ the following result is crucial in the proof of the duality.

Proposition 1.9. *Let Σ be an artin algebra, and let T be a Wakamatsu cotilting module over Σ. Let $\mathcal{Y}$ be a subcategory of $\mathcal{X}_T$ which is closed under cokernels of left* $\operatorname{add} T$*-approximations. Let $\mathcal{Z} = \{Z \in \operatorname{mod}\Sigma \mid \operatorname{Ext}^1_\Sigma(\mathcal{Y}, Z) = (0)\}$, and let $\Delta = \operatorname{End}_\Sigma(T)$. Then there is an isomorphism*

$$\phi\colon \operatorname{Hom}_\Sigma(T,\omega) \otimes_\Delta \operatorname{Hom}_\Sigma(X,T) \to \operatorname{Hom}_\Sigma(X,\omega)$$

given by $\phi(f \otimes g) = f \circ g$ for f in $\operatorname{Hom}_\Sigma(T,\omega)$ *and g in* $\operatorname{Hom}_\Sigma(X,T)$*, where X is in $\mathcal{Y}$ and ω is in $\mathcal{Z}$. The homomorphism ϕ is functorial in both X and ω.*

Proof. There is a homomorphism

$$\phi\colon \operatorname{Hom}_\Sigma(T,\omega) \otimes_\Delta \operatorname{Hom}_\Sigma(X,T) \simeq \operatorname{Hom}_\Sigma(X,\omega)$$

given by $\phi(f \otimes g) = f \circ g$ for $f\colon T \to \omega$ and $g\colon X \to T$. The homomorphism ϕ is clearly an isomorphism for $X = T$. Since ϕ commutes with direct sums, it follows that ϕ is an isomorphism for any X in $\operatorname{add} T$. It is easy to check that ϕ is natural in both X and ω.

Let X be in $\mathcal{Y}$. Then we have an exact sequence $0 \to X \to T_0 \xrightarrow{h} T_1$ of left $\operatorname{add} T$-approximations, where by assumption $\operatorname{Im} h = X'$ and $\operatorname{Coker} h = X''$ are in $\mathcal{Y}$. This gives rise to the exact sequence $(T_1,T) \to (T_0,T) \to (X,T) \to 0$. Tensoring this sequence with (T,ω) we obtain the following commutative diagram

$$\begin{tikzcd}
(T,\omega) \otimes_\Delta (T_1,T) \arrow[r] \arrow[d] & (T,\omega) \otimes_\Delta (T_0,T) \arrow[r] \arrow[d] & (T,\omega) \otimes_\Delta (X,T) \arrow[r] \arrow[d] & 0 \\
(T_1,\omega) \arrow[r] & (T_0,\omega) \arrow[r] & (X,\omega) \arrow[r] & \operatorname{Ext}^1_\Sigma(X',\omega)
\end{tikzcd}$$

The two leftmost vertical maps are isomorphisms, and $\operatorname{Ext}^1_\Sigma(X',\omega) = (0)$ by assumption. It follows from this that we have our desired isomorphism. □

The following step is also essential.

LEMMA 1.10. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting module over Λ_0. Suppose Λ satisfies the $\otimes$-Tor-conditions with respect to T.*

If M is in $\operatorname{gr}_0 \Lambda$ with M_i in $\mathcal{X}_T$ for $i = 0, 1$ and $M^1[-1]$ is in $\operatorname{gr}_0 \Lambda$, then

$$\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Hom}_\Lambda(M,T) = \operatorname{Ext}^1_\Lambda(M,T).$$

PROOF. For T we have that $T^1 = \Lambda_{\geq 1} \otimes_{\Lambda_0} T$, and this module is generated in the highest degree. Therefore by Lemma 1.5 we have that $\operatorname{Ext}^1_\Lambda(T,T) \simeq \operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$ and that $\operatorname{Ext}^1_\Lambda(M,T) \simeq \operatorname{Hom}_{\Lambda_0}(M^1_1, T)$. Note that $\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$ is a right Γ_0-module via the Γ_0-module structure on T in the tensor product $\Lambda_1 \otimes_{\Lambda_0} T$, where $\Gamma_0 = \operatorname{End}_{\Lambda_0}(T)$. We want to show that the map $\phi\colon \operatorname{Ext}^1_\Lambda(T,T)\otimes_{\Gamma_0}\operatorname{Hom}_\Lambda(M,T) \to \operatorname{Ext}^1_\Lambda(M,T)$ given by multiplication is onto.

Let $\theta\colon M^1_1 \to \Lambda_1 \otimes_{\Lambda_0} M_0$ be the inclusion. Then ϕ is given by $\phi(f \otimes g) = f \circ (\mathrm{id}_{\Lambda_1} \otimes g) \circ \theta\colon M^1_1 \to T$ when using the above identifications for f in $\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$ and g in $\operatorname{Hom}_{\Lambda_0}(M_0, T)$.

Since $\operatorname{Ext}^1_{\Lambda_0}(\mathcal{X}_T, \operatorname{Hom}_{\Lambda_0}(\Lambda_1, T)) = (0)$ by Lemma 1.7, we have the following isomorphisms by Proposition 1.9

$$\begin{aligned}
\operatorname{Ext}^1_\Lambda(T,T) \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(M,T) &\simeq \operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T) \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(M_0, T)\\
&\simeq \operatorname{Hom}_{\Lambda_0}(T, \operatorname{Hom}_{\Lambda_0}(\Lambda_1, T)) \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(M_0, T)\\
&\simeq \operatorname{Hom}_{\Lambda_0}(M_0, \operatorname{Hom}_{\Lambda_0}(\Lambda_1, T))\\
&\simeq \operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} M_0, T)
\end{aligned}$$

This implies that we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^1_\Lambda(T,T) \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(M,T) \arrow[r] \arrow[d, "\wr"] & \operatorname{Ext}^1_\Lambda(M,T) \arrow[d, "\wr"] & & \\
(\Lambda_1 \otimes_{\Lambda_0} M_0, T) \arrow[r, "\theta^*"] & (M^1_1, T) \arrow[r] & \operatorname{Ext}_{\Lambda_0}(M_1, T) \arrow[r] & 0
\end{tikzcd}$$

where θ^* is onto, since $\operatorname{Ext}_{\Lambda_0}(M_1, T) = (0)$ by assumption. This completes the proof. $\square$

Now proving that $\mathcal{K}_T(\Lambda)$ is contained in $\mathcal{G}_T(\Lambda)$ is an easy induction argument.

PROPOSITION 1.11. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting module over Λ_0. Suppose Λ satisfies the $\otimes$-Tor-conditions with respect to T.*

(a) *$\mathcal{K}_T(\Lambda)$ is a subcategory of $\mathcal{G}_T(\Lambda)$.*
(b) *Let M be in $\mathcal{K}_T(\Lambda)$, and let*

$$\cdots \to \Lambda \otimes M^2_2[2] \xrightarrow{d_2} \Lambda \otimes M^1_1[1] \xrightarrow{d_1} \Lambda \otimes M_0 \xrightarrow{d_0} M \to 0$$

be the linear fat resolution of M, where $M^i = \operatorname{Ker} d_{i-1}$ for $i \geq 1$. Then $\operatorname{Ext}^i_\Lambda(M,T) \simeq \operatorname{Hom}_{\Lambda_0}(M^i_i, T)$ for all $i \geq 0$.
(c) *$\Lambda \otimes_{\Lambda_0} \mathcal{X}_T$ is in $\mathcal{K}_T(\Lambda)$ and $\mathcal{E}(\Lambda \otimes_{\Lambda_0} X) \simeq \operatorname{Hom}_\Lambda(X,T)$ for all X in $\mathcal{X}_T$.*

PROOF. Assume that M is in $\mathcal{K}_T(\Lambda)$, and let M have a linear fat resolution as above. Since $\Lambda_i \otimes \mathcal{X}_T \subset \mathcal{X}_T$ and $\mathcal{X}_T$ is resolving, the Λ_0-module M^1_i is in $\mathcal{X}_T$ for all

$i \geq 1$. Moreover, given the resolution of M via the modules $\Lambda \otimes_{\Lambda_0} M_i^i[i]$ and that $\Lambda \otimes_{\Lambda_0} M_i^i[i]$ is in ${}^{\perp}_{\Lambda}T$ for all i, we see that

$$\begin{aligned}\operatorname{Ext}^i_\Lambda(M,T) &= \operatorname{Hom}_\Lambda(M^i,T)\\ &= \operatorname{Hom}_{\Lambda_0}(M_i^i,T)\end{aligned}$$

for all $i \geq 0$, where $M^0 = M$. Then

$$\begin{aligned}\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Ext}^i_\Lambda(M,T) &= \operatorname{Ext}^1_\Lambda(T,T)\operatorname{Hom}_\Lambda(M^i,T)\\ &= \operatorname{Ext}^1_\Lambda(M^i,T)\\ &= \operatorname{Ext}^{i+1}_\Lambda(M,T)\end{aligned}$$

This proves (a) and (b).

The claim in (c) is a direct consequence of the definition of $\mathcal{K}_T(\Lambda)$ and (b). □

We end this section with some useful properties of the category $\mathcal{K}_T(\Lambda)$ of Koszul modules.

The functor $\mathcal{E}$ is far from being an exact functor, but the next result shows that in some situations it does preserve exactness. This is used in Proposition 2.2 to show that the image of $\mathcal{X}_T$ under $\mathcal{E}$ consists of fat modules.

PROPOSITION 1.12. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting module over Λ_0.*

(a) *If $0 \to A \to B \to C \to 0$ is an F^T-exact sequence in $\operatorname{gr}_0 \Lambda$ with A in $\mathcal{G}_T(\Lambda)$, then $0 \to \mathcal{E}(C) \to \mathcal{E}(B) \to \mathcal{E}(A) \to 0$ is an exact sequence.*
(b) *If $0 \to A \to B \to C \to 0$ is an F^T-exact sequence in $\operatorname{gr}_0 \Lambda$ with A and C in $\mathcal{G}_T(\Lambda)$, then B is in $\mathcal{G}_T(\Lambda)$.*

PROOF. (a) Denote the map $A \to B$ by f. Then we get an induced map $\mathcal{E}(f)\colon \mathcal{E}(B) \to \mathcal{E}(A)$. Since the sequence $0 \to A \to B \to C \to 0$ is F^T-exact and all modules are generated in degree 0, the map $\mathcal{E}(f)_0 = \operatorname{Hom}_\Lambda(f,T)\colon \operatorname{Hom}_\Lambda(B,T) \to \operatorname{Hom}_\Lambda(A,T)$ is onto. Since A is in $\mathcal{G}_T(\Lambda)$ and $\mathcal{E}(f)$ is onto in degree 0, the map $\mathcal{E}(f)$ is onto. The exactness of $0 \to \mathcal{E}(C) \to \mathcal{E}(B) \to \mathcal{E}(A) \to 0$ then follows immediately.

(b) This is an easy consequence of (a). □

PROPOSITION 1.13. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting module over Λ_0. Assume that Λ satisfies the $\otimes$-Tor-conditions with respect to T.*

(a) *Suppose that $0 \to A \to B \to C \to 0$ is an exact sequence in $\operatorname{gr}_0 \Lambda$ with C_i in $\mathcal{X}_T$ for all $i \geq 0$. If A and B are in $\mathcal{K}_T(\Lambda)$, then C is in $\mathcal{K}_T(\Lambda)$.*
(b) *If $0 \to A \to B \to C \to 0$ is an exact sequence with A and C in $\mathcal{K}_T(\Lambda)$, then B is in $\mathcal{K}_T(\Lambda)$.*

PROOF. (a) It is enough to show that C has a linear fat resolution. Consider the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^1 \arrow[r] \arrow[d] & B^1 \arrow[r] \arrow[d] & C^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \otimes_{\Lambda_0} A_0 \arrow[r] \arrow[d] & \Lambda \otimes_{\Lambda_0} B_0 \arrow[r] \arrow[d] & \Lambda \otimes_{\Lambda_0} C_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The middle row is exact, since C_0 is in $\mathcal{X}_T$ and Λ satisfies the $\otimes$-Tor-conditions with respect to T. Hence the upper row, which is the sequence of kernels, is exact. It follows that C^1 is generated in degree 1. Since Λ satisfies the $\otimes$-Tor-conditions with respect to T, we infer that C^1_i is in $\mathcal{X}_T$ for all $i \geq 1$. By assumption $A^1[-1]$ and $B^1[-1]$ are in $\mathcal{K}_T(\Lambda)$, so that by repeating the above arguments it follows that C has a linear fat resolution.

(b) Using the same diagram as in the proof of (a), the claim follows directly. $\square$

2. Duality

In this section we show that if Λ is Koszul with respect to T, then $E_T(\Lambda)$ has the same property. Further we show that $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras, and that there is a duality between the T-Koszul modules.

We first want to show that the $\otimes$-Tor-conditions hold for $E_T(\Lambda)$ when Λ is Koszul with respect to T.

For any M in $\operatorname{gr}_0 \Lambda$ denote as before by M^1 the graded Λ-module given by the exact sequence $0 \to M^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$.

PROPOSITION 2.1. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting module over Λ_0. Let $\Gamma = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(T,T)$. Suppose that Λ is Koszul with respect to T.*

(a) *$\Gamma_i \simeq \operatorname{Hom}_{\Lambda_0}(X^i_i \otimes_{\Lambda_0} T, T)$ as Γ_0-bimodules, where*

$$\cdots \to \Lambda \otimes_{\Lambda_0} X^2_2[2] \xrightarrow{d_2} \Lambda \otimes_{\Lambda_0} X^1_1[1] \xrightarrow{d_1} \Lambda \otimes_{\Lambda_0} \Lambda_0 \xrightarrow{d_0} \Lambda_0 \to 0$$

is the linear fat resolution of Λ_0.

(b) *Γ satisfies the $\otimes$-Tor-conditions with respect to ${}_{\Gamma_0}T$.*

PROOF. (a) Let

$$\eta\colon\ \cdots \to \Lambda \otimes_{\Lambda_0} X^2_2[2] \xrightarrow{d_2} \Lambda \otimes_{\Lambda_0} X^1_1[1] \xrightarrow{d_1} \Lambda \otimes_{\Lambda_0} \Lambda_0 \xrightarrow{d_0} \Lambda_0 \to 0$$

be the linear fat resolution of Λ_0 over Λ. Since Λ satisfies the $\otimes$-Tor-conditions with respect to T, the tensor product $\eta \otimes_{\Lambda_0} T$ is a linear fat resolution of T over Λ. By Proposition 1.11 we have that $\Gamma_i = \operatorname{Ext}^i_\Lambda(T,T) \simeq \operatorname{Hom}_{\Lambda_0}(X^i_i \otimes_{\Lambda_0} T, T)$ as Γ_0-bimodules.

(b) Let C be in $\mathcal{X}_T$. Then we have the following isomorphisms

$$\begin{aligned}\Gamma_i \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(C,T) &\simeq \operatorname{Hom}_{\Lambda_0}(X_i^i \otimes_{\Lambda_0} T, T) \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(C,T)\\ &\simeq \operatorname{Hom}_{\Lambda_0}(T,(X_i^i,T)) \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(C,T)\\ &\simeq \operatorname{Hom}_{\Lambda_0}(C,(X_i^i,T))\\ &\simeq \operatorname{Hom}_{\Lambda_0}(X_i^i \otimes_{\Lambda_0} C, T)\end{aligned}$$

by Proposition 1.9. Since X_i^i satisfies the $\otimes$-condition with respect to T by Corollary 1.4, it follows that Γ satisfies the $\otimes$-condition with respect to T.

Let C be in $\mathcal{X}_T$. Then the sequence of left $\operatorname{add} T$-approximations

$$0 \to C \to T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots$$

is an exact sequence, where in addition $C_i = \operatorname{Ker} d_i$ is in $\mathcal{X}_T$ for all $i \geq 0$. Then the sequence

$$\cdots \to {}_{\Lambda_0}(T_2,T) \to {}_{\Lambda_0}(T_1,T) \to {}_{\Lambda_0}(T_0,T) \to {}_{\Lambda_0}(C,T) \to 0$$

is a projective resolution of $\operatorname{Hom}_{\Lambda_0}(C,T)$ over Γ_0. Applying the functor $\Gamma_i \otimes_{\Gamma_0}$ to this exact sequence, we obtain the following commutative diagram using the same isomorphism as above.

$$\begin{tikzcd}\Gamma_i \otimes_{\Gamma_0} (T_1,T) \arrow[r] \arrow[d, "\wr"] & \Gamma_i \otimes_{\Gamma_0} (T_0,T) \arrow[r] \arrow[d, "\wr"] & \Gamma_i \otimes_{\Gamma_0} (C,T) \arrow[r] \arrow[d, "\wr"] & 0\\ (X_i^i \otimes_{\Lambda_0} T_1, T) \arrow[r] & (X_i^i \otimes_{\Lambda_0} T_0, T) \arrow[r] & (X_i^i \otimes_{\Lambda_0} C, T) \arrow[r] & 0\end{tikzcd}$$

Since X_i^i satisfies the $\otimes$-Tor-conditions with respect to T by Corollary 1.4, the sequence $0 \to X_i^i \otimes_{\Lambda_0} C_j \to X_i^i \otimes_{\Lambda_0} T_j \to X_i^i \otimes_{\Lambda_0} C_{j+1} \to 0$ is exact for all $i \geq 0$ and $j \geq 0$. Hence the lower row in the above diagram is exact, and therefore $\operatorname{Tor}_j^{\Gamma_0}(\Gamma_i, \mathcal{X}_{\Gamma_0 T}) = (0)$ for all $j \geq 1$, and Γ satisfies the $\otimes$-Tor-conditions with respect to ${}_{\Gamma_0}T$. $\square$

The missing piece in order to show that $E_T(\Lambda)$ is Koszul with respect to T when Λ is Koszul with respect to T, is to show that $E_T(\Lambda)_0$ has a linear fat resolution. Lemma 1.6 and Proposition 1.13 are our preliminary stepping stones to prove this.

Proposition 2.2. *Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting module over Λ_0. Assume that Λ is T-Koszul.*

(a) *If M is in $\mathcal{K}_T(\Lambda)$, then $M_{\geq 1}[-1]$ is in $\mathcal{K}_T(\Lambda)$.*

(b) *If X is in $\mathcal{X}_T$, then X is in $\mathcal{K}_T(\Lambda)$ and $\mathcal{E}(X) \simeq \Gamma \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(X,T)$, where $\Gamma = E_T(\Lambda)$. In particular, $\mathcal{E}(X)$ is a fat module.*

Proof. (a) Let M be in $\mathcal{K}_T(\Lambda)$. Then the sequence $0 \to M^1 \to \Lambda_{\geq 1} \otimes_{\Lambda_0} M_0 \to M_{\geq 1} \to 0$ is exact. By assumption $M^1[-1]$ is in $\mathcal{K}_T(\Lambda)$. Since Λ_0 is in $\mathcal{K}_T(\Lambda)$, we have that $(\Lambda_0)^1[-1] = \Lambda_{\geq 1}[-1]$ is in $\mathcal{K}_T(\Lambda)$. Since Λ satisfies the $\otimes$-Tor-conditions with respect to T, tensoring the linear fat resolution of $\Lambda_{\geq 1}$ with T gives rise to a linear fat resolution of $\Lambda_{\geq 1} \otimes_{\Lambda_0} T$. Hence $\Lambda_{\geq 1} \otimes_{\Lambda_0} T$ is also in $\mathcal{K}_T(\Lambda)$. Since C_i is in $\mathcal{X}_T$ for all $i \geq 1$, Proposition 1.13 implies that $M_{\geq 1}[-1]$ is in $\mathcal{K}_T(\Lambda)$.

(b) Let X be in $\mathcal{X}_T$. Since Λ_0 is in $\mathcal{K}_T(\Lambda)$ and Λ satisfies the $\otimes$-Tor-conditions with respect to T, tensoring the linear fat resolution of Λ_0 with X gives rise to a linear fat resolution of X. Hence $\mathcal{X}_T$ is in $\mathcal{K}_T(\Lambda)$.

Let $\phi_X \colon \Gamma \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(X,T) \to \mathcal{E}(X)$ be the map given by multiplication. The homomorphism ϕ clearly commutes with finite direct sums, and ϕ_T is an isomorphism. It follows that $\phi_{T'}$ is an isomorphism for all $X = T'$ in $\operatorname{add} T$. For X in $\mathcal{X}_T$ we have an exact sequence of left $\operatorname{add} T$-approximations $0 \to X \to T_0 \xrightarrow{h} T_1$, where $\operatorname{Im} h$ and $\operatorname{Coker} h$ are in $\mathcal{X}_T$. By Proposition 1.12 the sequence $\mathcal{E}(T_1) \to \mathcal{E}(T_0) \to \mathcal{E}(X) \to 0$ is exact. The sequence $(T_1,T) \to (T_0,T) \to (X,T) \to 0$ is exact, and tensoring this sequence with Γ gives rise to the following commutative diagram

$$\begin{tikzcd}
\mathcal{E}(T_1) \arrow[r] \arrow[d, "\wr"] & \mathcal{E}(T_0) \arrow[r] \arrow[d, "\wr"] & \mathcal{E}(X) \arrow[r] \arrow[d] & 0 \\
\Gamma \otimes_{\Gamma_0} (T_1,T) \arrow[r] & \Gamma \otimes_{\Gamma_0} (T_0,T) \arrow[r] & \Gamma \otimes_{\Gamma_0} (X,T) \arrow[r] & 0
\end{tikzcd}$$

We infer that $\mathcal{E}(X) \simeq \Gamma \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(X,T)$. □

Now we have the necessary tools to show that Λ Koszul with respect to T implies that $E_T(\Lambda)$ is Koszul with respect to T.

THEOREM 2.3. *If Λ is Koszul with respect to T, then $\Gamma = E_T(\Lambda)$ is Koszul with respect to T.*

PROOF. By Proposition 2.1 we infer that $\Gamma = E_T(\Lambda)$ satisfies the $\otimes$-Tor-conditions with respect to ${}_{\Gamma_0}T$.

We have that $\Gamma_0 = \mathcal{E}(\Lambda \otimes_{\Lambda_0} T)$ and $\Lambda \otimes_{\Lambda_0} T$ is in $\mathcal{K}_T(\Lambda)$ by Proposition 1.11. By Lemma 1.6 we have that the sequence $0 \to \mathcal{E}((\Lambda \otimes_{\Lambda_0} T)_{\geq 1})[1] \to \mathcal{E}(T) = \Gamma \to \mathcal{E}(\Lambda \otimes_{\Lambda_0} T) = \Gamma_0 \to 0$ is exact. Since $(\Lambda \otimes_{\Lambda_0} T)_{\geq 1}$ is in $\mathcal{K}_T(\Lambda)$, we inductively infer that the sequences

$$0 \to \mathcal{E}((\Lambda \otimes_{\Lambda_0} T)_{\geq i+1})[1] \to \mathcal{E}(\Lambda_i \otimes_{\Lambda_0} T) \to \mathcal{E}((\Lambda \otimes_{\Lambda_0} T)_{\geq i}) \to 0$$

are exact. Since $\mathcal{E}(Y) \simeq \Gamma \otimes_{\Gamma_0} (Y,T)$ for all Y in $\mathcal{X}_T$ by Proposition 2.2, we conclude that Γ_0 has a linear fat resolution. Hence Γ is Koszul with respect to T. □

Whenever $E_T(\Gamma) \simeq \Lambda$ as graded algebras for $\Gamma = E_T(\Lambda)$, we have the natural functors $\mathcal{E} = \oplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(\ ,T) \colon \operatorname{Gr} \Lambda \to \operatorname{Gr} \Gamma$ and $\mathcal{E}' = \oplus_{i \geq 0} \operatorname{Ext}^i_\Gamma(\ ,T) \colon \operatorname{Gr} \Gamma \to \operatorname{Gr} \Lambda$. Our next aim is to show that the functors $\mathcal{E}$ and $\mathcal{E}'$ induce inverse dualities on $\mathcal{K}_T(\Lambda)$ and $\mathcal{K}_T(\Gamma)$ when Λ is T-Koszul. We first show that the compositions of the functors $\mathcal{E}$ and $\mathcal{E}'$ are identities on objects, and for this we need the following.

LEMMA 2.4. *Assume that Λ is Koszul with respect to T. If M is in $\mathcal{K}_T(\Lambda)$, then*

(a) *$\mathcal{E}(M)$ is in $\mathcal{K}_T(\Gamma)$.*
(b) *$\mathcal{E}'\mathcal{E}(M)_i \simeq M_i$ as a Λ_0-bimodule.*
(c) *The identification $\phi_i \colon M_i \to \mathcal{E}'\mathcal{E}(M)_i$ is given in the following way. Let $\eta_i(M) \colon 0 \to \mathcal{E}(M_{\geq i+1})[1] \to \mathcal{E}(M_i) \to \mathcal{E}(M_{\geq i}) \to 0$. Then $\phi_i(m_i) = (\mathcal{E}(a_{m_i}) \cdot \eta_i(M)) \cdot \eta_{i-1}(M) \cdots \eta_1(M) \cdot \eta_0(M)$, where m_i is in M_i and $a_{m_i} \colon \Lambda \to M_{\geq i}$ is given by $a_{m_i}(\lambda) = \lambda m_i$.*

PROOF. (a) Let M be in $\mathcal{K}_T(\Lambda)$. By Lemma 1.6, Proposition 2.1 and Proposition 1.11 (c) the sequence

$$\cdots \to \mathcal{E}(M_2) \to \mathcal{E}(M_1) \to \mathcal{E}(M_0) \to \mathcal{E}(M) \to 0$$

is a linear fat resolution of $\mathcal{E}(M)$.

(b) Using Lemma 1.5 and the linear fat resolution of $\mathcal{E}(M)$ above it follows that

$$\begin{aligned}\mathrm{Ext}^i_\Gamma(\mathcal{E}(M),T) &\simeq \mathrm{Ext}^{i-1}_\Gamma(\mathcal{E}(M_{\geq 1}),T)\\ &\simeq \cdots\\ &\simeq \mathrm{Ext}^1_\Gamma(\mathcal{E}(M_{\geq i-1}),T)\\ &\simeq \mathrm{Hom}_\Gamma(\mathcal{E}(M_{\geq i}),T)\\ &\simeq \mathrm{Hom}_{\Gamma_0}(\mathrm{Hom}_{\Lambda_0}(M_i,T),T)\\ &\simeq M_i\end{aligned}$$

(c) It follows from (b) that the identification of $\mathrm{Ext}^i_\Lambda(M,T)$ with M_i is given by the composition of $i-1$ applications of the connecting homomorphism given by the sequences $\eta_i(M)$ and the identification between $\mathrm{Ext}^1_\Lambda(\mathcal{E}(M^{i-1},T)$ and $\mathrm{Hom}_{\Gamma_0}(\mathcal{E}(M_{\geq i})_0,T)$ given in Lemma 1.5. The identification of M_i with $\mathrm{Hom}_{\Gamma_0}(\mathrm{Hom}_\Lambda(M_i,T),T)$ and $\mathrm{Hom}_\Gamma(\mathcal{E}(M_{\geq i}),T)$ is given by sending an element m_i in M_i to the map $\mathcal{E}(a_{m_i})\colon \mathcal{E}(M_{\geq i})\to T=\mathcal{E}(\Lambda)$ where $a_{m_i}\colon \Lambda\to M_{\geq i}$ is given by $a_{m_i}(\lambda)=\lambda m_i$. □

We can now prove the following.

THEOREM 2.5. *Assume that Λ is a Koszul algebra with respect to T.*

(a) *$E_T(E_T(\Lambda))\simeq\Lambda$ as graded algebras.*
(b) *$\mathcal{E}'(\mathcal{E}(M))\simeq M$ as graded left Λ-modules for all M in $\mathcal{K}_T(\Lambda)$.*

PROOF. Before we start the proof of this result we introduce a functor that we need. Let M and N be in $\mathrm{Gr}(\Lambda)$ generated in degree 0 or higher, and let $f\colon M\to N$ be a morphism in $\mathrm{Gr}(\Lambda)$. Define $\mathrm{rej}_T(M)=M_{\geq 1}$ and $\mathrm{rej}_T(f)=f|_{M_{\geq 1}}$. Inductively define $\mathrm{rej}^i_T=\mathrm{rej}_T\circ\mathrm{rej}^{i-1}_T$ for $i\geq 2$.

Let M be in $\mathcal{K}_T(\Lambda)$ and let $\eta_i(M)$ denote the exact sequence $0\to\mathcal{E}(M_{\geq i+1})\to\mathcal{E}(M_i)\to\mathcal{E}(M_{\geq i})\to 0$ for $i\geq 0$. Let $\varphi_i(M)\colon M_i\to\mathcal{E}'\mathcal{E}(M)_i$ be the identifications given in Lemma 2.4 for all $i\geq 0$.

Let λ_r be in Λ_r for $r\geq 1$ and m_n in M_n for $n\geq 1$. Then

$$\varphi_r(\Lambda)(\lambda_r)=(\mathcal{E}(a_{\lambda_r})\eta_r(\Lambda))\eta_{r-1}(\Lambda)\eta_{r-2}(\Lambda)\cdots\eta_0(\Lambda)$$

and

$$\varphi_n(M)(m_n)=(\mathcal{E}(a_{m_n})\eta_n(M))\eta_{n-1}(M)\eta_{n-2}(M)\cdots\eta_0(M).$$

We want to show that $\varphi_r(\Lambda)(\lambda_r)\varphi_n(M)(m_n)=\varphi_{n+r}(M)(\lambda_r m_n)$. The product $\varphi_r(\Lambda)(\lambda_r)\varphi_n(M)(m_n)$ can be represented as

$$\Big((\mathcal{E}(a_{\lambda_r})\eta_{r-1}(\Lambda))\mathcal{E}(\mathrm{rej}^{r-1}_T(a_{m_n}))\Big)\eta_{n+r-1}(M)\cdots\eta_0(M).$$

The sequence $(\mathcal{E}(a_{\lambda_r})\eta_{r-1}(\Lambda))\mathcal{E}(\mathrm{rej}^{r-1}_T(a_{m_n}))$ is the same as

$$(\mathcal{E}(a_{\lambda_r})\mathcal{E}(\mathrm{rej}^r_T(a_{m_n})))\eta_{r-1}(M)=\mathcal{E}(a_{\lambda_r}a_{m_n})\eta_{r-1}(M)=\varphi_{n+r}(M)(\lambda_r m_n).$$

Let λ_0 be in Λ_0 and m_n in M_n for $n\geq 1$. Then $\varphi_0(\Lambda)(\lambda_0)\varphi_n(M)(m_n)$ is given by

$$\Big(\mathcal{E}(a_{\lambda_0})(\mathcal{E}(a_{m_n})\eta_n(M))\Big)\eta_{n-1}(M)\cdots\eta_0(M).$$

Since $\mathcal{E}(a_{\lambda_0})(\mathcal{E}(a_{m_n})\eta_n(M))$ is equal to $\mathcal{E}(a_{\lambda_0}a_{m_n})\eta_n(M)=\mathcal{E}(a_{\lambda_0 m_n})\eta_n(M)$, it follows that $\varphi_0(\Lambda)(\lambda_0)\varphi_n(M)(m_n)=\varphi_n(M)(\lambda_0 m_n)$.

Let λ_r be in Λ_r for $r \geq 1$ and m_0 in M_0. Then $\varphi_r(\Lambda)(\lambda_r)\varphi_0(M)(m_0)$ is given by

$$(\mathcal{E}(a_{\lambda_r})\eta_r(\Lambda))\eta_{r-1}(\Lambda)\cdots(\eta_0(\Lambda)\mathcal{E}(a_{m_0})).$$

Using the same arguments as above this element can be represented as

$$\Big((\mathcal{E}(a_{\lambda_r})\eta_r(\Lambda))\mathcal{E}(\operatorname{rej}_T^{r-1}(a_{m_0}))\Big)\eta_{r-1}(M)\cdots\eta_0(M).$$

Since $\Big((\mathcal{E}(a_{\lambda_r})\eta_r(\Lambda))\mathcal{E}(\operatorname{rej}_T^{r-1}(a_{m_0}))\Big)$ is equal to $\mathcal{E}(a_{\lambda_r})(\mathcal{E}(\operatorname{rej}_T^{r}(a_{m_0})\eta_r(M))) = \mathcal{E}(a_{\lambda_r m_0})\eta_r(M)$, it follows that $\varphi_r(\Lambda)(\lambda_r)\varphi_0(M)(m_0) = \varphi_r(M)(\lambda_r m_0)$.

Let λ_0 be in Λ_0 and m_0 in M_0. Then $\varphi_0(\Lambda)(\lambda_0)\varphi_0(M)(m_0) = \mathcal{E}(a_{\lambda_0})\mathcal{E}(a_{m_0}) = \mathcal{E}(a_{\lambda_0 m_0}) = \varphi_0(M)(\lambda_0 m_0)$.

This shows that $\Lambda \simeq E_T(E_T(\Lambda)) = \mathcal{E}'(T) = \mathcal{E}'(\mathcal{E}(\Lambda))$ as graded algebras, and that $M \simeq \mathcal{E}'(\mathcal{E}(M))$ as graded left Λ-modules if M is in $\mathcal{K}_T(\Lambda)$. □

To complete the proof that the functors $\mathcal{E}\colon \mathcal{K}_T(\Lambda) \to \mathcal{K}_T(\Gamma)$ and $\mathcal{E}'\colon \mathcal{K}_T(\Gamma) \to \mathcal{K}_T(\Lambda)$ are inverse dualities, we need to show that they are full and faithful.

THEOREM 2.6. *Assume that Λ is Koszul with respect to T and let $\Gamma = E_T(\Lambda)$. Then we have the following assertions.*

(a) *Γ is Koszul with respect to T, and $\Lambda \simeq E_T(E_T(\Lambda))$ as graded algebras.*
(b) *The functors*

$$\mathcal{E} = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\ ,T)\colon \operatorname{Gr}\Lambda \to \operatorname{Gr}\Gamma$$

and

$$\mathcal{E}' = \oplus_{i\geq 0} \operatorname{Ext}^i_\Gamma(\ ,T)\colon \operatorname{Gr}\Gamma \to \operatorname{Gr}\Lambda$$

induce inverse dualities between $\mathcal{K}_T(\Lambda)$ and $\mathcal{K}_T(\Gamma)$.
(c) *If X is in $\mathcal{X}_T$, then $\mathcal{E}(X)$ is a fat Γ-module. If Z is a fat Λ-module, then $\mathcal{E}(Z)$ is in $\mathcal{X}_{\Gamma_0 T}$.*

PROOF. The statements in (a) and (c) have been proved in Theorem 2.5, Proposition 2.2 (b) and Proposition 1.11 (c).

(b) By Theorem 2.5 (b) we have that $\mathcal{E}' \circ \mathcal{E} \simeq \operatorname{id}_{\mathcal{K}_T(\Lambda)}$ and $\mathcal{E} \circ \mathcal{E}' \simeq \operatorname{id}_{\mathcal{K}_T(\Gamma)}$ on objects.

Let M and N be in $\mathcal{K}_T(\Lambda)$. Then we have the following commutative diagram

$$\begin{tikzcd}
{}_{\operatorname{Gr}(\Lambda)}(M,N) \arrow[r, "\mathcal{E}"] \arrow[d] & {}_{\operatorname{Gr}(E_T(\Lambda))}(\mathcal{E}(N),\mathcal{E}(M)) \arrow[r, "\mathcal{E}'"] \arrow[d] & {}_{\operatorname{Gr}(\Lambda)}(\mathcal{E}'(\mathcal{E}(M)),\mathcal{E}'(\mathcal{E}(N))) \arrow[d] \\
{}_{\Lambda_0}(M_0,N_0) \arrow[r, equal] & {}_{E_T(\Lambda)_0}(\mathcal{E}(N)_0,\mathcal{E}(M)_0) \arrow[r, equal] & {}_{\Lambda_0}(M_0,N_0)
\end{tikzcd},$$

where the vertical maps are monomorphisms. It follows that $\mathcal{E}' \circ \mathcal{E}$ induces a monomorphism from $\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M,N)$ to $\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(\mathcal{E}'(\mathcal{E}(M)),\mathcal{E}'(\mathcal{E}(N))) = \operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M,N)$ for M and N in $\mathcal{K}_T(\Lambda)$. Since this homomorphism group is of finite length over the center of Λ_0, the composition $\mathcal{E}' \circ \mathcal{E}$ induces an isomorphism on $\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M,N)$ for all M and N in $\mathcal{K}_T(\Lambda)$. Similarly, we can show that $\mathcal{E} \circ \mathcal{E}'$ induces an isomorphism on $\operatorname{Hom}_{E_T(\Lambda)}(X,Y)$ for all X and Y in $\mathcal{K}_T(\Gamma)$. This shows that $\mathcal{E}$ and $\mathcal{E}'$ are full and faithful, so that $\mathcal{E}$ and $\mathcal{E}'$ are inverse dualities. □

3. T-Koszul algebras and quadratic algebras

In this section we prove that T-Koszul algebras are quadratic, and, that the relations for the Ext-algebra are orthogonal to those of the original algebra. We start by proving that any T-Koszul algebra is quadratic.

PROPOSITION 3.1. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a Koszul algebra with respect to T.*
(a) *$\Lambda \simeq T_{\Lambda_0}(\Lambda_1)/I$ with I contained in $T_{\Lambda_0}(\Lambda_1)_{\geq 2}$.*
(b) *I is a quadratic ideal in $T_{\Lambda_0}(\Lambda_1)$.*

PROOF. (a) Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a T-Koszul algebra. Then Λ is generated in degrees 0 and 1. Hence it is well-known that $\Lambda \simeq T_{\Lambda_0}(\Lambda_1)/I$, for some ideal I in the tensor algebra $T_{\Lambda_0}(\Lambda_1)$, with I contained in $T_{\Lambda_0}(\Lambda_1)_{\geq 2}$.

(b) Let $L = T_{\Lambda_0}(\Lambda_1)_{\geq 1} = T_{\Lambda_0}(\Lambda_1)\otimes_{\Lambda_0}\Lambda_1$ and $J = L/I$. Then $\Lambda_0 = \Lambda/J$, and we have the following exact sequence

$$0 \to I/IL \to L/IL \to \Lambda \to \Lambda_0 \to 0.$$

We have that

$$\begin{aligned} L/IL &= (T_{\Lambda_0}(\Lambda_1)\otimes_{\Lambda_0}\Lambda_1)/I(T_{\Lambda_0}(\Lambda_1)\otimes_{\Lambda_0}\Lambda_1) \\ &\simeq (T_{\Lambda_0}(\Lambda_1)/I)\otimes_{T_{\Lambda_0}(\Lambda_1)}(T_{\Lambda_0}(\Lambda_1)\otimes_{\Lambda_0}\Lambda_1) \\ &= ((T_{\Lambda_0}(\Lambda_1)/I)\otimes_{T_{\Lambda_0}(\Lambda_1)}T_{\Lambda_0}(\Lambda_1))\otimes_{\Lambda_0}\Lambda_1 \\ &\simeq T_{\Lambda_0}(\Lambda_1)/I\otimes_{\Lambda_0}\Lambda_1 = \Lambda\otimes_{\Lambda_0}\Lambda_1 \end{aligned}$$

This shows that the above exact sequence can be rewritten as

$$0 \to I/IL \to \Lambda\otimes_{\Lambda_0}\Lambda_1 \to \Lambda\otimes_{\Lambda_0}\Lambda_0 \to \Lambda_0 \to 0.$$

Since Λ is T-Koszul, the above exact sequence is the start of a fat resolution of Λ_0. By definition Λ is in $\mathcal{K}_T(\Lambda)$, in particular, Λ_0 is in $\mathcal{K}_T(\Lambda)$ and therefore Λ_0 has a linear fat resolution. In particular, there is an epimorphism $\Lambda\otimes_{\Lambda_0}(I/IL)_2[2] \to I/IL$, and therefore I/IL is generated by elements in

$$(I/IL)_2 = (I/IL)\cap(\Lambda\otimes_{\Lambda_0}\Lambda_1)_2 = (I/IL)\cap(\Lambda_1\otimes_{\Lambda_0}\Lambda_1).$$

Let x_i be in $\Lambda_1\otimes_{\Lambda_0}\Lambda_1$ for $i = 1,\dots,n$, such that $\{x_1, x_2, \dots, x_n\}$ are generators of the left Λ-module I/IL. The ideal $\langle x_1, x_2, \dots, x_n\rangle$ in $T_{\Lambda_0}(\Lambda_1)$ is a graded ideal since it is generated by homogeneous elements. Therefore the $T_{\Lambda_0}(\Lambda_1)$-module $I/\langle x_1, x_2, \dots, x_n\rangle$ is a graded $T_{\Lambda_0}(\Lambda_1)$-module. It follows immediately that $I/\langle x_1, x_2, \dots, x_n\rangle = (I/\langle x_1, x_2, \dots, x_n\rangle)L$. The graded Nakayama Lemma (the proof is similar to the ungraded case, see [**14**] Lemma I.7.4) now gives that $\{x_1, x_2, \dots, x_n\}$ is a generating set for I as a twosided ideal in $T_{\Lambda_0}(\Lambda_1)$, hence I is a quadratic ideal in $T_{\Lambda_0}(\Lambda_1)$. $\square$

When Λ is T-Koszul, then we have shown that $\Gamma = E_T(\Lambda)$ is T-Koszul. Hence Γ is also quadratic in the above sense. Furthermore we give a closed formula for Γ in terms of Λ.

THEOREM 3.2. *Let Λ be a T-Koszul algebra. Let $0 \to X^2 \to \Lambda\otimes_{\Lambda_0}\Lambda_1 \to \Lambda\otimes_{\Lambda_0}\Lambda_0 \to \Lambda_0 \to 0$ be the start of a linear fat resolution of Λ_0.*
(a) *$\Lambda \simeq T_{\Lambda_0}(\Lambda_1)/\langle X_2^2\rangle$.*
(b) *Denote by Γ_0 and Γ_1 the endomorphism ring $\mathrm{End}_{\Lambda_0}(T)$ and $\mathrm{Hom}_{\Lambda_0}(\Lambda_1\otimes_{\Lambda_0} T, T)$, respectively. Then*

$$E_T(\Lambda) = T_{\Gamma_0}(\Gamma_1)/\langle A_2\rangle,$$

where $A_2 = \{\sum \beta_i \otimes \alpha_i \in \Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \mid \sum \beta_i(\mathrm{id}_{\Lambda_1} \otimes \alpha_i)|_{X_2^2 \otimes_{\Lambda_0} T} = 0\}$.

PROOF. (a) By Proposition 3.1 we have that $\Lambda \simeq T_{\Lambda_0}(\Lambda_1)/I$ for some quadratic ideal I in $T_{\Lambda_0}(\Lambda_1)$. The ideal I is generated by the kernel of the multiplication map $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1 \to \Lambda_2$. The kernel is exactly given by X_2^2.

(b) Since Λ is T-Koszul, we know by Theorem 2.3 and Proposition 3.1 that $\mathcal{E}_T(\Lambda) = T_{E_T(\Lambda)_0}(E_T(\Lambda)_1)/\langle A_2 \rangle$ for some submodule A_2 of $E_T(\Lambda)_1 \otimes_{E_T(\Lambda)_0} E_T(\Lambda)_1$. We also have that $E_T(\Lambda)_i \simeq \Gamma_i$ for $i = 0, 1$. Using these identifications the submodule A_2 is given as the kernel of the multiplication map $\Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \to E_T(\Lambda)_2$. We have that $E_T(\Lambda)_2 \simeq \mathrm{Hom}_{\Lambda_0}(X_2^2 \otimes_{\Lambda_0} T, T)$. Using this identification in addition and that Λ_1 satisfies the Tor-condition we obtain the following commutative diagram by Proposition 1.9 and Lemma 1.7

$$\begin{tikzcd}
E_T(\Lambda)_1 \otimes_{E_T(\Lambda)_0} E_T(\Lambda)_1 \arrow[r] \arrow[d, "\wr"] & E_T(\Lambda)_2 \arrow[d, "\wr"] \\
(\Lambda_1 \otimes_{\Lambda_0} T, T) \otimes_{\Gamma_0} (\Lambda_1 \otimes_{\Lambda_0} T, T) \arrow[r] \arrow[d, "\wr"] & (X_2^2 \otimes_{\Lambda_0} T, T) \arrow[d, equal] \\
(T, (\Lambda_1, T)) \otimes_{\Gamma_0} (\Lambda_1 \otimes_{\Lambda_0} T, T) \arrow[r] \arrow[d, "\wr"] & (X_2^2 \otimes_{\Lambda_0} T, T) \arrow[d, equal] \\
(\Lambda_1 \otimes_{\Lambda_0} T, (\Lambda_1, T)) \arrow[r] \arrow[d, "\wr"] & (X_2^2 \otimes_{\Lambda_0} T, T) \arrow[d, equal] \\
(\Lambda_1 \otimes_{\Lambda_0} \Lambda_1 \otimes_{\Lambda_0} T, T) \arrow[r] & (X_2^2 \otimes_{\Lambda_0} T, T)
\end{tikzcd}$$

where the identification $\psi\colon (\Lambda_1 \otimes_{\Lambda_0} T, T) \otimes_{\Gamma_0} (\Lambda_1 \otimes_{\Lambda_0} T, T) \to (\Lambda_1 \otimes_{\Lambda_0} \Lambda_1 \otimes_{\Lambda_0} T, T)$ is given by $\psi(\beta \otimes \alpha) = \beta(\mathrm{id}_{\Lambda_1} \otimes \alpha)$ for α and β in $\Gamma_1 = \mathrm{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$. Moreover the last horizontal map is given by $\theta^* = \mathrm{Hom}_{\Lambda_0}(\theta, T)$, where $\theta\colon X_2^2 \otimes_{\Lambda_0} T \to \Lambda_1 \otimes_{\Lambda_0} \Lambda_1 \otimes_{\Lambda_0} T$ is the inclusion. We have that

$$\operatorname{Ker} \theta^* = \{\sum \beta_i \otimes \alpha_i \in \Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \mid \sum \beta_i(\mathrm{id}_{\Lambda_1} \otimes \alpha_i)|_{X_2^2 \otimes_{\Lambda_0} T} = 0\}.$$

This completes the proof. $\square$

CHAPTER III

Generalized T-Koszul algebras

In this chapter we discuss the second approach to our generalization of Koszul algebras, which we call generalized T-Koszul algebras. Our point of view is to focus on essential common properties of the category of Koszul modules for a Koszul algebra and the category $\mathcal{X}_T$ associated with a Wakamatsu cotilting module. At the same time we have in mind to get a duality theorem, which is the main result of this chapter. It turns out that the new class contains the class of T-Koszul algebras discussed in the previous chapter, along with the example at the end of Chapter I.

Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a positively $\mathbb{Z}$-graded ring, and let T be a Λ-module. Our main requirement on the pair (Λ, T) is that the functor $\mathcal{E} = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\ ,T)$ induces a duality between "nice" subcategories of $\operatorname{Gr}\Lambda$ containing Λ and T, and a corresponding type of subcategory of $\operatorname{Gr} E_T(\Lambda)$, where $E_T(\Lambda) = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(T,T)$. In order for this to be true, it is easy to see that T must be concentrated in degree 0, since $T = \mathcal{E}'(\mathcal{E}(T)) = \mathcal{E}'(E_T(\Lambda)) = \operatorname{Hom}_{E_T(\Lambda)}(\mathcal{E}(T), \mathcal{E}(\Lambda)) \oplus 0 \oplus \cdots$. Hence it is no restriction to assume that T is concentrated in degree 0.

In Section 1 we identify some essential properties of the subcategories we are interested in, and give some necessary conditions for having a reasonable duality theory.

We introduce the notion of generalized T-Koszul algebras in section 2, and show the relationship to the T-Koszul algebras discussed in Chapter II. In section 3 we compare with the classical cases. We see in section 4 that the $\otimes$-Tor-conditions, which occurred in the definition of a T-Koszul algebra, are almost necessary in order to have a good duality theory. In section 5 we investigate correspondences between various types of modules via the functor $\mathcal{E}$. The main duality theorem is discussed in section 6.

1. Natural properties of Koszul modules

In this section we identify essential common properties of the category of Koszul modules over a Koszul algebra and the subcategories $\mathcal{X}_T$ of $\operatorname{mod}\Lambda_0$ associated with Wakamatsu cotilting Λ_0-modules T. We also give some necessary conditions in order to have a duality theory.

For the category of Koszul modules $\mathcal{K}(\Lambda)$ over a Koszul algebra, we have that if M is in $\mathcal{K}(\Lambda)$, then $\operatorname{rad} M[-1]$ and $\Omega_\Lambda(M)[-1]$ are again in $\mathcal{K}(\Lambda)$. When T is a Wakamatsu cotilting Λ-module, and M is in $\mathcal{X}_T$, then $\operatorname{Coker} f$ is again in $\mathcal{X}_T$, when $f\colon M \to M_T$ is a minimal left $\operatorname{add} T$-approximation, and $\Omega_\Lambda(M)$ is also in $\mathcal{X}_T$. For a Koszul algebra $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$, the module $T = \Lambda_0$ is a cotilting Λ_0-module. In this case the natural epimorphism $f\colon M \to M/\operatorname{rad} M$ for a Koszul module M is a minimal left $\operatorname{add} T$-approximation, and in this case it is the kernel $\operatorname{rad} M$ which, up to shift, is again a Koszul module. There is then a common formulation involving minimal left $\operatorname{add} T$-approximations. So we first need to recall and develop some background on this topic.

Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a left finite graded ring generated in degrees 0 and 1 with Λ_0 an artin algebra. We will assume throughout that T is concentrated in degree 0, and we assume that T is a finitely generated module over Λ_0 to ensure that $E_T(\Lambda)_0 = \operatorname{End}_{\Lambda_0}(T)$ is an artin algebra.

Let $\mathcal{X}$ be a category and $\mathcal{C}$ a subcategory of $\mathcal{X}$. For X in $\mathcal{X}$ a morphism $f \colon X \to C_X$ with C_X in $\mathcal{C}$ is called a *left $\mathcal{C}$-approximation* of X, if for any morphism $g \colon X \to C'$ with C' in $\mathcal{C}$ there exists a morphism $h \colon C_X \to C'$ such that $hf = g$. A left approximation $f \colon X \to C_X$ is called *minimal* if whenever $gf = f$ for a morphism $g \colon C_X \to C_X$, then g is an isomorphism.

Let $\psi_\Lambda \colon \operatorname{Gr} \Lambda \to \operatorname{Mod} \Lambda$ denote the forgetful functor. In the following let T be in $\operatorname{gr} \Lambda$ such that $T = T_0$. Moreover, let $\sigma T = \operatorname{add}\{T[j]\}_{j \in \mathbb{Z}}$. Next we show that the modules in $\operatorname{gr} \Lambda$ have minimal left σT-approximations.

LEMMA 1.1. (a) *If M is in $\operatorname{gr} \Lambda$, then $\psi_\Lambda(M)$ has a minimal left $\operatorname{add} T$-approximation in $\operatorname{mod} \Lambda$.*

(b) *If M is in $\operatorname{gr} \Lambda$, then M has a minimal left σT-approximation $M \xrightarrow{\alpha} M_T$ in $\operatorname{gr} \Lambda$, such that $\psi(M) \xrightarrow{\psi(\alpha)} \psi(M_T)$ is a minimal left $\operatorname{add} \psi(T)$-approximation of $\psi(M)$.*

PROOF. (a) Suppose M is generated in the degrees $i_0, i_1, \ldots, i_t$, where $i_0 < i_1 < \cdots < i_t$. Let $M_{i_{-1}} = (0)$. For each Λ_0-submodule U of M let $\langle U \rangle$ denote the Λ-submodule of M generated by U. Then by using Lemma 0.4 we have that

$$\operatorname{Hom}_\Lambda(M, T) = \oplus_{j=0}^t \operatorname{Hom}_{\Lambda_0}((M/\langle M_{i_{-1}} + \cdots + M_{i_{j-1}} \rangle)_{i_j}, T) \tag{1}$$

Let X_j be the Λ_0-module $(M/\langle M_{i_{-1}} + \cdots + M_{i_{j-1}} \rangle)_{i_j}$. Let $X_j \to T(j)$ be the minimal left $\operatorname{add} T$-approximation in $\operatorname{mod} \Lambda_0$ for $j = 0, \ldots, t$. Let f_j denote the composition $M \to (M/\langle M_{i_{-1}} + \cdots + M_{i_{j-1}} \rangle)_{i_j} = X_j \to T(j)$ for $j = 0, \ldots, t$ and $f = (f_j)_{j=0}^t \colon M \to \oplus_{j=0}^t (T(j))[i_j]$ in $\operatorname{mod} \Lambda$. Given any morphism $g \colon M \to T$, the morphism g can be decomposed according to the decomposition in (1), and therefore g factors through f. Hence we have constructed a left $\operatorname{add} T$-approximation in $\operatorname{mod} \Lambda$.

Choose a left $\operatorname{add} T$-approximation $f \colon M \to M_T$ in $\operatorname{mod} \Lambda$ with M_T of minimal length as a Λ_0-module. Assume there is a morphism $g \colon M_T \to M_T$ such that $gf = f$. Then $g^n f = f$ for all $n \geq 1$. Since M_T has finite length over Λ_0, the Fitting Lemma applies, and therefore there exists an n such that $M_T = \operatorname{Ker} g^n \oplus \operatorname{Im} g^n$. Then it follows that all maps $h \colon M \to T$ factor through $M \xrightarrow{g^n f} \operatorname{Im} g^n$, and it is a left $\operatorname{add} T$-approximation in $\operatorname{mod} \Lambda$. If $\operatorname{Ker} g^n \neq (0)$, we have a contradiction. Therefore $\operatorname{Ker} g^n = (0)$, and g is an isomorphism.

(b) We continue to use the notation from (a). As in (a), construct the maps $X_j \to T(j)$. We view X_j and $T(j)$ as graded Λ-modules concentrated in degree i_j, and hence we can view the Λ_0-morphism $X_j \to T(j)$ as a degree 0 map. As above let $f_j \colon M \to T(j)$ be the induced map in $\operatorname{Hom}_{\operatorname{gr} \Lambda}(M, T(j))$. Let $f = (f_j)_{j=0}^t \colon M \to \oplus_{j=0}^t T(j)$.

It suffices to show that any degree 0 homomorphism $g \colon M \to T[i]$ factors through $f \colon M \to \oplus_{j=0}^t T(j)$. Since M is generated in the degrees $i_0, \ldots, i_t$, the map g is 0 if $i \neq i_j$ for some j, and hence factors through f. If $i = i_j$ for some j, then $g \colon M \to T[i_j]$ factors through $M \to X_j \to T[i_j]$ (since $T[i_j]$ has support in degree i_j). But as Λ_0-modules $X_j \to \psi(T[i_j])$ factors through $X_j \to T(j)$, and hence we get

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "f_j"] \arrow[dd, "f"'] & & T[i_j] \\ & T(j) \arrow[ur] & \\ \oplus_{j=1}^t T(j) \arrow[ur] & & \end{tikzcd}$$

in $\operatorname{gr}\Lambda$. The minimality we prove similarly as in (a).

Since $\psi(M)$ and $\psi(M_T)$ are finitely generated, then we have the following exact commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_\Lambda(\psi(M_T), T) \arrow[r] \arrow[d, "\wr"] & \operatorname{Hom}_\Lambda(\psi(M), T) \arrow[d, "\wr"] & \\ \oplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{gr}\Lambda}(M_T, T[j]) \arrow[r] & \oplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{gr}\Lambda}(M, T[j]) \arrow[r] & 0 \end{tikzcd}$$

It follows directly from this diagram that $\psi(M) \to \psi(M_T)$ is a left $\operatorname{add} T$-approximation in $\operatorname{mod}\Lambda$. $\square$

Remark 1.2. Note that the approximation of a module M in $\operatorname{gr}\Lambda$ has support in at most the degrees of the generators of M.

We now go back to discussing common properties of $\mathcal{K}(\Lambda)$ and $\mathcal{X}_T$. Let T be a module in $\operatorname{gr}\Lambda$ concentrated in degree zero, and let M be a module in $\operatorname{gr}\Lambda$. Denote by $\operatorname{rej}_T(M)$ the submodule $\cap_{f\in\operatorname{Hom}_{\operatorname{gr}\Lambda}(M,T)} \operatorname{Ker} f$ of M and by $\overline{M}$ the factor module $M/\operatorname{rej}_T(M)$ of M. Let $\alpha_M\colon M \to M_T$ denote the minimal left σT-approximation of M in $\operatorname{gr}\Lambda$. Furthermore, denote by $\operatorname{corej}_T(M)$ the module $M_T/\overline{M}$. Then we have the exact sequences

$$0 \to \operatorname{rej}_T(M) \to M \to \overline{M} \to 0$$

and

$$0 \to \overline{M} \to M_T \to \operatorname{corej}_T(M) \to 0.$$

The last of the above sequences has an additional property which we now explain.

As in the introduction, let $F^{\sigma T}$ be the additive subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ for the subcategory σT of $\operatorname{gr}\Lambda$. For simplicity we denote $F^{\sigma T}$ by F^T. Then one observes that the sequence $0 \to \overline{M} \to M_T \to \operatorname{corej}_T(M) \to 0$ is F^T-exact. By construction all these modules and sequences are in $\operatorname{Gr}\Lambda$. The following observation follows directly from the previous result.

Lemma 1.3. *Let M be in $\operatorname{gr}\Lambda$. Then*

$$\psi(\operatorname{rej}_T(M)) = \operatorname{rej}_T \psi(M) = \cap_{f\in\operatorname{Hom}_\Lambda(\psi(M),\psi(T))} \operatorname{Ker} f,$$

$$\psi(\overline{M}) = \overline{\psi(M)}$$

and

$$\psi(\operatorname{corej}_T(M)) = \operatorname{corej}_T(\psi(M)).$$

When Λ is Koszul, we have $\operatorname{rej}_T(M) = \operatorname{rad} M$ and $\operatorname{corej}_T(M) = (0)$, so that $\operatorname{rej}_T(M)[-1]$ and $\operatorname{corej}_T(M)$ are Koszul modules when M is. Also $\overline{M} = M/\operatorname{rad} M$ is a Koszul module. When T is a Wakamatsu cotilting module for an artin algebra Λ, then, as we have observed, $\operatorname{corej}_T(M)$ is in $\mathcal{X}_T$ when M is, and $\operatorname{rej}_T(M) = (0)$ in this case. Further $\overline{M} = M$.

If M is in $\mathcal{K}(\Lambda)$ or in $\mathcal{X}_T$, the property of syzygies can be formulated in a common way to say that $\langle \Omega(M)_0 \rangle$ and $\Omega(M)/\langle \Omega(M)_0 \rangle[-1]$ are also in $\mathcal{K}(\Lambda)$ or $\mathcal{X}_T$, respectively. For in the case of Koszul modules, $\langle \Omega(M)_0 \rangle = (0)$ and $\Omega(M)/\langle \Omega(M)_0 \rangle[-1] = \Omega(M)[-1]$, and in the second case $\langle \Omega(M)_0 \rangle = \Omega(M)$ and $\Omega(M)/\langle \Omega(M)_0 \rangle = (0)$. In addition, any Λ-module of the form $\Lambda \otimes_{\Lambda_0} N$ for a Λ_0-module N is projective, and is hence in $\mathcal{K}(\Lambda)$, and in the second case $\Lambda \otimes_{\Lambda_0} N \simeq N$ since $\Lambda = \Lambda_0$.

We shall use the above observations as inspiration for our definitions in the next section. We end this section by deducing some necessary conditions in order to have a reasonable duality theory. In particular we give necessary conditions on the modules in the exact sequences associated with M given above. In order for a duality to make sense, we must assume that $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras.

LEMMA 1.4. *Assume that $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras. Suppose M is in $\mathcal{G}_T(\Lambda)$ and that $\mathcal{E}'\mathcal{E}(M) \simeq M$ as graded Λ-modules. Then*

(a) *M_0 is dualized by T as a Λ_0-module.*
(b) *$\operatorname{corej}_T(M)$ is in $\operatorname{Sub}(T)$.*
(c) *$\overline{M} = M_0$.*
(d) *$\operatorname{rej}_T(M) = M_{\geq 1}$.*

PROOF. We have that

$$\begin{aligned} M_0 &\simeq \operatorname{Hom}_\Gamma(\mathcal{E}(M), \operatorname{Hom}_\Lambda(\Lambda, T)) \simeq \operatorname{Hom}_{\Gamma_0}(\mathcal{E}(M)_0, \operatorname{Hom}_{\Lambda_0}(\Lambda_0, T)) \\ &\simeq \operatorname{Hom}_{\Gamma_0}(\operatorname{Hom}_{\Lambda_0}(M, T), T) \end{aligned}$$

Consider the minimal left $\operatorname{add} T$-approximation $\operatorname{corej}_T(M) \to \operatorname{corej}_T(M)_T = T^1$. Then the sequence

$$(T^1, T) \to (M_T, T) \to (\overline{M}, T) \simeq (M, T) \to 0$$

is exact. This gives rise to the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & ((M,T),(\Lambda,T)) \arrow[r] \arrow[d, dash, "\wr"] & ((M_T,T),(\Lambda,T)) \arrow[r] \arrow[d, dash, "\wr"] & ((T^1,T),(\Lambda,T)) \arrow[d, dash, "\wr"] & \\
0 \arrow[r] & M_0 \arrow[r] & M_T \arrow[r] & T^1 & \\
0 \arrow[r] & \overline{M} \arrow[r] \arrow[u, dashed, "f"] & M_T \arrow[r] \arrow[u, equal] & \operatorname{corej}_T(M) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

It follows that f is a monomorphism, and that M_0 is in $\operatorname{Sub}(T)$.

Let $p\colon M \to M_0$ be the natural epimorphism. The map $M \to \overline{M}$ factors through p. Hence $\overline{M}$ is a factor module of M_0. Since $\overline{M}$ is also a submodule of M_0 and the modules have finite length over Λ_0, it follows that $\overline{M}$ and M_0 are isomorphic.

It is an immediate consequence of the above exact commutative diagram that $\operatorname{corej}_T(M)$ is in $\operatorname{Sub}(T)$. Recall from *Notation and general preliminary results*, that M_0 is then said to be dualized by T as a Λ_0-module. □

We shall now see that assuming T to be a Wakamatsu cotilting module is natural in order to have a reasonable duality theory. Assuming $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras we would like to have subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{G}_T(\Lambda)$ and $\mathcal{G}_T(E_T(\Lambda))$ respectively, such that the functor $\mathcal{E}$ induces a duality between $\mathcal{X}$ and $\mathcal{Y}$. Due to the central role of the modules Λ and T, it is natural to require that they lie in $\mathcal{X}$. Under one extra assumption, that $\mathcal{X}$ and $\mathcal{Y}$ are closed under taking coreject with respect to T, we obtain the following necessary condition for having duality.

PROPOSITION 1.5. *Assume that $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras, and that there are subcategories $\mathcal{X}$ in $\mathcal{G}_T(\Lambda)$ and $\mathcal{Y}$ in $\mathcal{G}_T(\Gamma)$ such that they are dual subcategories via the functors $\mathcal{E}$ and $\mathcal{E}'$. Furthermore, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are closed under taking coreject with respect to ${}_\Lambda T$ and ${}_\Gamma T$, respectively, and that Λ and T are in $\mathcal{X}$. Then*

(a) *${}_{\Lambda_0}T$ is a Wakamatsu cotilting module.*
(b) *${}_{\Gamma_0}T$ is a Wakamatsu cotilting module.*
(c) *If M is in $\mathcal{X}$, then M_0 is in $\mathcal{X}_{{}_{\Lambda_0}T}$.*
(d) *If N is in $\mathcal{Y}$, then N_0 is in $\mathcal{X}_{{}_{\Gamma_0}T}$.*

PROOF. Let M be in $\mathcal{X}$. Since $\mathcal{X}$ is closed under corejects, then by Lemma 1.4 there exists an exact sequence

$$0 \to M_0 \to T^0 \to T^1 \to T^2 \to \cdots$$

of minimal left $\operatorname{add} T$-approximations.

We have that $\mathcal{E}(M)$ is in $\mathcal{Y}$. Using that T is in $\mathcal{X}$, we infer that $\Gamma = \mathcal{E}(T)$ is in $\mathcal{Y}$, and therefore $\mathcal{E}\mathcal{E}'(\Gamma) \simeq \Gamma$. Hence we can apply Lemma 1.4 and use that $\mathcal{Y}$ is closed under corejects to obtain an exact sequence

$$0 \to \mathcal{E}(M)_0 \to T^0 \to T^1 \to T^2 \to \cdots$$

of minimal left $\operatorname{add} T$-approximations. Applying $\operatorname{Hom}_{\Gamma_0}(\ ,T)$ gives rise to the exact sequence

$$\cdots \to (T^2,T) \to (T^1,T) \to (T^0,T) \to (\mathcal{E}(M)_0,T) \to 0,$$

which is a projective Λ_0-resolution of $(\mathcal{E}(M)_0,T)$. Since M is generated in degree 0, we have that

$$(\mathcal{E}(M)_0,T) \simeq (\operatorname{Hom}_{\Lambda_0}(M_0,T), \operatorname{Hom}_{\Lambda_0}(\Lambda_0,T)) \simeq M_0$$

by Lemma 1.4. Applying $\operatorname{Hom}_{\Lambda_0}(\ ,T)$ we obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & ((\mathcal{E}(M)_0,T),T) \arrow[r] \arrow[d, "\wr"] & ((T^0,T),T) \arrow[r] \arrow[d, "\wr"] & ((T^1,T),T) \arrow[r] \arrow[d, "\wr"] & \cdots \\
0 \arrow[r] & \mathcal{E}(M)_0 \arrow[r] & T^0 \arrow[r] & T^1 \arrow[r] & \cdots
\end{tikzcd}$$

The lower row is exact, hence $\operatorname{Ext}^i_{\Lambda_0}(M_0,T) = (0)$ for all $i > 0$. This shows the statements in (a) and (c), and the claims in (b) and (d) follow by symmetry. □

2. Generalized T-Koszul algebras

Motivated by the connections between the categories $\mathcal{K}(\Lambda)$ and $\mathcal{X}_T$ discussed in the previous section, we introduce our notion of generalized T-Koszul algebras, and the corresponding category of Koszul modules. We also compare with our earlier definitions.

We have pointed out six operations under which the categories $\mathcal{K}(\Lambda)$ and $\mathcal{X}_T$ are closed. Actually, the first three conditions are related to the last three via duality, as now discuss.

Let M be in $\operatorname{gr}\Lambda$ and generated in degree zero. Then we have the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V_M \arrow[r, equal] \arrow[d] & V_M \arrow[d] & & \\
0 \arrow[r] & \Omega_\Lambda(M) \arrow[r] \arrow[d] & P(M)=\Lambda\otimes_{\Lambda_0}P(M)_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M^1 \arrow[r] \arrow[d] & \Lambda\otimes_{\Lambda_0}M_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $P(M)_0$ is the projective cover of M_0 as a Λ_0-module. The module V_M is generated in degree 0, since $\Lambda\otimes_{\Lambda_0}\Omega_{\Lambda_0}(M_0)\to V_M$ is an epimorphism. The module M^1 is generated in degree 1 and higher, since $(\Lambda\otimes_{\Lambda_0}M_0)_0=\Lambda_0\otimes_{\Lambda_0}M_0\simeq M_0$. It follows from this that $V_M=\langle\Omega_\Lambda(M)_0\rangle\subset\Omega_\Lambda(M)$ and $M^1=\Omega_\Lambda(M)/\langle\Omega_\Lambda(M)_0\rangle$. We show in Section 5 (see Proposition 5.4 and Proposition 5.5) that the concept of $\operatorname{rej}_T(M)$ is dual to $\Omega_\Lambda(M)/\langle\Omega_\Lambda(M)_0\rangle$, $\overline{M}$ is dual to $\Lambda\otimes_{\Lambda_0}M_0$, and $\operatorname{corej}_T(M)$ is dual to $\langle\Omega_\Lambda(M)_0\rangle$ via the functor $\mathcal{E}$, under some additional assumptions.

Let Λ be a $\mathbb{Z}$-graded ring as above, and let now T be some module in $\operatorname{mod}\Lambda_0$. Denote as usual $\oplus_{i\geq 0}\operatorname{Ext}^i_\Lambda(T,T)$ by $E_T(\Lambda)$. For ease of notation we also denote $E_T(\Lambda)$ by Γ. Define $\mathcal{E}\colon \operatorname{Gr}\Lambda\to\operatorname{Gr}\Gamma$ by $\mathcal{E}(\ )=\oplus_{i\geq 0}\operatorname{Ext}^i_\Lambda(\ ,T)$. Then $\mathcal{E}(\Lambda)=\operatorname{Hom}_\Lambda(\Lambda,T)$, which we also denote by T, is a module also over Γ. Define $\mathcal{E}'\colon \operatorname{Gr}\Gamma\to\operatorname{Gr}E_T(\Gamma)$ by $\mathcal{E}'=\oplus_{i\geq 0}\operatorname{Ext}^i_\Gamma(\ ,T)$.

We now define the concept of generalized T-Koszul algebras.

Definition 2.1. Let $\Lambda=\oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting Λ_0-module.

(a) We define $\mathcal{K}_T(\Lambda)$ to be the largest full subcategory of $\mathcal{G}_T(\Lambda)$ consisting of modules M with M_0 in $\mathcal{X}_T$ such that $\mathcal{K}_T(\Lambda)$ is closed under the operations $\operatorname{rej}_T(\ )[-1]$, $\overline{(\ )}$, $\operatorname{corej}_T(\ )$ $\Omega_\Lambda(\ )/\langle\Omega_\Lambda(\ )_0\rangle[-1]$, $\Lambda\otimes_{\Lambda_0}(\ )_0$ and $\langle\Omega_\Lambda(\ )_0\rangle$.

(b) The modules in $\mathcal{K}_T(\Lambda)$ are called the *Koszul modules* over Λ with respect to T.

(c) The algebra Λ is said to be a *generalized T-Koszul algebra* if Λ and T are in $\mathcal{K}_T(\Lambda)$.

(d) The algebra Λ is said to be a *generalized strongly T-Koszul algebra* if $\mathcal{K}_T(\Lambda) = \mathcal{G}_T(\Lambda)$.

For any subcategory $\mathcal{Y}$ of $\mathcal{X}_T$ define $\mathcal{FR}_\Lambda(\mathcal{Y})$ to be the full subcategory of $\mathrm{Gr}(\Lambda)$ consisting of all graded modules M such that M_i is in $\mathcal{Y}$ for all i and such that there is an exact sequence

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where for each $n \geq 0$ the modules $F_n \simeq \Lambda \otimes_{\Lambda_0} Y_n[n]$ for some Y_n in $\mathcal{Y}$. Note that for a T-Koszul algebra Λ, the category of T-Koszul modules is $\mathcal{FR}_\Lambda(\mathcal{X}_T)$. Immediately from the definition of $\mathcal{K}_T(\Lambda)$ for a generalized T-Koszul algebra we show that in this more general situation the category of T-Koszul modules has similar properties as the category of modules with linear fat resolutions $\mathcal{FR}_\Lambda(\mathcal{X}_T)$ studied in the previous chapter.

LEMMA 2.2. *If M is in $\mathcal{K}_T(\Lambda)$, then the following assertions hold.*
(a) *M_i is in $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T$ for all $i \geq 0$.*
(b) *There exists an exact sequence*

$$\cdots \to \Lambda \otimes_{\Lambda_0} M_2^2[2] \to \Lambda \otimes_{\Lambda_0} M_1^1[1] \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$$

where M_2^2 is in $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T$ for all $i \geq 0$.

PROOF. (a) Let M be in $\mathcal{K}_T(\Lambda)$. Then it follows immediately from the definition that $\overline{M} = M_0$ and that M_0 is in $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T$. A consequence of this is that $\mathrm{rej}_T(M) = M_{\geq 1}$, where, by definition of $\mathcal{K}_T(\Lambda)$, the module $\mathrm{rej}_T(M)[-1]$ is in $\mathcal{K}_T(\Lambda)$. Hence by using the same arguments as above, $\mathrm{rej}_T(M)[-1]_0 = M_1$ is in $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T$. Continuing this way we infer that M_i is in $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T$ for all $i \geq 0$.

(b) Let M be in $\mathcal{K}_T(\Lambda)$. Then the sequence $0 \to M^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$ is exact with $M^1[-1]$ in $\mathcal{K}_T(\Lambda)$ and M_0 in $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T$ by (a). The claim in (b) then follows by induction. □

Next we show that a T-Koszul algebra as defined in the previous chapter is a special case of the notion of a generalized T-Koszul algebra.

PROPOSITION 2.3. *Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, with Λ_0 an artin algebra. Let T be a Wakamatsu cotilting Λ_0-module. Then Λ is a T-Koszul algebra if and only if Λ satisfies the $\otimes$-Tor-conditions with respect to T and Λ is a generalized T-Koszul algebra.*

PROOF. Assume now that Λ is a T-Koszul algebra as defined in the previous chapter. We need to show that the category $\mathcal{FR}_\Lambda(\mathcal{X}_T)$ is closed under our six operations. For the reject it follows from Proposition 2.2. We leave the other cases to the reader. Hence we have that $\mathcal{FR}_\Lambda(\mathcal{X}_T)$ is contained in $\mathcal{K}_T(\Lambda)$. Since T and $\Lambda \cong \Lambda \otimes_{\Lambda_0} \Lambda_0$ are in $\mathcal{FR}_\Lambda(\mathcal{X}_T)$, they are in $\mathcal{K}_T(\Lambda)$, so that Λ is a generalized T-Koszul algebra. By the previous result receive that $\mathcal{FR}_\Lambda(\mathcal{X}_T)$ is contained in $\mathcal{K}_T(\Lambda)$, and consequently $\mathcal{K}_T(\Lambda) = \mathcal{FR}_\Lambda(\mathcal{X}_T)$.

Assume that Λ satisfies the $\otimes$-Tor-conditions with respect to $\mathcal{X}_T$ and that Λ is a generalized T-Koszul algebra. Since Λ is a generalized T-Koszul algebra, Λ is in $\mathcal{K}_T(\Lambda)$. By definition we have that $\Lambda_0 = \overline{\Lambda}$ is in $\mathcal{K}_T(\Lambda)$. Then by Lemma 2.2 we immediately get that Λ_0 has a linear fat resolution. Therefore Λ_0 is in $\mathcal{FR}_\Lambda(\mathcal{X}_T)$ and Λ is T-Koszul. □

3. The classical cases revisited

In this section we take a closer look at how the classical cases fit in with the concept of generalized T-Koszul algebras.

We start with the classical Koszul case. Note that for a semisimple ring Λ_0 the only Wakamatsu cotilting module up to multiplicity of direct summands is Λ_0.

PROPOSITION 3.1. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, where Λ_0 is semisimple. Let $T = \Lambda_0$. Then the following are equivalent.*

(a) *Λ is a generalized Koszul algebra with respect to T.*
(b) *Λ is a generalized strongly Koszul algebra with respect to T.*
(c) *Λ is a Koszul algebra.*

PROOF. Given the assumptions on Λ, it is immediate that Λ satisfies the $\otimes$-Tor-conditions with respect to ${}^{\perp}T = \operatorname{add}\Lambda_0 = \operatorname{mod}\Lambda_0$. Using Proposition 2.3 we conclude that Λ is a generalized T-Koszul algebra if and only if Λ is a T-Koszul algebra. We already observed in Proposition 4.3 that under these assumptions that Λ is a T-Koszul algebra if and only if Λ is a Koszul algebra. This proves that (a) and (c) are equivalent.

We clearly have that (b) implies (a).

We have that $\mathcal{G}_T(\Lambda)$ is closed under all six operations, and hence $\mathcal{K}_T(\Lambda) = \mathcal{G}_T(\Lambda)$, so that Λ is strongly Λ_0-Koszul. This shows that (a) implies (b), and this completes the proof of the proposition. $\square$

We next treat the (Wakamatsu) cotilting case.

PROPOSITION 3.2. *Let $\Lambda = \Lambda_0$ be an artin algebra, and let T be a Wakamatsu cotilting module.*

(a) *Λ is a generalized T-Koszul algebra.*
(b) *The following are equivalent.*
   (i) *T is a cotilting module.*
   (ii) *Λ is a generalized strongly T-Koszul algebra with $\operatorname{id}_\Lambda T < \infty$.*

PROOF. (a) When Λ is an artin algebra and T is a Wakamatsu cotilting module, $\mathcal{G}_T(\Lambda)$ is clearly ${}^{\perp}T$. We have observed that the subcategory $\mathcal{X}_T$ is closed under the six operations. Since $\mathcal{K}_T(\Lambda)$ is supposed to be closed under corejects, it follows from the definition of $\mathcal{X}_T$ that $\mathcal{K}_T(\Lambda) = \mathcal{X}_T$. Since T and Λ are in $\mathcal{X}_T$, we have that Λ is a generalized T-Koszul algebra.

(b) If T is a cotilting module, then $\operatorname{id}_\Lambda T < \infty$ by definition. By the above $\mathcal{K}_T(\Lambda) = \mathcal{X}_T$ and $\mathcal{G}_T(\Lambda) = {}^{\perp}T$. If T is a cotilting module, then $\mathcal{X}_T = {}^{\perp}T$. Hence Λ is strongly T-Koszul.

Assume that Λ is generalized strongly T-Koszul with $\operatorname{id}_\Lambda T < \infty$. Then by assumption $\mathcal{K}_T(\Lambda) = \mathcal{X}_T = {}^{\perp}T = \mathcal{G}_T(\Lambda)$. The subcategory $\mathcal{X}_T = {}^{\perp}T$ is a resolving subcategory of $\operatorname{mod}\Lambda$ with an Ext-injective cogenerator T. Since $\operatorname{id}_\Lambda T < \infty$, we infer that $\widehat{{}^{\perp}T} = \operatorname{mod}\Lambda$. Then by [**2**, Theorem 2.3] $\widehat{{}^{\perp}T}$ is contravariantly finite in $\operatorname{mod}\Lambda$ and $({}^{\perp}T)^{\perp} = \widehat{\operatorname{add}T}$. It is clear that ${}^{\perp}T \cap ({}^{\perp}T)^{\perp} = \operatorname{add}T$, which by [**3**] is the additive category generated by a cotilting module. Hence T is a cotilting module. $\square$

4. Necessity of the $\otimes$-Tor-conditions

In our original definition of a T-Koszul algebra the $\otimes$-Tor-conditions appeared, which at first sight look artificial. In this section we try to remove this mystery by showing that they are almost necessary in order to have a duality between nice subcategories. We start with the $\otimes$-condition.

PROPOSITION 4.1. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting Λ_0-module T, and let $\mathcal{X}_T$ and $\mathcal{K}_T(\Lambda)$ be defined as before. Then $\Lambda_i \otimes_{\Lambda_0} (\mathcal{K}_T(\Lambda)\cap\mathcal{X}_T) \subset \mathcal{K}_T(\Lambda)\cap\mathcal{X}_T$ for all $i \geq 0$.*

PROOF. Let X be in $\mathcal{K}_T(\Lambda)\cap\mathcal{X}_T$. Then $\Lambda\otimes_{\Lambda_0} X$ is in $\mathcal{K}_T(\Lambda)$, and hence $(\Lambda\otimes_{\Lambda_0} X)_i$ is in $\mathcal{K}_T(\Lambda)\cap\mathcal{X}_T$ for all i by Lemma 2.2. $\square$

To treat the Tor-condition we need the following preliminary result.

LEMMA 4.2. *Let M_0 be in $\mathcal{K}_T(\Lambda)\cap\mathcal{X}_T$. Then the exact sequence*

$$\eta\colon 0 \to \langle\Omega_\Lambda(M)_0\rangle \to P(M) \to \Lambda\otimes_{\Lambda_0} M_0 \to 0$$

is F^T-exact.

PROOF. Applying $\operatorname{Hom}_\Lambda(\ ,T)$ to the sequence η induces the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(\Lambda\otimes_{\Lambda_0} M_0, T) \to \operatorname{Hom}_\Lambda(P(M),T) \to \operatorname{Hom}_{\Lambda_0}(\langle\Omega_\Lambda(M)_0\rangle, T).$$

This sequence is isomorphic to the sequence

$$0 \to \operatorname{Hom}_{\Lambda_0}(M_0,T) \to \operatorname{Hom}_{\Lambda_0}(P(M)_0,T) \to \operatorname{Hom}_{\Lambda_0}(\Omega_{\Lambda_0}(M_0),T).$$

The cokernel of the last morphism in the above sequence is $\operatorname{Ext}^1_{\Lambda_0}(M_0,T)$, which is zero since M_0 is in $\mathcal{X}_T$. Hence η is F^T-exact. $\square$

We now have the following.

PROPOSITION 4.3. *Let $\Lambda = \bigoplus_{i\geq 0}\Lambda_i$ be a graded artin algebra and T a Wakamatsu cotilting module. Then $\operatorname{Tor}_1^{\Lambda_0}(\Lambda_i, \mathcal{X}_T\cap\mathcal{K}_T(\Lambda)) = (0)$ for $i \geq 0$, and hence $\operatorname{Ext}^1_{\Lambda_0}(\mathcal{X}_T\cap\mathcal{K}_T(\Lambda), D(\Lambda_i)) = (0)$.*

PROOF. Let M_0 be in $\mathcal{K}_T(\Lambda)\cap\mathcal{X}_T$. We have the exact sequences

$$0 \to \langle\Omega_\Lambda(M)_0\rangle \to \Lambda\otimes_{\Lambda_0} P_{\Lambda_0}(M_0) \to \Lambda\otimes_{\Lambda_0} M_0 \to 0$$

and

$$0 \to \operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0) \to \Lambda\otimes_{\Lambda_0}\Omega_{\Lambda_0}(M_0) \to \langle\Omega_\Lambda(M)_0\rangle \to 0.$$

Since $\langle\Omega_\Lambda(M)_0\rangle$ is in $\mathcal{K}_T(\Lambda)$ and $\langle\Omega_\Lambda(M_0)_0\rangle_0 = \Omega_{\Lambda_0}(M_0)$, it follows that $\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0) = \langle\Omega_\Lambda(M)_0\rangle^1$, which is in $\mathcal{K}_T(\Lambda)$. Applying $\operatorname{Hom}_\Lambda(\ ,T)$ to the short exact sequence starting in $\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0)$ we obtain the following exact sequence

$$\begin{aligned} 0 \to (\langle\Omega_\Lambda(M_0)_0\rangle, T) \xrightarrow{f} (\Lambda\otimes_{\Lambda_0}\Omega_{\Lambda_0}(M_0), T) \to \\ (\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0), T) \to \operatorname{Ext}^1_\Lambda(\langle\Omega_\Lambda(M_0)_0\rangle, T), \end{aligned}$$

where f is an isomorphism, since $\langle\Omega_\Lambda(M)_0\rangle_0 = (\Lambda\otimes_{\Lambda_0}\Omega_{\Lambda_0}(M_0))_0$. It follows from Lemma 4.2 that $\operatorname{Ext}^1_\Lambda(\langle\Omega_\Lambda(M_0)_0\rangle, T) = (0)$. Therefore $\operatorname{Hom}_\Lambda(\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0), T) = (0)$.

Since $\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0)[-1]$ is in $\mathcal{K}_T(\Lambda)$ (and therefore generated in degree 0), the module $(\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0)[1])_0$ is in $\mathcal{X}_T \subset \operatorname{Sub}(T)$ by definition of $\mathcal{K}_T(\Lambda)$. Hence we infer that $\operatorname{Tor}_1^{\Lambda_0}(\Lambda, M_0) = (0)$. In particular, $\operatorname{Tor}_1^{\Lambda_0}(\Lambda_j, M_0) = (0)$ for all $j \geq 0$. $\square$

5. Properties of the Ext-functor

In this section we investigate the category $\mathcal{K}_T(\Lambda)$ of T-Koszul modules for generalized T-Koszul algebras Λ more closely, especially the behavior under the functor $\mathcal{E}$. Motivated by the property that a module goes to a module concentrated in degree 0 under $\mathcal{E}$, we introduce the notion of fat modules, generalizing the notion from Chapter I. This leads to linear fat resolutions, which we show that all modules in $\mathcal{K}_T(\Lambda)$ have.

Throughout this section let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring with Λ_0 an artin algebra, and let T be a Wakamatsu cotilting module over Λ_0.

Now we turn to the behavior of modules under the functor $\mathcal{E}$.

LEMMA 5.1. *The graded projective cover of the $E_T(\Lambda)$-module $\mathcal{E}(M)$ for M in $\mathcal{K}_T(\Lambda)$ is isomorphic to $\mathcal{E}(M_T)$.*

PROOF. Let M be in $\mathcal{K}_T(\Lambda)$. Then $\mathcal{E}(M)$ is generated in degree 0, and we have an induced map $\alpha_M^*\colon \mathcal{E}(M_T)\to\mathcal{E}(M)$. Since the map $\alpha_M\colon M\to M_T$ is a left $\operatorname{add}T$-approximation, the induced map

$$(\alpha_M^*)_0\colon\ \operatorname{Hom}_\Lambda(M_T,T)\to\operatorname{Hom}_\Lambda(M,T)$$

is surjective. By assumption $\mathcal{E}(M)$ is generated in degree 0. Therefore the map $\alpha_M^*\colon\mathcal{E}(M_T)\to\mathcal{E}(M)$ is surjective. Since $\alpha_M\colon M\to M_T$ is a left minimal morphism, $(\alpha_M^*)_0$ is a projective cover as an $\operatorname{End}_\Lambda(T)$-module. This implies that $\alpha_M^*\colon\mathcal{E}(M_T)\to\mathcal{E}(M)$ is a projective cover. $\square$

The next result is the dual of the above result, showing that the projective cover of a module M in $\mathcal{K}_T(\Lambda)$ induces a minimal $\operatorname{add}T$-approximation of $\mathcal{E}(M)$.

LEMMA 5.2. *If M is a module in $\mathcal{K}_T(\Lambda)$, then $\mathcal{E}(M)\to\mathcal{E}(P(M))$ is the minimal* $\operatorname{add}T$*-approximation of $\mathcal{E}(M)$, where $P(M)\to M$ is the projective cover of M.*

PROOF. It is clear that $\mathcal{E}(P(M))$ is in $\operatorname{add}T$. First we show that $\mathcal{E}(M)\to\mathcal{E}(P(M))$ is a left $\operatorname{add}T$-approximation. For this it is enough to show that $\operatorname{Hom}_\Gamma(\mathcal{E}(P(M)),T)\to\operatorname{Hom}_\Gamma(\mathcal{E}(M),T)$ is onto. We have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_\Gamma(\mathcal{E}(P(M)),T) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_\Gamma(\mathcal{E}(M),T) \arrow[d, equal] & \\
\operatorname{Hom}_{\Gamma_0}(\mathcal{E}(P(M))_0,T) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\Gamma_0}(\mathcal{E}(M)_0,T) \arrow[d, equal] & \\
\operatorname{Hom}_{\Gamma_0}((P(M)_0,T),(\Lambda_0,T)) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\Gamma_0}((M_0,T),(\Lambda_0,T)) \arrow[d, equal] & \\
P(M)_0 \arrow[r] & M_0 \arrow[r] & 0
\end{tikzcd}$$

Hence $\mathcal{E}(M)\to\mathcal{E}(P(M))$ is a left $\operatorname{add}T$-approximation of $\mathcal{E}(M)$.

The minimal left $\operatorname{add}T$-approximation of $\mathcal{E}(M)$ is a direct summand of $\mathcal{E}(P(M))$. Therefore there is a direct summand Q of $P(M)$ such that $\mathcal{E}(M)\to\mathcal{E}(Q)$ is the minimal left $\operatorname{add}T$-approximation. This follows from the fact that $\operatorname{add}\Lambda$ is dual to $\operatorname{add} T$ when these subcategories are viewed as subcategories of the category of graded modules with degree 0 morphisms. Using the same isomorphisms and the diagram above, it follows that $Q_0 \to M_0$ is an epimorphism. This is a contradiction unless $Q_0 = P(M)_0$ and therefore $Q = P(M)$. We conclude that $\mathcal{E}(M) \to \mathcal{E}(P(M))$ is the minimal left $\operatorname{add} T$-approximation. $\square$

To see what $\mathcal{E}$ of $\overline{M}$, $\operatorname{rej}_T(M)$ and $\operatorname{corej}_T(M)$ are mapped to via the functor $\mathcal{E}$, we need the following lemma.

LEMMA 5.3. *For any T_0 in $\operatorname{add} T$ there is a natural isomorphism*

$$\Gamma \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(T_0, T) \to \mathcal{E}(T_0).$$

PROOF. We clearly have a natural isomorphism $\Gamma \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(T,T) \simeq \Gamma = \mathcal{E}(T)$, and also we have $\Gamma \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(T^n, T) \simeq \Gamma^n = \mathcal{E}(T^n)$. The morphism is additive, hence

$$\Gamma \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(T_0, T) \simeq \mathcal{E}(T_0)$$

for any T_0 in $\operatorname{add} T$. $\square$

Next we show that the syzygy of $\mathcal{E}(M)$ for M in $\mathcal{K}_T(\Lambda)$ is an extension of $\mathcal{E}(\operatorname{corej}_T(M))$ and $\mathcal{E}(\operatorname{rej}_T(M))[1]$, in addition to showing that corej, $\overline{(\ )}$ and rej are mapped to $\langle \Omega_\Gamma(\ )_0\rangle$, $\Gamma \otimes_{\Gamma_0} \mathcal{E}(\ )_0$ and $\Omega_\Gamma(\ )/\langle \Omega_\Gamma(\ )_0\rangle$, respectively.

PROPOSITION 5.4. *Let M be in $\mathcal{K}_T(\Lambda)$. Then the following assertions hold.*

(a) *We have the following exact commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{E}(\operatorname{corej}_T(M)) \arrow[r, equal] \arrow[d] & \mathcal{E}(\operatorname{corej}_T(M)) \arrow[d] & & \\
0 \arrow[r] & \Omega_{E_T(\Lambda)}(\mathcal{E}(M)) \arrow[r] \arrow[d] & \mathcal{E}(M_T) \arrow[r] \arrow[d] & \mathcal{E}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}(\operatorname{rej}_T(M))[1] \arrow[r] \arrow[d] & \mathcal{E}(\overline{M}) \arrow[r] \arrow[d] & \mathcal{E}(M) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

(b) (i) $\langle \Omega_\Gamma(\mathcal{E}(M))_0\rangle \simeq \mathcal{E}(\operatorname{corej}_T(M))$.
(ii) $\Gamma \otimes_{\Gamma_0} \mathcal{E}(M)_0 = \Gamma \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(M, T) \simeq \mathcal{E}(\overline{M})$.
(iii) $\Omega_\Gamma(\mathcal{E}(M))/\langle \Omega_\Gamma(\mathcal{E}(M))_0\rangle \simeq \mathcal{E}(\operatorname{rej}_T(M))[1]$, *moreover is generated in degree one.*

PROOF. (a) By Lemma 1.6 the sequence $0 \to \mathcal{E}(\operatorname{rej}_T(M))[1] \to \mathcal{E}(M_0) \to \mathcal{E}(M) \to 0$ is exact. The sequence $0 \to M_0 \to M^T \to \operatorname{corej}_T(M) \to 0$ is F^T-exact, so that by Proposition 1.12 the sequence $0 \to \mathcal{E}(\operatorname{corej}_T(M)) \to \mathcal{E}(M^T) \to \mathcal{E}(M_0) \to 0$ is exact. Using that $\mathcal{E}(M^T) \simeq \Gamma \otimes_{\Gamma_0} \operatorname{Hom}_{\Lambda_0}(M^T, T)$ is the projective cover of $\mathcal{E}(M)$ by Lemma 5.1, the desired diagram is just putting these sequences together.

(b) It follows directly from (a) that $\langle \Omega_\Gamma(\mathcal{E}(M))_0\rangle \simeq \mathcal{E}(\operatorname{corej}_T(M))$, since $\mathcal{E}(\operatorname{corej}_T(M))$ and $\mathcal{E}(\operatorname{rej}_T(M))$ are generated in degree 0 by assumption. It follows also that $\Omega_\Gamma(\mathcal{E}(M))/\langle\Omega_\Gamma(\mathcal{E}(M))_0\rangle \simeq \mathcal{E}(\operatorname{rej}_T(M))[1]$, and for the same reason as above, is generated in degree 1.

The sequence

$$0 \to (\operatorname{corej}_T(M), T) \xrightarrow{t} (M_T, T) \to (M, T) \to 0$$

is exact. Hence we obtain the following exact sequence

$$\Gamma \otimes_{\Gamma_0} (\operatorname{corej}_T(M), T) \xrightarrow{\mathrm{id}\otimes t} \Gamma \otimes_{\Gamma_0} (M_T, T) \to \Gamma \otimes_{\Gamma_0} (M, T) \to 0.$$

Since $\Gamma \otimes_{\Gamma_0} (M_T, T) \simeq \mathcal{E}(M_T)$ by Lemma 5.3 and the image of the map $\mathrm{id}\otimes t$ is $\langle \Omega_\Gamma(\mathcal{E}(M))_0\rangle$, the assertion in (a) implies that $\mathcal{E}(\overline{M}) \simeq \Gamma \otimes_{\Gamma_0} \operatorname{Hom}_\Lambda(M, T) = \Gamma \otimes_{\Gamma_0} \mathcal{E}(M)_0$. □

Now we consider the dual situation; namely, we show that $\langle \Omega_\Lambda(\ )_0\rangle$, $\Lambda \otimes_{\Lambda_0} (\ )_0$ and $\Omega_\Lambda(\ )/\langle\Omega_\Lambda(\ )_0\rangle$ are mapped to corej, $\overline{(\ )}$ and rej, respectively.

PROPOSITION 5.5. *Let M be in $\mathcal{K}_T(\Lambda)$. Then we have the following.*

(a) *The exact sequences*

$$\eta\colon 0 \to \langle \Omega_\Lambda(M)_0\rangle \to P(M) \to \Lambda \otimes_{\Lambda_0} M_0 \to 0$$

and

$$\eta'\colon 0 \to \langle \Omega_\Lambda(M)_0\rangle \to \Omega_\Lambda(M) \to \Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0\rangle \to 0$$

are F^T-exact.

(b) (i) $\mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) \simeq \mathcal{E}(M)_0 = \overline{\mathcal{E}(M)}$.
(ii) $\mathcal{E}(\langle \Omega_\Lambda(M)_0\rangle) \simeq \operatorname{corej}_T \mathcal{E}(M)$.
(iii) $\mathcal{E}(\Omega_\Lambda(M)/\langle\Omega_\Lambda(M)_0\rangle) \simeq \operatorname{rej}_T \mathcal{E}(M)[-1]$.
(iv)

$$\begin{aligned}\mathcal{E}(\Omega_\Lambda(M)) &\simeq \mathcal{E}(\Omega_\Lambda(M)/\langle\Omega_\Lambda(M)_0\rangle) \oplus \mathcal{E}(\langle\Omega_\Lambda(M)_0\rangle)\\ &\simeq \operatorname{rej}_T \mathcal{E}(M) \oplus \operatorname{corej}_T \mathcal{E}(M),\end{aligned}$$

PROOF. (a) Applying $\operatorname{Hom}_\Lambda(\ , T)$ to the sequence η induces the following exact sequence.

$$0 \to \operatorname{Hom}_\Lambda(\Lambda \otimes_{\Lambda_0} M_0, T) \to \operatorname{Hom}_\Lambda(P(M), T) \to \operatorname{Hom}_\Lambda(\langle\Omega_\Lambda(M)_0\rangle, T).$$

This sequence is isomorphic to the sequence

$$0 \to \operatorname{Hom}_{\Lambda_0}(M_0, T) \to \operatorname{Hom}_{\Lambda_0}(P(M)_0, T) \to \operatorname{Hom}_{\Lambda_0}(\Omega_{\Lambda_0}(M_0), T).$$

The cokernel of the last morphism in the above sequence is $\operatorname{Ext}^1_{\Lambda_0}(M_0, T)$, which is zero since M_0 is in $\mathcal{X}_T$. Hence η is F^T-exact, and therefore η' is also F^T-exact since it is a pullback of η.

(b) (i) The sequence $\eta\colon 0 \to \langle\Omega_\Lambda(M)_0\rangle \to P(M) \to \Lambda\otimes_{\Lambda_0} M_0 \to 0$ is F^T-exact, so that by Proposition 1.12 in Chapter II the sequence $0 \to \mathcal{E}(\Lambda\otimes_{\Lambda_0} M_0) \to \mathcal{E}(P(M)) \to \mathcal{E}(\langle\Omega_\Lambda(M)_0\rangle) \to 0$ is exact. Since $\mathcal{E}(P(M)) = \operatorname{Hom}_\Lambda(P(M), T)$, we infer that $\mathcal{E}(\Lambda\otimes_{\Lambda_0} M_0) = \operatorname{Hom}_\Lambda(\Lambda\otimes_{\Lambda_0} M_0, T) = \operatorname{Hom}_{\Lambda_0}(M_0, T) = \mathcal{E}(M)_0$.

(ii) By Lemma 5.2 we have that $\mathcal{E}(\Lambda\otimes_{\Lambda_0} M_0) = \mathcal{E}(M)_0 \to \mathcal{E}(P(M))$ is the minimal left $\operatorname{add} T$-approximation of $\mathcal{E}(M)$. Hence $\mathcal{E}(\langle\Omega_\Lambda(M)_0\rangle) = \operatorname{corej}_T \mathcal{E}(M)$.

(iii) We have the exact sequence $0 \to \Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$. This sequence gives rise to the exact sequence

$$0 \to \mathcal{E}(\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle)[1] \to \mathcal{E}(M) \to \mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) = \mathcal{E}(M)_0 \to 0$$

Hence $\operatorname{rej}_T \mathcal{E}(M) = \mathcal{E}(\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle)[1]$.

(iv) Using the above sequences we have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{E}(M)_0 \arrow[r, equal] \arrow[d, "\wr"] & \mathcal{E}(M)_0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) \arrow[r] \arrow[d, "0"] & \mathcal{E}(P(M)) \arrow[r] \arrow[d] & \mathcal{E}(\langle \Omega_\Lambda(M)_0 \rangle) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}(\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle) \arrow[r] & \mathcal{E}(\Omega_\Lambda(M)) \arrow[r] & \mathcal{E}(\langle \Omega_\Lambda(M)_0 \rangle) \arrow[r] & 0
\end{tikzcd}$$

This shows that the lower row is split, hence the claim in (iv) follows. □

We have the following basic property of the category $\mathcal{K}_T(\Lambda)$.

Proposition 5.6. *The category $\mathcal{K}_T(\Lambda)$ is closed under extensions.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence with A and C in $\mathcal{K}_T(\Lambda)$. Since this sequence is F^T-exact and $\mathcal{G}_T(\Lambda)$ is closed under F^T-extensions, B is also in $\mathcal{G}_T(\Lambda)$. Moreover all the sequences

$$0 \to A_0 \to B_0 \to C_0 \to 0$$
$$0 \to \operatorname{corej}_T A \to \operatorname{corej}_T B \oplus T' \to \operatorname{corej}_T C \to 0$$
$$0 \to \operatorname{rej}_T A \to \operatorname{rej}_T B \to \operatorname{rej}_T C \to 0$$
$$0 \to \Lambda \otimes_{\Lambda_0} A_0 \to \Lambda \otimes_{\Lambda_0} B_0 \to \Lambda \otimes_{\Lambda_0} C_0 \to 0$$
$$0 \to \langle \Omega_\Lambda(A)_0 \rangle \to \langle \Omega_\Lambda(B)_0 \rangle \to \langle \Omega_\Lambda(C_0) \rangle \to 0$$
$$0 \to A^1 \to B^1 \to C^1 \to 0$$

are F^T-exact with the end terms in $\mathcal{K}_T(\Lambda)$. Since $\mathcal{G}_T(\Lambda)$ is closed under F^T-extensions, it follows that all the middle terms are in $\mathcal{G}_T(\Lambda)$. From this we infer that B is in $\mathcal{K}_T(\Lambda)$. □

It will be convenient to introduce the following definition, which turns out to generalize the corresponding one from Chapter I.

Definition 5.7. A graded Λ-module X is called *fat* if X is in $\operatorname{gr}_0 \Lambda$ and $\mathcal{E}(X) = \operatorname{Hom}_\Lambda(X, T)$, or equivalently X is in ${}^{\perp}_{\Lambda}T$.

An immediate consequence of the definitions involved is that any fat module is in $\mathcal{G}_T(\Lambda)$. If Λ is a Koszul algebra and $T = \Lambda_0$, then the fat modules are nothing else than the projective Λ-modules generated in degree zero. If $\Lambda = \Lambda_0$ and T is a cotilting module, then the full subcategory of fat modules is all of $\mathcal{K}_T(\Lambda) = {}^{\perp}_{\Lambda}T$.

Proposition 5.5 (b) shows that $\Lambda \otimes_{\Lambda_0} M_0$ is a fat module for all M in $\mathcal{K}_T(\Lambda)$. The fat modules inside $\mathcal{K}_T(\Lambda)$ have the following properties.

Proposition 5.8. *Suppose that Λ and T are as above.*

(a) *Let X be in $\mathcal{K}_T(\Lambda)$. Then X is fat if and only if $X \simeq \Lambda \otimes_{\Lambda_0} X_0$.*
(b) *If X is in $\mathcal{K}_T(\Lambda)$, then $\Lambda \otimes_{\Lambda_0} X_0$ is a fat module, and all fat modules in $\mathcal{K}_T(\Lambda)$ are of this form.*
(c) *The full subcategory $\mathcal{C}$ of fat modules in $\mathcal{K}_T(\Lambda)$ is contravariantly finite in $\mathcal{K}_T(\Lambda)$.*

PROOF. (a) Assume that X in $\mathcal{K}_T(\Lambda)$ is fat. Consider the exact sequence $0 \to \Omega_\Lambda(X)/\langle \Omega_\Lambda(X)_0\rangle \to \Lambda \otimes_{\Lambda_0} X_0 \to X \to 0$. We saw in the proof of Proposition 5.5 (b) (iii) that it gives rise to the following exact sequence

$$0 \to \mathcal{E}(\Omega_\Lambda(X)/\langle \Omega_\Lambda(X)_0\rangle)[1] \to \mathcal{E}(X) \to \mathcal{E}(X)_0 \to 0.$$

By assumption $\mathcal{E}(X) = \mathcal{E}(X)_0$, and hence $\mathcal{E}(\Omega_\Lambda(X)/\langle \Omega_\Lambda(X)_0\rangle) = (0)$. Since $\Omega_\Lambda(X)/\langle \Omega_\Lambda(X)_0\rangle[-1]$ is in $\mathcal{K}_T(\Lambda)$, it follows that $\Omega_\Lambda(X)/\langle \Omega_\Lambda(X)_0\rangle = (0)$ and therefore $X \simeq \Lambda \otimes_{\Lambda_0} X_0$.

Assume that X is in $\mathcal{K}_T(\Lambda)$ and that $X \simeq \Lambda \otimes_{\Lambda_0} X_0$. Since X is in $\mathcal{K}_T(\Lambda)$, Proposition 1.9 (c) implies that $\mathcal{E}(X) = \mathcal{E}(\Lambda \otimes_{\Lambda_0} X_0) = \mathcal{E}(X)_0 = \operatorname{Hom}_\Lambda(X, T)$. Hence X is a fat module.

(b) The statement in (b) follows directly from (a) and Proposition 1.9 (c).

(c) Let M be in $\mathcal{K}_T(\Lambda)$. We claim that $\Lambda \otimes_{\Lambda_0} M_0 \to M$ is a left $\mathcal{C}$-approximation of M. To show this we first prove that $\operatorname{Hom}_{\operatorname{gr}\Lambda}(\Lambda \otimes_{\Lambda_0} X_0, Y) \simeq \operatorname{Hom}_{\Lambda_0}(X_0, Y_0)$ for X_0 in $\operatorname{mod}\Lambda_0$ and Y in $\operatorname{gr}\Lambda$ generated in degree 0.

Define $\varphi\colon \operatorname{Hom}_{\operatorname{gr}\Lambda}(\Lambda \otimes_{\Lambda_0} X_0, Y) \to \operatorname{Hom}_{\Lambda_0}(X_0, Y_0)$ and $\psi\colon \operatorname{Hom}_{\Lambda_0}(X_0, Y_0) \to \operatorname{Hom}_{\operatorname{gr}\Lambda}(\Lambda \otimes_{\Lambda_0} X_0, Y)$ by $\varphi(f) = f_0$ and $\psi(g)$ is the composition of the maps $\Lambda \otimes_{\Lambda_0} X_0 \xrightarrow{\Lambda \otimes_{\Lambda_0} g} \Lambda \otimes_{\Lambda_0} Y_0 \xrightarrow{\text{mult}} Y$. Then it is easy to see that $\varphi\psi = \mathrm{id}$ and $\psi\varphi = \mathrm{id}$, and the above isomorphism is proved.

Let $X = \Lambda \otimes_{\Lambda_0} X_0$ be a fat module. Then we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\operatorname{gr}\Lambda}(\Lambda \otimes_{\Lambda_0} X_0, \Lambda \otimes_{\Lambda_0} M_0) \arrow[r] \arrow[d, dash, "\wr"] & \operatorname{Hom}_{\operatorname{gr}\Lambda}(\Lambda \otimes_{\Lambda_0} X_0, M) \arrow[d, dash, "\wr"] \\
\operatorname{Hom}_{\Lambda_0}(X_0, M_0) \arrow[r, equal] & \operatorname{Hom}_{\Lambda_0}(X_0, M_0)
\end{tikzcd}$$

and therefore the map $\Lambda \otimes_{\Lambda_0} M_0 \to M$ is a left $\mathcal{C}$-approximation of M. □

It is clear that the fat modules introduced in Section 3 also are fat modules in the above sense. There we saw that the notion a linear fat resolution was important. In the setting of this section we define a linear fat resolution as follows.

DEFINITION 5.9. A module M in $\operatorname{gr}\Lambda$ is said to have a *linear fat resolution* if there is an exact sequence in $\operatorname{gr}\Lambda$ of the form

$$\cdots \to X^2[2] \to X^1[1] \to X^0 \to M \to 0,$$

where X^i is a fat module for all i.

Since $\Lambda \otimes_{\Lambda_0} M_0$ is fat for any M in $\mathcal{K}_T(\Lambda)$, it is an immediate consequence of the definition of $\mathcal{K}_T(\Lambda)$ that any M in $\mathcal{K}_T(\Lambda)$ has a linear fat resolution.

Next we observe that $\mathcal{E}(M_0)$ for M in $\mathcal{K}_T(\Lambda)$ is a fat module over $\Gamma = E_T(\Lambda)$. Recall that $\mathcal{E}' = \oplus_{i\geq 0} \operatorname{Ext}^i_{E_T(\Lambda)}(\ , T)\colon \operatorname{Gr} E_T(\Lambda) \to \operatorname{Gr}\Lambda$.

LEMMA 5.10. *If M is in $\mathcal{K}_T(\Lambda)$, then $\mathcal{E}(M_0)$ is in ${}^{\perp}_{\Gamma}T$. In particular, $\mathcal{E}'\mathcal{E}(M_0) \simeq M_0$ as abelian groups.*

PROOF. Since M is in $\mathcal{K}_T(\Lambda)$, then M_0 is in $\mathcal{X}_T$ and we have an exact sequence of minimal left $\operatorname{add} T$-approximations

$$0 \to M_0 \to T_0 \xrightarrow{d_1} T_1 \xrightarrow{d_2} T_2 \xrightarrow{d_3} T_3 \to \cdots .$$

Then $\operatorname{Im} d_i$ is in $\mathcal{K}_T(\Lambda)$ for all $i \geq 1$.

By Proposition 1.12 the sequence $\cdots \to \mathcal{E}(T_2) \to \mathcal{E}(T_1) \to \mathcal{E}(T_0) \to \mathcal{E}(M_0) \to 0$ is exact and a projective resolution of $\mathcal{E}(M_0)$ over Γ. Applying $\operatorname{Hom}_\Gamma(\ ,T)$ to this sequence we obtain the following exact diagram using that $\operatorname{Hom}_\Gamma(\mathcal{E}(X),T) \simeq \operatorname{Hom}_{\Gamma_0}(\operatorname{Hom}_{\Lambda_0}(X,T),T) \simeq X_0$ for any module X in $\mathcal{K}_T(\Lambda)$.

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{E}(M_0),T) \arrow[r] \arrow[d, "\wr"] & (\mathcal{E}(T_0),T) \arrow[r] \arrow[d, "\wr"] & (\mathcal{E}(T_0),T) \arrow[r] \arrow[d, "\wr"] & \cdots \\
0 \arrow[r] & M_0 \arrow[r] & T_0 \arrow[r] & T_1 \arrow[r] & \cdots
\end{tikzcd}$$

Since the lower row is exact, we conclude that $\mathcal{E}'\mathcal{E}(M_0) \simeq M_0$ and that $\mathcal{E}(M_0)$ is in ${}^{\perp}_{\Gamma}T$. □

We have seen that a module M in $\mathcal{K}_T(\Lambda)$ has a linear fat resolution. Next we see how a linear fat resolution of $\mathcal{E}(M)$ is given when M is in $\mathcal{K}_T(\Lambda)$.

COROLLARY 5.11. *Let M be in $\mathcal{K}_T(\Lambda)$. Then*

$$\cdots \to \mathcal{E}(M_2)[2] \to \mathcal{E}(M_1)[1] \to \mathcal{E}(M_0) \to \mathcal{E}(M) \to 0$$

is a linear fat resolution of $\mathcal{E}(M)$.

PROOF. The claim follows directly from Lemma 1.6 in Chapter II and Lemma 5.10. □

6. Duality

In this section we generalize the central results on T-Koszul algebras to generalized T-Koszul algebras. In particular we show that the property of generalized T-Koszul is transferred to the Ext-algebra, and that $\mathcal{E}$ induces a duality between the Koszul modules for the original algebra and its Ext-algebra.

We start with a preliminary result for proving the duality.

LEMMA 6.1. *Assume that Λ is generalized T-Koszul. If M is in $\mathcal{K}_T(\Lambda)$, then*

(a) *$\mathcal{E}'\mathcal{E}(M)_i \simeq M_i$ as Λ_0-bimodules.*

(b) *The identification $\phi_i\colon M_i \to \mathcal{E}'\mathcal{E}(M)_i$ is given in the following way. Let $\eta_i(M)\colon 0 \to \mathcal{E}(M_{\geq i+1})[1] \to \mathcal{E}(M_i) \to \mathcal{E}(M_{\geq i}) \to 0$. Then $\phi_i(m_i) = (\mathcal{E}(a_{m_i})\cdot\eta_i(M))\cdot\eta_{i-1}(M)\cdots\eta_1(M)\cdot\eta_0(M)$, where m_i is in M_i and $a_{m_i}\colon \Lambda \to M_{\geq i}$ is given by $a_{m_i}(\lambda) = \lambda m_i$.*

PROOF. By Corollary 5.11 the module $\mathcal{E}(M)$ has the following linear fat resolution

$$\cdots \to \mathcal{E}(M_2)[2] \to \mathcal{E}(M_1)[1] \to \mathcal{E}(M_0) \to \mathcal{E}(M) \to 0.$$

Now the same proof as for the corresponding result (Lemma 2.4) can be carried out, and this completes the proof. □

The proof of the following result is the same as the one presented in Section 4.

THEOREM 6.2. *Assume that Λ is a generalized T-Koszul algebra.*

(a) *$E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras.*

(b) $\mathcal{E}'(\mathcal{E}(M)) \simeq M$ *as graded left* Λ*-modules for all* M *in* $\mathcal{K}_T(\Lambda)$*. In particular,* $\mathcal{E}(\mathcal{K}_T(\Lambda))$ *is contained in* $\mathcal{G}_T(\Gamma)$*.*

We now prove that the image under $\mathcal{E}$ of a Koszul module is again a Koszul module.

PROPOSITION 6.3. *If* M *is in* $\mathcal{K}_T(\Lambda)$*, then* $\mathcal{E}(M)$ *is in* $\mathcal{K}_T(\Gamma)$*.*

PROOF. Let M be in $\mathcal{K}_T(\Lambda)$. First we show that $\mathcal{E}(M)$ is in $\operatorname{fgsyz}\Gamma$. By Proposition 5.4 we know that $\Omega_\Gamma(\mathcal{E}(M))$ is an extension of $\mathcal{E}(\operatorname{corej}_T(M))$ and $\mathcal{E}(\operatorname{rej}_T(M))[1]$. Since M is in $\mathcal{K}_T(\Lambda)$, we have that $\mathcal{E}(\operatorname{corej}_T(M))$ and $\mathcal{E}(\operatorname{rej}_T(M))$ are generated in degree zero, hence $\Omega_\Gamma(\mathcal{E}(M))$ is generated in degree 0 and 1. By induction we get that $\Omega^i_\Gamma(\mathcal{E}(M))$ is generated in at most the degrees in the set $\{0, 1, \ldots, i\}$. Consequently $\mathcal{E}(M)$ is in $\operatorname{fgsyz}\Gamma$. Moreover, by Theorem 6.2 (b) the module $\mathcal{E}(M)$ is in $\mathcal{G}_T(\Gamma)$. Since M_0 is in $\mathcal{X}_{{}_{\Lambda_0}T}$, the module $\mathcal{E}(M)_0 = \operatorname{Hom}_{\Lambda_0}(M_0, T)$ is in $\mathcal{X}_{{}_{\Gamma_0}T}$.

By Proposition 5.4 (b) we have that

$$\begin{aligned} \langle \Omega_\Gamma(\mathcal{E}(M))_0\rangle &\simeq \mathcal{E}(\operatorname{corej}_T(M)) \\ \Omega_\Gamma(\mathcal{E}(M))/\langle \Omega_\Gamma(\mathcal{E}(M))_0\rangle[1] &\simeq \mathcal{E}(\operatorname{rej}_T(M)) \\ \Gamma \otimes_{\Gamma_0} \mathcal{E}(M)_0 &\simeq \mathcal{E}(\overline{M}) \end{aligned}$$

which all are in $\mathcal{G}_T(\Gamma)$.

By Proposition 5.5 (b) we have that

$$\begin{aligned} \operatorname{rej}_T \mathcal{E}(M) &\simeq \mathcal{E}(\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0\rangle) \\ \overline{\mathcal{E}(M)} &\simeq \mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) \\ \operatorname{corej}_T \mathcal{E}(M) &\simeq \mathcal{E}(\langle \Omega_\Lambda(M)\rangle) \end{aligned}$$

which all are in $\mathcal{G}_T(\Gamma)$.

Since the six operations only produce modules of the form $\mathcal{E}(X)$ for some X in $\mathcal{K}_T(\Lambda)$ and $\mathcal{E}(X)$ is in $\mathcal{G}_T(\Lambda)$, we infer that $\mathcal{E}(M)$ is in $\mathcal{K}_T(\Gamma)$ for all M in $\mathcal{K}_T(\Lambda)$. $\square$

Now we can prove that the property of being generalized T-Koszul is preserved by the Ext-algebra.

THEOREM 6.4. *If* Λ *is generalized* T*-Koszul, then* $\Gamma = E_T(\Lambda)$ *is generalized* T*-Koszul.*

PROOF. Assume that Λ is T-Koszul. Since Λ and T are in $\mathcal{K}_T(\Lambda)$, it follows immediately from Proposition 6.3 that Γ and T are in $\mathcal{K}_T(\Gamma)$.

Since $\Gamma_0 = \operatorname{Hom}_\Lambda(T, T) = \operatorname{Hom}_{\Lambda_0}(T, T)$, we have that Γ_0 is an artin algebra. By assumption T is in $\mathcal{K}_T(\Lambda)$, so that Γ is generated in degree 0 and 1 as an algebra. Moreover, T is a Wakamatsu cotilting Γ_0-module.

Let $\eta\colon \cdots \to \Lambda \otimes_{\Lambda_0} X^2_2[2] \to \Lambda \otimes_{\Lambda_0} X^1_1[1] \to \Lambda \otimes_{\Lambda_0} \Lambda_0 \to \Lambda_0 \to 0$ be the linear fat resolution of Λ_0. Since T is in $\mathcal{K}_T(\Lambda)$, tensoring η with T gives rise to a linear fat resolution of T. Then we get that $\Gamma_i = \operatorname{Ext}^i_\Lambda(T, T) \simeq \operatorname{Hom}_{\Lambda_0}(X^i_i \otimes_{\Lambda_0} T, T)$ as left Γ_0-modules. Since X^i_i are finitely generated Λ_0-modules due to the fact that Λ_i is a finitely generated Λ_0-module for all $i \geq 0$; that is, $X^i_i \otimes_{\Lambda_0} T$ is a finitely generated Λ-module, and therefore Γ_i is a finitely generated left Γ_0-module for all $i \geq 0$. Hence Γ is a left finite graded ring. This shows that Γ is a T-Koszul algebra, and this completes the proof. $\square$

We are now in the position to give the main result on duality between Koszul modules.

THEOREM 6.5. *Assume that Λ is generalized T-Koszul and let $\Gamma = E_T(\Lambda)$. Then we have the following assertions.*

(a) $\Lambda \simeq E_T(E_T(\Lambda))$ *as graded algebras.*

(b) *The functors*

$$\mathcal{E} = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\ ,T)\colon \operatorname{Gr}\Lambda \to \operatorname{Gr}\Gamma$$

and

$$\mathcal{E}' = \oplus_{i\geq 0} \operatorname{Ext}^i_\Gamma(\ ,T)\colon \operatorname{Gr}\Gamma \to \operatorname{Gr}\Lambda$$

induce inverse dualities between $\mathcal{K}_T(\Lambda)$ and $\mathcal{K}_T(\Gamma)$.

(c) *The subcategory $\mathcal{X}_{\Lambda_0 T} \cap \mathcal{K}_T(\Lambda)$ and the subcategory of fat modules in $\mathcal{K}_T(\Gamma)$ are dual subcategories via the functor $\mathcal{E}$. The subcategory of fat modules in $\mathcal{K}_T(\Lambda)$ and the subcategory $\mathcal{X}_{\Gamma_0 T} \cap \mathcal{K}_T(\Gamma)$ are dual subcategories via the functor $\mathcal{E}$.*

PROOF. (a) This is a restatement of Theorem 6.2 (a).

(b) By Theorem 6.2, Proposition 6.3 and (a) it follows that the compositions $\mathcal{E}'\mathcal{E}$ and $\mathcal{E}\mathcal{E}'$ are isomorphic to the identity functor on objects. To prove that $\mathcal{E}$ and $\mathcal{E}'$ are full and faithful the arguments used in Theorem 2.6 in Chapter II can be applied.

(c) The claims follow from Lemma 5.10, (a) and (b). □

We saw above that the subcategory of fat modules in $\mathcal{K}_T(\Lambda)$ and $\mathcal{X}_{\Gamma_0 T} \cap \mathcal{K}_T(\Gamma)$ are dual subcategories via the functor $\mathcal{E}$. Next we identify this subcategory with yet another subcategory and find further subcategories of $\mathcal{K}_T(\Lambda)$ and $\mathcal{K}_T(\Gamma)$ which correspond to each other under the duality.

PROPOSITION 6.6. *Let Λ be a generalized T-Koszul algebra.*

(a) *The full subcategories*

$$\mathcal{X} = \{M \in \mathcal{K}_T(\Lambda) \mid \Omega^i_\Lambda(M) \text{ generated in degree } 0 \text{ for all } i \geq 0\}$$

of $\mathcal{K}_T(\Lambda)$ and $\mathcal{X}_{\Gamma_0 T} \cap \mathcal{K}_T(\Gamma)$ of $\mathcal{K}_T(\Gamma)$ are dual subcategories via the functors $\mathcal{E}$ and $\mathcal{E}'$. In particular, $\mathcal{X}$ and the subcategory of fat modules in $\mathcal{K}_T(\Lambda)$ are equal.

(b) *The full subcategories*

$$\mathcal{Y} = \{M \in \mathcal{K}_T(\Lambda) \mid \Omega^i_\Lambda(M) \text{ generated in degree } i \text{ for all } i \geq 0\}$$

of $\mathcal{K}_T(\Lambda)$ and $\mathcal{Y}' = \{Y \in \mathcal{K}_T(\Gamma) \mid Y_i \in \operatorname{add} T \text{ for all } i\}$ of $\mathcal{K}_T(\Gamma)$ are dual subcategories via the functors $\mathcal{E}$ and $\mathcal{E}'$.

PROOF. (a) Let M be in $\mathcal{X}$. Then it follows immediately from the definition of $\mathcal{K}_T(\Lambda)$ that $M \simeq \Lambda \otimes_{\Lambda_0} M_0$. By Proposition 5.5 (b) (ii) we have that $\mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) = \operatorname{Hom}_{\Lambda_0}(M_0, T)$. Moreover by Proposition 6.3 the module $\operatorname{Hom}_{\Lambda_0}(M_0, T)$ is in $\mathcal{K}_T(\Gamma)$, and consequently it is in $\mathcal{X}_{\Gamma_0 T} \cap \mathcal{K}_T(\Gamma)$.

Let X_0 be in $\mathcal{X}_{\Gamma_0 T} \cap \mathcal{K}_T(\Gamma)$. Then we have an F^T-exact sequence $0 \to X_0 \xrightarrow{f_0} T^0 \xrightarrow{f_1} T^1 \xrightarrow{f_2} T^2 \xrightarrow{f_3} \cdots$ with T^i in $\operatorname{add}_{\Gamma_0} T$. Since X_0 is in $\mathcal{K}_T(\Gamma)$, Proposition 1.12 gives that the sequence

$$\cdots \to \mathcal{E}'(T^2) \to \mathcal{E}'(T^1) \to \mathcal{E}'(T^0) \to \mathcal{E}'(X_0) \to 0$$

is exact. It is clearly a projective resolution of $\mathcal{E}'(X_0)$ and $\Omega^i_\Lambda(\mathcal{E}'(X_0))$ is generated in degree 0 for all $i \geq 0$. It follows from Theorem 6.5 that full subcategories $\mathcal{X}$ and $\mathcal{X}_{\Gamma_0 T} \cap \mathcal{K}_T(\Gamma)$ are dual subcategories. Also we get that the subcategory of fat modules in $\mathcal{K}_T(\Lambda)$ is equal to $\mathcal{X}$.

(b) Let Y be in $\mathcal{Y}$. Then we have that $\Omega_\Lambda(Y)_0 = (0)$, so that $\Lambda \otimes_{\Lambda_0} P(Y)_0 \simeq \Lambda \otimes_{\Lambda_0} Y_0$ and in particular that Y_0 is in $\mathcal{P}(\Lambda_0)$. We also have that $\Omega_\Lambda(Y)[-1]$ is in $\mathcal{Y}$, and therefore by induction we have that $\Omega^n_\Lambda(Y)_n$ is in $\mathcal{P}(\Lambda_0)$. Given that Y has a linear projective resolution,

$$\begin{aligned}\operatorname{Ext}^n_\Lambda(Y,T) &\simeq \operatorname{Hom}_\Lambda(\Omega^n_\Lambda(Y),T)\\ &\simeq \operatorname{Hom}_{\Lambda_0}(\Omega^n_\Lambda(Y)_n,T),\end{aligned}$$

where $\operatorname{Hom}_{\Lambda_0}(\Omega^n_\Lambda(Y),T)$ is in $\operatorname{add}_{\Gamma_0} T$ by the above for all $n \geq 0$. This shows that $\mathcal{E}(Y)$ is in $\mathcal{Y}'$ for all Y in $\mathcal{Y}$.

Let Y' be in $\mathcal{Y}'$. Then $\operatorname{corej}_T(Y) = (0)$, and by Proposition 5.4 (b) we have that $\langle \Omega_\Lambda(\mathcal{E}'(Y'))_0\rangle \simeq \mathcal{E}'(\operatorname{corej}_T(Y)) = (0)$. Hence $\Omega_\Lambda(\mathcal{E}'(Y))$ is generated in degree 1. Furthermore, $\Omega_\Lambda(\mathcal{E}'(Y)) \simeq \mathcal{E}'(\operatorname{rej}_T(Y))[1]$. Since $\operatorname{rej}^i_T(Y)[-i]$ is in $\mathcal{Y}'$ for all $i \geq 0$, we obtain by induction that $\Omega^n_\Lambda(\mathcal{E}'(Y)) \simeq \mathcal{E}'(\operatorname{rej}^n_T(Y))[n]$ for all $n \geq 0$. This shows that $\mathcal{E}(Y')$ is in $\mathcal{Y}$ for all Y' in $\mathcal{Y}'$. Therefore the full subcategories $\mathcal{Y}$ and $\mathcal{Y}'$ are dual to each other. □

The following result have the same proof as in the special case treated in Section 4 of Chapter II.

THEOREM 6.7. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a generalized T-Koszul algebra.*

(a) *$\Lambda \simeq T_{\Lambda_0}(\Lambda_1)/I$ with I contained in $T_{\Lambda_0}(\Lambda_1)_{\geq 2}$.*

(b) *I is a quadratic ideal in $T_{\Lambda_0}(\Lambda_1)$.*

Let $0 \to X^2 \to \Lambda \otimes_{\Lambda_0} \Lambda_1 \to \Lambda \otimes_{\Lambda_0} \Lambda_0 \to \Lambda_0 \to 0$ be the start of a linear fat resolution of Λ_0.

(c) *$\Lambda \simeq T_{\Lambda_0}(\Lambda_1)/\langle X^2_2\rangle$.*

(d) *Denote by Γ_0 and Γ_1 the endomorphism ring $\operatorname{End}_{\Lambda_0}(T)$ and $\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$, respectively. Then*

$$E_T(\Lambda) = T_{\Gamma_0}(\Gamma_1)/\langle A_2\rangle,$$

where $A_2 = \{\sum \beta_i \otimes \alpha_i \in \Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \mid \sum \beta_i(\operatorname{id}_{\Lambda_1} \otimes \alpha_i)|_{X^2_2 \otimes_{\Lambda_0} T} = 0\}$.

CHAPTER IV

Further results and questions

The category of Koszul modules for Λ Koszul or generalized Koszul with respect to a Wakamatsu cotilting module T is by definition a subcategory of $w\text{-}\mathcal{G}_T(\Lambda)$, where $w\text{-}\mathcal{G}_T(\Lambda)$ is the full subcategory of $\operatorname{fgsyz}\Lambda$ consisting of modules M such that $\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Ext}^i_\Lambda(M,T) = \operatorname{Ext}^{i+1}_\Lambda(M,T)$ for all $i \geq 0$. The latter category is also an interesting object of study, and section 1 is devoted to giving some basic properties. The description of the Koszul modules when Λ is Koszul with respect to T is much nicer when Λ is strongly T-Koszul. In the next two sections we give sufficient conditions for a T-Koszul algebra to be strongly T-Koszul. In Section 4 we show that we recover are the two classical cases of T-Koszul by requiring $\operatorname{rej}_T = 0$ or $\operatorname{corej}_T = 0$ on the Koszul modules. Finally we pose some questions in Section 5, along with some results to put the questions into the proper context.

1. Modules generated in 0

Throughout this section let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, where Λ_0 is an artin algebra. We assume that T is a Wakamatsu cotilting Λ_0-module. Many results in this section remain true even without this assumption on T.

In this section we give elementary properties of the subcategories $w\text{-}\mathcal{G}_T(\Lambda)$ and $\mathcal{G}_T(\Lambda)$. We start with giving a useful description of the modules in $w\text{-}\mathcal{G}_T(\Lambda)$, and for this we need the following definition.

Definition 1.1. A graded projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

for a module M in $\operatorname{fgsyz}\Lambda$ is said to be T-*linear* if

$$\operatorname{Ext}^1_\Lambda((\Omega^i_\Lambda(M))_T, T) \to \operatorname{Ext}^1_\Lambda(\Omega^i_\Lambda(M), T)$$

is surjective for all $i \geq 0$.

Proposition 1.2. *The following are equivalent for a module M in* $\operatorname{fgsyz}\Lambda$.

(a) *M is in $w\text{-}\mathcal{G}_T(\Lambda)$.*

(b) *M has a T-linear resolution*

$$\cdots \to P_n \xrightarrow{g_n} P_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} P_0 \xrightarrow{g_0} M \to 0$$

(c) *The following properties are satisfied for all $i \geq 0$.*

  (i) *The sequence $0 \to \Omega^{i+1}_\Lambda(M) \to g^{-1}_{i-1}(\operatorname{rej}_T(\Omega^i_\Lambda(M))) \to \operatorname{rej}_T(\Omega^i_\Lambda(M)) \to 0$ is F^T-exact.*

  (ii) *$\operatorname{Ext}^1_\Lambda((\Omega^i_\Lambda(M))_T, T) \to \operatorname{Ext}^1_\Lambda(\overline{\Omega^i_\Lambda(M)}, T)/\operatorname{Hom}_\Lambda(\operatorname{rej}_T(\Omega^i_\Lambda(M)), T)$ is surjective.*

We show that the above proposition follows immediately from the following lemma.

LEMMA 1.3. *The following are equivalent for a Λ-module M in* $\operatorname{gr}\Lambda$.

(a) $\operatorname{Ext}^1_\Lambda(T,T)\cdot\operatorname{Hom}_\Lambda(M,T)=\operatorname{Ext}^1_\Lambda(M,T)$.

(b) $\operatorname{Ext}^1_\Lambda(M_T,T)\to\operatorname{Ext}^1_\Lambda(M,T)$ *is surjective, where the map is induced by the homomorphism* $\alpha_M\colon M\to M_T$.

(c) (i) *The sequence* $0\to\Omega_\Lambda(M)\to g_0^{-1}(\operatorname{rej}_T(M))\to\operatorname{rej}_T(M)\to 0$ *is* F^T-*exact.*

(ii) $\operatorname{Ext}^1_\Lambda(M_T,T)\to\operatorname{Ext}^1_\Lambda(\overline{M},T)/\operatorname{Hom}_\Lambda(\operatorname{rej}_T(M),T)$ *is surjective.*

PROOF. (a) implies (b). Assume that $\operatorname{Ext}^1_\Lambda(T,T)\cdot\operatorname{Hom}_\Lambda(M,T)=\operatorname{Ext}^1_\Lambda(M,T)$. Let $\eta\colon 0\to T\to E\to M\to 0$ be in $\operatorname{Ext}^1_\Lambda(M,T)$. Then there exists an extension $\eta'\colon 0\to T\to X\to T^n\to 0$ in $\operatorname{Ext}^1_\Lambda(T^n,T)$ for some n and a homomorphism $f\colon M\to T^n$ such that we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & T \arrow[r] & X \arrow[r] & T^n \arrow[r] & 0
\end{tikzcd}$$

Since $\alpha_M\colon M\to M_T$ is a left $\operatorname{add}T$-approximation, there exists a homomorphism $g\colon M_T\to T^n$, such that $f=g\circ\alpha_M$. This gives rise to the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^1_\Lambda(T^n,T) \arrow[r, "g^*"] \arrow[d, equal] & \operatorname{Ext}^1_\Lambda(M_T,T) \arrow[d, "\alpha_M^*"] \\
\operatorname{Ext}^1_\Lambda(T^n,T) \arrow[r, "f^*"] & \operatorname{Ext}^1_\Lambda(M,T)
\end{tikzcd}$$

Hence, $f^*(\eta')=\eta=\alpha_M^*\circ g^*(\eta)$ and therefore the claim follows.

(b) implies (a). Assume that $\operatorname{Ext}^1_\Lambda(M_T,T)\xrightarrow{\alpha_M^*}\operatorname{Ext}^1_\Lambda(M,T)$ is surjective. Since M_T is in $\operatorname{add}T$, the claim follows immediately.

(b) implies (c). Assume that $\operatorname{Ext}^1_\Lambda(M_T,T)\xrightarrow{\alpha_M^*}\operatorname{Ext}^1_\Lambda(M,T)$ is surjective. Since α_M factors through the natural projection $\pi_M\colon M\to\overline{M}$, the morphism $\operatorname{Ext}^1_\Lambda(\overline{M},T)\to\operatorname{Ext}^1_\Lambda(M,T)$ is surjective. Hence the morphism

$$\operatorname{Ext}^1_\Lambda(\operatorname{rej}_T(M),T)\to\operatorname{Ext}^2_\Lambda(\overline{M},T)\simeq\operatorname{Ext}^1_\Lambda(g_0^{-1}(\operatorname{rej}_T(M)),T)$$

is a monomorphism. Using long exact sequences, the claim in (i) follows.

Since we have the exact sequence

$$\operatorname{Hom}_\Lambda(\operatorname{rej}_T(M),T)\to\operatorname{Ext}^1_\Lambda(\overline{M},T)\xrightarrow{\pi_M^*}\operatorname{Ext}^1_\Lambda(M,T)\to 0$$

and α_M^* is surjective, the claim in (ii) is a direct consequence of this.

(c) implies (b). Assume that the conditions in (i) and (ii) are satisfied. The statement in (i) means that the morphism $\operatorname{Ext}^1_\Lambda(\operatorname{rej}_T(M),T)\xrightarrow{\varphi}\operatorname{Ext}^2_\Lambda(\overline{M},T)\simeq\operatorname{Ext}^1_\Lambda(g_0^{-1}(\operatorname{rej}_T(M)),T)$ is a monomorphism. Using the long exact sequence induced from the exact sequence $0\to\operatorname{rej}_T(M)\to M\to\overline{M}\to 0$ it follows that the map $\pi_M^*\colon\operatorname{Ext}^1_\Lambda(\overline{M},T)\to\operatorname{Ext}^1_\Lambda(M,T)$ is an epimorphism. By the condition (ii) the claim in (b) is immediate. $\square$

We now explain how Proposition 1.2 follows from the above lemma.

Proof of Proposition 1.2. The equivalence of (b) and (c) follows immediately from Lemma 1.3.

We have the following commutative diagram, where all the maps are the natural maps.

$$\begin{tikzcd}
\operatorname{Ext}^1_\Lambda(T,T)\otimes_{\mathbb{Z}}\operatorname{Hom}_\Lambda(\Omega^i_\Lambda(M),T) \arrow[r] \arrow[d] & \operatorname{Ext}^1_\Lambda(\Omega^i_\Lambda(M),T) \arrow[d, "\wr"] \\
\operatorname{Ext}^1_\Lambda(T,T)\otimes_{\mathbb{Z}}\operatorname{Ext}^i_\Lambda(M,T) \arrow[r] & \operatorname{Ext}^{i+1}_\Lambda(M,T)
\end{tikzcd}$$

The equivalence of (a) and (b) follows directly from this commutative diagram and Lemma 1.3. ☐

We have the following immediate consequences.

Proposition 1.4. (a) *$\mathcal{P}(\Lambda)$ is contained in w-$\mathcal{G}_T(\Lambda)$.*

(b) *The module T is in w-$\mathcal{G}_T(\Lambda)$ if and only if $E_T(\Lambda)=\oplus_{i\geq 0}\operatorname{Ext}^i_\Lambda(T,T)$ as a graded algebra is generated in degrees 0 and 1 and T is in* $\operatorname{fgsyz}\Lambda$.

(c) *If a module M is in w-$\mathcal{G}_T(\Lambda)$, then $\Omega_\Lambda(M)$ is in w-$\mathcal{G}_T(\Lambda)$.*

(d) *A pair of modules A and B are in w-$\mathcal{G}_T(\Lambda)$ if and only if $A\oplus B$ is in w-$\mathcal{G}_T(\Lambda)$.*

Note that for a module M in $\operatorname{fgsyz}\Lambda$ it is not the case that M is in w-$\mathcal{G}_T(\Lambda)$ if and only if $\mathcal{E}(M)$ as an $E_T(\Lambda)$-module is generated in degree 0. This is only true if $E_T(\Lambda)$ is generated in degrees 0 and 1 as a graded algebra.

The category w-$\mathcal{G}_T(\Lambda)$ is not necessarily closed under all extensions, but we see next that it is closed under all F^T-extensions.

Proposition 1.5. *Let $\eta\colon 0\to A\to B\to C\to 0$ be an F^T-exact sequence in* $\operatorname{gr}\Lambda$ *with A and C in w-$\mathcal{G}_T(\Lambda)$. Then B is in w-$\mathcal{G}_T(\Lambda)$. In particular, w-$\mathcal{G}_T(\Lambda)$ is closed under F^T-extensions.*

Proof. It is clear that B is in $\operatorname{fgsyz}\Lambda$. Since η is F^T-exact, we have the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_T \arrow[r, "\delta"] & A_T\oplus C_T \arrow[r] & C_T \arrow[r] & 0
\end{tikzcd}$$

where $g\colon B\to A_T\oplus C_T$ is an $\operatorname{add}T$-approximation. Therefore there exists a homomorphism $h\colon B_T\to A_T\oplus C_T$, such that $h\circ\alpha_B=g$. Hence to show that $\alpha_B^*\colon \operatorname{Ext}^1_\Lambda(B_T,T)\to\operatorname{Ext}^1_\Lambda(B,T)$ is an epimorphism, it is enough to show that $g^*\colon \operatorname{Ext}^1_\Lambda(A_T\oplus C_T,T)\to\operatorname{Ext}^1_\Lambda(B,T)$ is an epimorphism. By an extended version of Proposition 1.12 in Chapter II and the above diagram we obtain the following exact commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_\Lambda(C_T,T) \arrow[r] \arrow[d] & \operatorname{Ext}^1_\Lambda(A_T\oplus C_T,T) \arrow[r] \arrow[d, "g^*"] & \operatorname{Ext}^1_\Lambda(A_T,T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ext}^1_\Lambda(C,T) \arrow[r] \arrow[d] & \operatorname{Ext}^1_\Lambda(B,T) \arrow[r] & \operatorname{Ext}^1_\Lambda(A,T) \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 &
\end{tikzcd}$$

By the Five Lemma the morphism g^* is an epimorphism. Since the syzygy sequences induced by η have the same properties as η, the claim follows. $\square$

In Chapter II we introduced some fundamental sequences involving the projective cover and the minimal left $\operatorname{add} T$-approximation for a module M in $\operatorname{gr}_0 \Lambda$. The correspondence between the projective cover and the minimal left $\operatorname{add} T$-approximation we found in Lemma 5.1 and Lemma 5.2 in Chapter III for modules in $\mathcal{K}_T(\Lambda)$ is still true for modules in $\mathcal{G}_T(\Lambda)$. We leave it to the reader to state and prove the analogous results for $\mathcal{G}_T(\Lambda)$.

We give some additional properties of modules in w-$\mathcal{G}_T(\Lambda)$. First we show that for a module M in w-$\mathcal{G}_T(\Lambda)$ the degree zero part M_0 is in ${}^{\perp}T$.

PROPOSITION 1.6. *Let M be in w-$\mathcal{G}_T(\Lambda)$. Then M_0 is in ${}^{\perp}T$.*

PROOF. Let M be in w-$\mathcal{G}_T(\Lambda)$. Then we have that

$$\operatorname{Ext}^i_\Lambda(M,T) = \oplus_{j\in\mathbb{Z}} \operatorname{Ext}^i_{\operatorname{gr}(\Lambda)}(M,T[j]) = \oplus_{j\geq 0} \operatorname{Ext}^i_{\operatorname{gr}(\Lambda)}(M,T[j])$$

Since

$$\big(\oplus_{j\geq 0} \operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T[j])\big) \cdot \big(\oplus_{j\geq 0} \operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T[j])\big) = \oplus_{j\geq 0} \operatorname{Ext}^{i+1}_{\operatorname{gr}\Lambda}(M,T[j])$$

and the product of two elements in degree j_1 and j_2 is in degree $j_1 + j_2$, we have that

$$\operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T) \cdot \operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T) = \operatorname{Ext}^{i+1}_{\operatorname{gr}\Lambda}(M,T).$$

Furthermore, since $\operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T) = \operatorname{Ext}^1_{\Lambda_0}(T,T)$, $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T) = \operatorname{Ext}^i_{\Lambda_0}(M,T)$ and $\operatorname{Ext}^1_{\Lambda_0}(T,T) = (0)$, it follows that $\operatorname{Ext}^i_{\Lambda_0}(M,T) = (0)$ for all $i > 0$. Hence M_0 is in ${}^{\perp}T$. $\square$

The next result shows that the operations defining the subcategory $\mathcal{K}_T(\Lambda)$ are not independent.

LEMMA 1.7. *Assume that M is in $\mathcal{G}_T(\Lambda)$. Then $\Lambda \otimes_{\Lambda_0} M_0$ is in $\mathcal{G}_T(\Lambda)$ if and only if $\langle \Omega_\Lambda(M)_0\rangle$ is in $\mathcal{G}_T(\Lambda)$.*

PROOF. Assume that $\Lambda \otimes_{\Lambda_0} M_0$ is in $\mathcal{G}_T(\Lambda)$. The module $\langle \Omega_\Lambda(M)_0\rangle$ is a syzygy of $\Lambda \otimes_{\Lambda_0} M_0$. Since $\langle \Omega_\Lambda(M)_0\rangle$ is generated in degree 0, it is in $\mathcal{G}_T(\Lambda)$.

Conversely, assume that $\langle \Omega_\Lambda(M)_0\rangle$ is in $\mathcal{G}_T(\Lambda)$. The sequence

$$0 \to \langle \Omega_\Lambda(M)_0\rangle \to P(M) \to \Lambda \otimes_{\Lambda_0} M_0 \to 0$$

is F^T-exact by Proposition 1.12 in Chapter II. If $\langle \Omega_\Lambda(M)_0\rangle$ is in $\mathcal{G}_T(\Lambda)$, then by Proposition 1.12 in Chapter II the sequence

$$0 \to \mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) \to \mathcal{E}(P(M)) \to \mathcal{E}(\langle \Omega_\Lambda(M)_0\rangle) \to 0$$

is exact. Therefore $\mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0) = \operatorname{Hom}_\Lambda(\Lambda \otimes_{\Lambda_0} M_0, T)$, and in particular $\mathcal{E}(\Lambda \otimes_{\Lambda_0} M_0)$ is generated in degree 0. This completes the proof. $\square$

2. T-Koszul algebras in degrees 0 and 1, revisited

In this section we show that a T-Koszul algebra concentrated in degrees 0 and 1, with T being a cotilting Λ_0-module must be strongly T-Koszul.

For $M = M_0 + M_1$ generated in degree 0, the essential point will be to show that M_1 is in ${}^{\perp}_{\Lambda_0}T$ when $\mathcal{E}(M)$ is generated in degree 0. For this we start with the following.

LEMMA 2.1. *Suppose that Λ is a left finite graded algebra generated in degrees 0 and 1 with Λ_0 an artin algebra, and let T be a cotilting Λ_0-module. Suppose that Λ satisfies the $\otimes$-Tor-conditions. If M is in $\mathcal{G}_T(\Lambda)$, then $\operatorname{Ext}^1_{\Lambda_0}(M_1, T) = (0)$.*

PROOF. Let $M \to M_T$ be the minimal $\operatorname{add} T$-approximation of M and let $P \to M$ and $Q \to M_T$ be Λ-projective covers.

Then, after applying the functor $\operatorname{Hom}_\Lambda(\ , T)$ to the obvious diagram, we obtain the following exact commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & {}_\Lambda(M_T, T) \arrow[r] \arrow[d] & {}_\Lambda(\Lambda \otimes Q, T) \arrow[r] \arrow[d] & {}_\Lambda(\Omega_\Lambda(M_T), T) \arrow[r] \arrow[d] & \operatorname{Ext}^1_\Lambda(M_T, T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_\Lambda(M, T) \arrow[r] & {}_\Lambda(\Lambda \otimes P, T) \arrow[r] & {}_\Lambda(\Omega_\Lambda(M), T) \arrow[r] & \operatorname{Ext}^1_\Lambda(M, T) \arrow[r] & 0
\end{tikzcd}$$

Next we note that ${}_\Lambda(\Omega_\Lambda(M_T, T)) = {}_{\Lambda_0}(\Omega_{\Lambda_0}(M_T, T)) \oplus {}_{\Lambda_0}(M_T^1, T)$ and ${}_\Lambda(\Omega_\Lambda(M), T) = {}_{\Lambda_0}(\Omega_{\Lambda_0}(M_0), T) \oplus {}_{\Lambda_0}(M^1, T)$ using the $\otimes$-Tor-conditions. Furthermore we get short exact sequences

$$0 \to {}_\Lambda(M_T, T) \to {}_\Lambda(\Lambda \otimes Q, T) \to {}_{\Lambda_0}(\Omega_{\Lambda_0}(M_T), T) \to 0$$

$$0 \to {}_\Lambda(M, T) \to {}_\Lambda(\Lambda \otimes P, T) \to {}_{\Lambda_0}(\Omega_{\Lambda_0}(M), T) \to 0$$

using that if X is generated in degree 0, then $\operatorname{Hom}_\Lambda(X, T) = \operatorname{Hom}_{\Lambda_0}(X_0, T)$. Hence, we conclude that ${}_\Lambda(M_T^1, T) \simeq \operatorname{Ext}^1_\Lambda(M_T, T)$ and ${}_\Lambda(M^1, T) \simeq \operatorname{Ext}^1_\Lambda(M, T)$.

Next we consider the commutative diagram

$$\begin{tikzcd}
{}_\Lambda(M_T^1, T) \arrow[r, "\sim"] \arrow[d, "b"] & \operatorname{Ext}^1_\Lambda(M_T, T) \arrow[d, "a"] \\
{}_\Lambda(M^1, T) \arrow[r, "\sim"] & \operatorname{Ext}^1_\Lambda(M, T)
\end{tikzcd}$$

Since we are assuming that $\mathcal{E}(M)$ is generated in degree 0, by Lemma 1.3, the map a is an epimorphism. Since M_T is concentrated in degree 0, we have that $M_T^1 = \Lambda_{\geq 1} \otimes M_T$. Hence ${}_\Lambda(M_T^1, T) \simeq_{\Lambda_0} (\Lambda_1 \otimes M_T, T)$. We note that ${}_\Lambda(M^1, T) = {}_{\Lambda_0}(M_1^1, T) \oplus {}_\Lambda(M^1/\langle M_1^1 \rangle, T)$ by Lemma 0.4. Finally, we note that the image of b is in ${}_{\Lambda_0}(M_1^1, T)$ and that a being an epimorphism implies that ${}_\Lambda(M^1/\langle M^1 \rangle, T) = 0$. We have proved that ${}_{\Lambda_0}(\Lambda_1 \otimes M_T, T) \xrightarrow{b} {}_{\Lambda_0}(M_1^1, T)$ is an epimorphism.

Let $L \to M_1$ be a Λ_0-projective cover. Next consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{\Lambda_0}(M_1) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1^1 \arrow[r, "\alpha"] & \Lambda_1 \otimes M_0 \arrow[r] \arrow[d, "\beta"] & M_1 \arrow[r] & 0 \\
 & & \Lambda_1 \otimes M_T & &
\end{tikzcd}$$

Here, of course, β is obtained by tensoring the add(T)-approximation $M_0 \to M_T$. Apply the functor $\mathrm{Hom}_\Lambda(\ ,T)$ to get

$$\begin{tikzcd}
 & & (\Lambda_1\otimes M_T,T) \arrow[r, equal] \arrow[d, "\beta^*"] & (\Lambda_1\otimes M_T,T) \arrow[d, "\alpha^*\beta^*"] & & \\
0 \arrow[r] & (M_1,T) \arrow[r] \arrow[d, equal] & (\Lambda_1\otimes M_0,T) \arrow[r, "\alpha^*"] \arrow[d] & (M_1^1,T) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\Lambda_0}(M_1,T) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (M_1,T) \arrow[r] & (L,T) \arrow[r] & (\Omega_{\Lambda_0}(M_1),T) \arrow[r] & \mathrm{Ext}^1_{\Lambda_0}(M_1,T) \arrow[r] & 0
\end{tikzcd}$$

The 0 at the right side of the middle row comes from the fact that $\mathrm{Ext}^1_{\Lambda_0}(\Lambda_1\otimes M_0,T)=0$ since we are assuming that T is a cotilting module and that Λ_1 satisfies the $\otimes$-condition. Also, α^* denotes (α,T) and β^* denotes (β,T).

We have shown that $\alpha^*\beta^*\colon {}_{\Lambda_0}(\Lambda_1\otimes M_T,T)\to {}_{\Lambda_0}(M_1^1,T)$ is an epimorphism. A diagram chase shows that $\mathrm{Ext}^1_{\Lambda_0}(M_1,T)=(0)$. $\square$

As an immediate consequence, we have the following result.

Proposition 2.2. *Let Λ be a left finite graded algebra generated in degrees 0 and 1, with Λ_0 an artin algebra, and let T be a cotilting Λ_0-module. Assume that Λ satisfies the $\otimes$-*Tor*-conditions. If M is in $\mathcal{G}_T(\Lambda)$, then M_0 and M_1 are in ${}^\perp T$.*

Proof. We have shown that M_0 is in ${}^\perp T$ and that $\mathrm{Ext}^1_{\Lambda_0}(M_1,T)=(0)$. We have a short exact sequence $0\to M_1^1\to\Lambda_1\otimes M_0\to M_1\to 0$. By an extension of Proposition 5.5 (b) (iii) in Chapter III the module M^1 has the property that $\mathcal{E}(M^1)$ is generated in degree 0. By Proposition 1.6 the module M_1^1 is in ${}^\perp T$. Applying the long exact sequence obtained by applying the functor $\mathrm{Hom}_\Lambda(\ ,T)$ yields the result. $\square$

Our main result in this section completes the proof of Theorem 3.1 in Chapter I.

Theorem 2.3. *Suppose that $\Lambda=\Lambda_0+\Lambda_1+0+0+\cdots$ is a T-Koszul algebra with respect to a cotilting Λ_0-module T. Then Λ is strongly T-Koszul.*

Proof. We show that the category $\mathcal{G}_T(\Lambda)$ contained in $\mathcal{K}_T(\Lambda)$. Let M be in $\mathcal{G}_T(\Lambda)$. By Proposition 2.2 the modules M_0 and M_1 are in ${}^\perp T$. It now follows that $\overline{M}=M_0, \mathrm{rej}_T(M)=M_1, \mathrm{corej}_T(M)$ and $M^1=\ker(\Lambda_1\otimes M_0\to M_1)$ are all in ${}^\perp T$. But ${}^\perp T$ is a subcategory of $\mathcal{G}_T(\Lambda)$. Finally, using the $\otimes$-Tor-conditions, we see that $\Lambda\otimes M_0$ and $\Lambda\otimes\Omega_{\Lambda_0}(M_0)$ are also in ${}^\perp T$. Thus, $\mathcal{G}_T(\Lambda)$ is closed under the necessary operations and we have our result. $\square$

3. Ext-algebras for injective cotilting modules

In this section we give other sufficient conditions for a T-Koszul algebra to be strongly T-Koszul when T is an injective Λ_0-module. We also give sufficient conditions for Λ T-Koszul to be equivalent to $E_T(E_T(\Lambda))\simeq\Lambda$ as graded algebras, when Λ_0 is selfinjective.

The following preliminary result is useful.

Lemma 3.1. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ where Λ is generated in degrees 0 and 1, and let $T = D(\Lambda_0)$. Let M be in $\operatorname{fgsyz}(\Lambda)$, and assume that the module M^1 in the exact sequence*

$$\eta\colon 0 \to M^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$$

is generated in at least two degrees, 1 and r, where r is chosen smallest possible with $r > 1$. Then

$$\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Hom}_\Lambda(M,T) \neq \operatorname{Ext}^1_\Lambda(M,T)$$

Proof. We first produce a nonsplit extension in $\operatorname{Ext}^1_\Lambda(M,T)$. We have that

$$\operatorname{Hom}_\Lambda(M^1,T) \simeq \operatorname{Hom}_{\Lambda_0}(M^1_1,T) \oplus \operatorname{Hom}_{\Lambda_0}((M^1/\langle M^1_1\rangle)_r,T) \oplus X$$

for some X. Let $i\colon (M^1/\langle M^1_1\rangle)_r \to T_0$ be the injective envelope, and let $f\colon M^1 \to M^1/\langle M^1_1\rangle \to (M^1/\langle M^1_1\rangle)_r \to T_0$ be the composition of the natural projections and i. Then we have the following pushout diagram.

$$\begin{tikzcd}
\eta\colon 0 \arrow[r] & M^1 \arrow[r] \arrow[d, "f"] & \Lambda \otimes_{\Lambda_0} M_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta'\colon 0 \arrow[r] & T_0 \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The homomorphism f is a degree zero morphism when we view T_0 as a graded module concentrated in degree r. Since η is an exact sequence in $\operatorname{gr}\Lambda$, the pushout η' is also an exact sequence in $\operatorname{gr}\Lambda$. Assume that η' is split exact. Then $\operatorname{Hom}_{\operatorname{gr}\Lambda}(\Lambda\otimes_{\Lambda_0} M_0, T_0) \neq (0)$. But this is a contradiction, since T_0 is concentrated in degree $r > 1$. Hence η' is not split.

Next we show that η' is not in $\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Hom}_\Lambda(M,T)$.

We have that $\operatorname{Ext}^1_\Lambda(T,T) = \oplus_{j\geq 0}\operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T[j])$, and that $\operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T[j]) = (0)$ for $j \neq 1$, since $T = D(\Lambda_0)$ for $j = 0$ and Λ is generated in degree 0 and 1 for $j > 1$. Hence $\operatorname{Ext}^1_\Lambda(T,T) = \operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T[1])$.

Let $\theta\colon 0 \to T \to E' \to T \to 0$ be any extension in $\operatorname{Ext}^1_\Lambda(T,T)$. Then by the above we can view θ as an exact sequence in $\operatorname{gr}\Lambda$ where the right end term T is viewed as a graded module concentrated in degree 0 and the other end term as a graded module concentrated in degree 1.

Since M is generated in degree 0, all Λ-homomorphisms $g\colon M \to T$ can be viewed as a homomorphism of degree 0, when we consider T as a graded Λ-module concentrated in degree 0. Hence the pullback of θ by g, $0 \to T \to E' \to M \to 0$, is a graded extension with M generated in degree 0 and T concentrated in degree 1. It follows from this that our original nonsplit sequence η can not be in $\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Hom}_\Lambda(M,T)$. Therefore

$$\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Hom}_\Lambda(M,T) \neq \operatorname{Ext}^1_\Lambda(M,T).$$

□

The next result gives a sufficient condition for the Koszul modules to be described as the modules having fat resolutions.

Proposition 3.2. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded algebra generated in degrees 0 and 1, with Λ_0 an artin algebra. Let $T = D(\Lambda_0)$, and assume that $(\Lambda_i)_{\Lambda_0}$*

is projective for all $i \geq 0$. *Let* M *be in* $\mathrm{fgsyz}(\Lambda)$. *Then* M *is in* $\mathcal{G}_T(\Lambda)$ *if and only if the complex*

$$\cdots \to \Lambda \otimes_{\Lambda_0} X_2^2[2] \xrightarrow{d_2} \Lambda \otimes_{\Lambda_0} X_1^1[1] \xrightarrow{d_1} \Lambda \otimes_{\Lambda_0} M_0 \xrightarrow{d_0} M \to 0$$

is exact, where $X^i = \operatorname{Ker} d_{i-1}$.

PROOF. The claim follows from the previous result and Lemma 1.10 in Chapter II. □

We now get our main result on the relationship between T-Koszul and strongly T-Koszul algebras.

THEOREM 3.3. *Let* $\Lambda = \oplus_{i\geq 0}\Lambda_i$ *be a left finite graded algebra generated in degrees* 0 *and* 1 *with* Λ_0 *an artin algebra. Let* $T = D(\Lambda_0)$, *and assume that* $(\Lambda_i)_{\Lambda_0}$ *is projective for all* $i \geq 0$. *Furthermore assume that* Λ_0 *is in* $\mathrm{fgsyz}(\Lambda)$. *Then* Λ *is* T*-Koszul if and only if* Λ_0 *is in* $\mathcal{G}_T(\Lambda)$.

Moreover, if Λ *is* T*-Koszul, then* Λ *is strongly* T*-Koszul.*

PROOF. If Λ is T-Koszul, then Λ_0 must be in $\mathcal{K}_T(\Lambda)$ and therefore certainly in $\mathcal{G}_T(\Lambda)$.

Conversely, assume that Λ_0 is in $\mathcal{G}_T(\Lambda)$. By Proposition 3.2 there exists an exact sequence

$$\cdots \to \Lambda \otimes_{\Lambda_0} X_2^2[2] \xrightarrow{d_2} \Lambda \otimes_{\Lambda_0} X_1^1[1] \xrightarrow{d_1} \Lambda \otimes_{\Lambda_0} \Lambda_0 \xrightarrow{d_0} \Lambda_0 \to 0.$$

As we observed in the proof of Proposition 4.3 in Chapter I the algebra Λ satisfies the $\otimes$-Tor-conditions. Since $T = D(\Lambda_0)$, we have that $\mathcal{X}_T = \operatorname{mod}\Lambda_0$ and therefore the above exact sequence is a linear fat resolution of Λ_0. This shows that Λ is a T-Koszul algebra by definition.

Assume now that Λ is T-Koszul. We want to prove that Λ is strongly T-Koszul. Since Λ satisfies the $\otimes$-Tor-conditions, tensoring the above exact sequence with any module M_0 in $\mathcal{X}_T$ induces a linear fat resolution of M_0. By Proposition 3.2 the subcategory $\mathcal{X}_T$ is in $\mathcal{G}_T(\Lambda)$. Moreover it is also immediate that $\Lambda \otimes_{\Lambda_0} \mathcal{X}_T$ is contained in $\mathcal{G}_T(\Lambda)$.

Given M in $\mathcal{G}_T(\Lambda)$ we have that

$$\begin{aligned}
\operatorname{corej}_T(M) &= \Omega_{\Lambda_0}^{-1}(M_0) \\
\overline{M} &= M_0 \\
\operatorname{rej}_T(M) &= M_{\geq 1} \\
\langle \Omega_\Lambda(M)_0 \rangle &= \Lambda \otimes_{\Lambda_0} \Omega_{\Lambda_0}(M_0) \\
\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle[1] &= \operatorname{Ker}(\Lambda \otimes_{\Lambda_0} M_0 \to M)[1]
\end{aligned}$$

By the above remarks we have that $\operatorname{corej}_T(M)$, $\overline{M}$, $\Lambda \otimes_{\Lambda_0} M_0$ and $\langle \Omega_\Lambda(M)_0 \rangle$ are in $\mathcal{G}_T(\Lambda)$ again. The module $\Omega_\Lambda(M)/\langle \Omega_\Lambda(M)_0 \rangle[1]$ is in $\mathcal{G}_T(\Lambda)$ by the previous result. We want to show that $\mathcal{G}_T(\Lambda)$ is closed under all these operations, and therefore we would have $\mathcal{K}_T(\Lambda) = \mathcal{G}_T(\Lambda)$. So it remains to prove that $M_{\geq 1}$ is in $\mathcal{G}_T(\Lambda)$.

We have the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M^1 \arrow[r, equal] \arrow[d] & M^1 \arrow[d] & & \\
0 \arrow[r] & \Lambda_{\geq 1} \otimes_{\Lambda_0} M_0 \arrow[r] \arrow[d] & \Lambda \otimes_{\Lambda_0} M_0 \arrow[r] \arrow[d] & M_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{\geq 1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Hence $0 \to M^1 \to \Lambda_{\geq 1} \otimes_{\Lambda_0} M_0 \to M_{\geq 1} \to 0$ is exact, where M^1 and $\Lambda_{\geq 1} \otimes_{\Lambda_0} M_0$ are in $\mathcal{G}_T(\Lambda)$. This gives rise to the following commutative exact diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M^2 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda_{\geq 1} \otimes_{\Lambda_0} M_1^1 \arrow[r] \arrow[d] & \Lambda \otimes_{\Lambda_0} \Lambda_1 \otimes_{\Lambda_0} M_0 \arrow[r] \arrow[d] & \Lambda \otimes_{\Lambda_0} M_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^1 \arrow[r] \arrow[d] & \Lambda_{\geq 1} \otimes_{\Lambda_0} M_0 \arrow[r] \arrow[d] & M_{\geq 1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where M^2 and X^1 are in $\mathcal{G}_T(\Lambda)$, the middle row is exact since $(\Lambda_j)_{\Lambda_0}$ is a projective Λ_0^{op}-module and the upper row is the exact sequence of the kernels. It now follows by induction that $M_{\geq 1}$ has a fat resolution and therefore it is in $\mathcal{G}_T(\Lambda)$ by Proposition 3.2.

This shows that $\mathcal{G}_T(\Lambda)$ is closed under all the operations $\mathcal{K}_T(\Lambda)$ should be closed under. Moreover Λ and T are in $\mathcal{G}_T(\Lambda)$, hence $\mathcal{K}_T(\Lambda) = \mathcal{G}_T(\Lambda)$ and Λ is strongly T-Koszul. $\square$

For classical Koszul algebras a characterization of a Koszul algebra Λ is that Λ and the Ext-algebra of the Ext-algebra are isomorphic as graded algebras. As a consequence of the next result, we obtain a generalization of this.

PROPOSITION 3.4. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring with Λ_0 selfinjective, where Λ is generated in degrees 0 and 1. Let $T = \Lambda_0$, and let $\Gamma = E_T(\Lambda) = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(T,T)$. Assume that $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras. Then T is in $\mathcal{G}_T(\Lambda)$.*

PROOF. Since $\Lambda_{\geq 1}$ is generated in degree 1, it follows that $\Lambda_{\geq 1} \otimes_{\Lambda_0} T$ is generated in degree 1. By Lemma 1.5 in Chapter II we get that $\Gamma_1 = \operatorname{Ext}^1_\Lambda(T,T) = \operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T)$ and that $E_T(\Gamma)_1 = \operatorname{Ext}^1_\Gamma(T,T) = \operatorname{Hom}_\Gamma(\Gamma_{\geq 1} \otimes_{\Gamma_0} T, T)$.

We claim that $\Gamma_{\geq 1} \otimes_{\Gamma_0} T/\langle \Gamma_1 \otimes_{\Gamma_0} T\rangle = (0)$. To this end consider the exact sequence

$$0 \to \langle \Gamma_1 \otimes_{\Gamma_0} T\rangle \to \Gamma_{\geq 1} \otimes_{\Gamma_0} T \to \Gamma_{\geq 1} \otimes_{\Gamma_0} T/\langle \Gamma_1 \otimes_{\Gamma_0} T\rangle \to 0.$$

Applying $\operatorname{Hom}_\Gamma(\ ,T)$ to this sequence and observing that the middle term by assumption is isomorphic to Λ_1, we get the following exact sequence

$$\eta\colon 0 \to \operatorname{Hom}_\Gamma(\Gamma_{i\geq 1} \otimes_{\Gamma_0} T/\langle \Gamma_1 \otimes_{\Gamma_0} T\rangle, T) \to \Lambda_1 \to \operatorname{Hom}_{\Gamma_0}(\Gamma_1 \otimes_{\Gamma_0} T, T) \to 0,$$

which is a split exact sequence by Lemma 0.4. We have that

$$\begin{aligned}\operatorname{Hom}_{\Gamma_0}(\Gamma_1 \otimes_{\Gamma_0} T, T) &\simeq \operatorname{Hom}_{\Gamma_0}(\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T) \otimes_{\Gamma_0} T, T)\\ &\simeq \operatorname{Hom}_{\Gamma_0}(T, \operatorname{Hom}_{\Gamma_0}(\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T), T))\\ &\simeq \operatorname{Hom}_{\Gamma_0}(T, \Lambda_1 \otimes_{\Lambda_0} T)\end{aligned}$$

since $\operatorname{Hom}_{\Gamma_0}(\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T), T) \simeq \Lambda_1 \otimes_{\Lambda_0} T$.

Let $P_1^* \xrightarrow{f^*} P_0^* \to \Lambda_1 \to 0$ be a projective presentation of Λ_1 as a right Λ_0-module, where $f\colon P_0 \to P_1$ is in $\mathcal{P}(\Lambda_0)$. Then we have the following commutative diagram.

$$\begin{tikzcd}
P_1^* \otimes_{\Lambda_0} T \arrow[r, "f^*\otimes \mathrm{id}"] \arrow[d] & P_0^* \otimes_{\Lambda_0} T \arrow[r] \arrow[d] & \Lambda_1 \otimes_{\Lambda_0} T \arrow[r] \arrow[d, equal] & 0\\
\operatorname{Hom}_{\Lambda_0}(P_1, T) \arrow[r, "\operatorname{Hom}_{\Lambda_0}(f,T)"] & \operatorname{Hom}_{\Lambda_0}(P_0, T) \arrow[r] & \Lambda_1 \otimes_{\Lambda_0} T \arrow[r] & 0
\end{tikzcd}$$

Since ${}_{\Gamma_0}T = \operatorname{Hom}_{\Lambda_0}(\Lambda_0, T)$ is projective, the above commutative diagram gives rise to the following commutative exact diagram.

$$\begin{tikzcd}
{}_{\Gamma_0}((\Lambda_0,T),(P_1,T)) \arrow[r, "((\Lambda_0,T),(f,T))"] \arrow[d] & {}_{\Gamma_0}((\Lambda_0,T),(P_0,T)) \arrow[r] \arrow[d] & {}_{\Gamma_0}((\Lambda_0,T),\Lambda_1 \otimes_{\Lambda_0} T)) \arrow[r] \arrow[d] & 0\\
P_1^* \arrow[r, "f^*"] & P_0^* \arrow[r] & \Lambda_1 \arrow[r] & 0
\end{tikzcd}$$

where the two first vertical maps are isomorphisms. Hence it follows that $\operatorname{Hom}_{\Gamma_0}(T, \Lambda_1 \otimes_{\Lambda_0} T) \simeq \Lambda_1$. Using the sequence η above we infer that $\operatorname{Hom}_\Gamma(\Gamma_{\geq 1} \otimes_{\Gamma_0} T/\langle \Gamma_1 \otimes_{\Gamma_0} T\rangle, T) = (0)$. Since T contains a copy of every indecomposable injective Λ_0-module, we have that $\Gamma_{\geq 1} \otimes_{\Gamma_0} T/\langle \Gamma_1 \otimes_{\Gamma_0} T\rangle = (0)$.

Consider the sequence

$$0 \to \langle \Gamma_1\rangle \to \Gamma_{\geq 1} \to \Gamma_{\geq 1}/\langle \Gamma_1\rangle \to 0,$$

which gives rise to the exact sequence

$$\langle \Gamma_1\rangle \otimes_{\Gamma_0} T \to \Gamma_{\geq 1} \otimes_{\Gamma_0} T \to (\Gamma_{\geq 1}/\langle \Gamma_1\rangle) \otimes_{\Gamma_0} T \to 0.$$

We have that $\langle \Gamma_1\rangle \otimes_{\Gamma_0} T = \langle \Gamma_1 \otimes_{\Gamma_0} T\rangle$, so that $(\Gamma_{\geq 1}/\langle \Gamma_1\rangle) \otimes_{\Gamma_0} T = (0)$. Since ${}_{\Gamma_0}T = \Gamma_0$, it follows that $\Gamma_{\geq 1}/\langle \Gamma_1\rangle = (0)$. Then we have that $\Gamma_1\Gamma_1 = \Gamma_2$ and inductively $\Gamma_1\Gamma_i = \Gamma_{i+1}$ for all $i \geq 1$. Hence Γ is generated in degrees 0 and 1, and T is in $\mathcal{G}_T(\Lambda)$. $\square$

As a consequence we get sufficient conditions for $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras to imply Λ being T-Koszul.

COROLLARY 3.5. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring with Λ_0 a selfinjective artin algebra, $T = \Lambda_0$ and where Λ_i is projective as a right Λ_0-module for all i. Then Λ is T-Koszul if and only if $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras.*

PROOF. If Λ is T-Koszul, then by Theorem 2.6 (a) in Chapter II the algebras Λ and $E_T(E_T(\Lambda))$ are isomorphic as graded algebras.

Conversely, assume that $E_T(E_T(\Lambda)) \simeq \Lambda$ as graded algebras. By Proposition 3.4 the module $T = \Lambda_0$ is in $\mathcal{G}_T(\Lambda)$, which in turn by Theorem 3.3 implies that Λ is T-Koszul. $\square$

4. $\operatorname{rej}_T = 0$ and $\operatorname{corej}_T = 0$

In this section we show for a T-Koszul algebra Λ that the classical Koszul and cotilting case occur as special cases of algebras having the property that $\operatorname{corej}_T = 0$ on $\mathcal{K}_T(\Lambda)$ and $\operatorname{rej}_T = 0$ on $\mathcal{K}_T(\Lambda)$, respectively.

We start by considering the case $\operatorname{corej}_T = 0$. Here we let $\mathcal{P}(\Sigma)$ denote the full subcategory of $\operatorname{mod}\Sigma$ consisting of the finitely generated projective Σ-modules for an artin algebra Σ.

PROPOSITION 4.1. *Let Λ be a generalized T-Koszul algebra, and let $\Gamma = E_T(\Lambda)$.*

(a) *If $\operatorname{corej}_T = 0$ on $\mathcal{K}_T(\Lambda)$ and Λ_0 satisfies the Generalized Nakayama conjecture, then $\operatorname{add} T = \mathcal{P}(\Lambda_0)$ and M_i is in $\mathcal{P}(\Lambda_0)$ for all M in $\mathcal{K}_T(\Lambda)$.*

(b) *Assume that Λ_0 and Γ_0 satisfy the Generalized Nakayama conjecture. Then the following are equivalent.*
   (i) $\operatorname{corej}_T = 0$ *on* $\mathcal{K}_T(\Lambda)$.
   (ii) $\operatorname{corej}_T = 0$ *on* $\mathcal{K}_T(\Gamma)$.
   (iii) *All modules in $\mathcal{K}_T(\Lambda)$ have linear projective resolutions.*
   (iv) *All modules in $\mathcal{K}_T(\Gamma)$ have linear projective resolutions.*

PROOF. (a) Suppose that $\operatorname{corej}_T(M) = (0)$ for all M in $\mathcal{K}_T(\Lambda)$. Then M_i is in $\operatorname{add} T$ for all $i \geq 0$ for a module M in $\mathcal{K}_T(\Lambda)$. In particular we get that Λ_0 is in $\operatorname{add} T$. By our assumption on Λ_0 we must have that $\operatorname{add} T = \mathcal{P}(\Lambda_0)$. Hence M_i is in $\mathcal{P}(\Lambda_0)$ for all M in $\mathcal{K}_T(\Lambda)$.

(b) We first prove that (i) implies (iii). Assume that $\operatorname{corej}_T = 0$ on $\mathcal{K}_T(\Lambda)$. We have in general that all modules M in $\mathcal{K}_T(\Lambda)$ have a linear fat resolution of the form

$$\cdots \to \Lambda \otimes_{\Lambda_0} M_2^2[2] \xrightarrow{d_2} \Lambda \otimes_{\Lambda_0} M_1^1[1] \xrightarrow{d_1} \Lambda \otimes_{\Lambda_0} M^0 \xrightarrow{d_0} M \to 0,$$

where $M^i = \operatorname{Ker} d_{i-1}$. From (a) it follows that this is a projective resolution. Therefore all modules in $\mathcal{K}_T(\Lambda)$ have a linear projective resolution.

Assume that all modules in $\mathcal{K}_T(\Lambda)$ have a linear projective resolution. By Proposition 5.5 (b) (ii) in Chapter III we have that $\operatorname{corej}_T(\mathcal{E}(M)) = \mathcal{E}(\langle \Omega_\Lambda(M)_0 \rangle)$. Since Λ is generalized T-Koszul it follows that $\operatorname{corej}_T = 0$ in $\mathcal{K}_T(\Gamma)$. This shows that (iii) implies (ii).

By symmetry we have also proved the other implications, and this completes the proof of the proposition. $\square$

If follows from the above result that when Λ is generalized T-Koszul, $\operatorname{corej}_T = 0$ on $\mathcal{K}_T(\Lambda)$ and Λ_0 satisfies the Generalized Nakayama conjecture, then Λ_i is in $\mathcal{P}(\Lambda_0)$ for all i. The converse is not true, as we can see from Theorem 3.1 in Chapter I by choosing $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$ and $T = \Lambda_0$, where Λ_0 is a selfinjective artin algebra. In particular, in this case $\mathcal{K}_T(\Lambda) = \operatorname{gr}_0 \Lambda$.

Now we address the case when $\operatorname{rej}_T = 0$ on $\mathcal{K}_T(\Lambda)$.

PROPOSITION 4.2. *Let Λ be a T-Koszul algebra, and let $\Gamma = E_T(\Lambda)$. Then the following are equivalent.*

(i) $\operatorname{rej}_T = 0$ *on* $\mathcal{K}_T(\Lambda)$.
(ii) $\operatorname{rej}_T = 0$ *on* $\mathcal{K}_T(\Gamma)$.
(iii) $\Lambda = \Lambda_0$.
(iv) $\Gamma = \Gamma_0$.

PROOF. Assume that $\operatorname{rej}_T = 0$ on $\mathcal{K}_T(\Lambda)$. We have that Λ is in $\mathcal{K}_T(\Lambda)$, so that $\operatorname{rej}_T(\Lambda) = \oplus_{i\geq 1}\Lambda_i = (0)$ and therefore $\Lambda = \Lambda_0$. This show that (i) implies (iii).

Suppose $\Lambda = \Lambda_0$. Then it is clear that $\Omega_\Lambda(M)$ is generated in degree 0 for all modules M in $\mathcal{K}_T(\Lambda)$. By Proposition 5.5 (b) (iii) in Chapter III we have that $\operatorname{rej}_T(\mathcal{E}(M))[-1] = \mathcal{E}(\Omega_\Lambda(M)/\langle\Omega_\Lambda(M)_0\rangle)$. Since Λ is T-Koszul, it follows that $\operatorname{rej}_T = 0$ on $\mathcal{K}_T(\Gamma)$. This proves that (iii) implies (ii). Using the same arguments as above we leave it to the reader to complete the proof of the proposition. $\square$

We end this section with a result useful for showing that algebras are not T-Koszul.

PROPOSITION 4.3. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a T-Koszul algebra with $\Lambda_1 \neq (0)$. Then $\Lambda_1 \otimes_{\Lambda_0} T \neq (0)$ or equivalently $\operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T) \neq (0)$.*

PROOF. Assume that Λ is T-Koszul. Then T is in $\mathcal{K}_T(\Lambda)$, and therefore $0 \to \Lambda_{i\geq 1}\otimes_{\Lambda_0} T \to \Lambda\otimes_{\Lambda_0} T \to T \to 0$ is the start of a fat resolution of T and $\Lambda_{i\geq 1}\otimes_{\Lambda_0} T$ is again in $\mathcal{K}_T(\Lambda)$. Hence the Λ_0-module $\Lambda_1\otimes_{\Lambda_0} T$ is dualized by T, so that $\Lambda_1\otimes_{\Lambda_0} T = (0)$ if and only if $\operatorname{Hom}_{\Lambda_)}(\Lambda_1\otimes_{\Lambda_0} T, T) = (0)$. Assume that $\Lambda_1\otimes_{\Lambda_0} T = (0)$. Given the start of the fat resolution of T, we get that $\operatorname{Ext}^1_\Lambda(T,T) \simeq \operatorname{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} T, T) = (0)$. Since Λ is T-Koszul, this implies that $\Gamma = E_T(\Lambda) = \operatorname{Hom}_{\Lambda_0}(T,T) = \Gamma_0$. Then it follows from Proposition 4.2 that $\Lambda = \Lambda_0$. This is a contradiction, hence $\Lambda_1 \otimes_{\Lambda_0} T \neq (0)$. $\square$

5. Questions

In this section we discuss some open problems related to our presentation of Koszul algebras with respect to T. We have emphasized the study when the functor $\mathcal{E}\colon \operatorname{gr}_0 \Lambda \to \operatorname{gr}_0 \Gamma$ where Γ is the Ext-algebra $\oplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(T,T)$, restricts to a duality on a subcategory of $\operatorname{gr}_0 \Lambda$ closed under corejects. In Chapter III, in the introduction and Proposition 1.5, we showed that T must be a graded module concentrated in degree 0 and a Wakamatsu cotilting module for Λ_0. Furthermore, we have seen (Theorem 6.7 in Chapter III) that Λ must be a quadratic algebra in a natural sense. What is less clear is when there is some duality for some subcategories.

In the first part of the paper, we assumed that the category $\mathcal{X}_T$ satisfied the $\otimes$-Tor-conditions. In this case we showed there was a duality between the categories of modules with linear fat resolutions. The definitions and results are closely related to the classical Koszul case. One difference was that the category of modules with linear fat resolutions need not be all the graded modules generated in degree 0 such that $\mathcal{E}(M)$ is generated in degree 0; see Example 4.1 in Chapter I . This leads to the following questions.

QUESTION 5.1. Let Λ be a T-Koszul algebra such that $\mathcal{X}_T$ satisfies the $\otimes$-Tor-conditions. Is there a largest subcategory of $\mathrm{gr}_0 \Lambda$ containing the Koszul modules with respect to T such that $\mathcal{E}$ restricts to a duality? Does $\mathcal{E}$ restrict to a duality on the subcategory $\mathcal{G}_T(\Lambda)$?

Next we turn our attention to the more general case of generalized T-Koszul algebras. By Example 4.9 in Chapter I, we see that in this general setting, it may be the case that $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ can be properly smaller than $\mathcal{X}_T$. If $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ were a resolving subcategory of $\mathrm{mod}\,\Lambda_0$, then we can show that $\mathcal{K}_T(\Lambda) = \mathcal{FR}(\mathcal{X}_T \cap \mathcal{K}_T(\Lambda))$. Here, if $\mathcal{Y}$ is a subcategory of $\mathrm{mod}\,\Lambda_0$, then $\mathcal{FR}(\mathcal{Y})$ is the full subcategory of $\mathrm{gr}_0 \Lambda$ of graded modules M such that M_i is in $\mathcal{Y}$ for $i \geq 0$ and M has a linear fat resolution by modules of the form $\Lambda \otimes_{\Lambda_0} Y$ where Y is in $\mathcal{Y}$.

PROPOSITION 5.2. *Let Λ be a generalized Koszul algebra with respect to a Wakamatsu cotilting module T. Suppose that $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is resolving. Then $\mathcal{K}_T(\Lambda) = \mathcal{FR}(\mathcal{X}_T \cap \mathcal{K}_T(\Lambda))$.*

PROOF. First we show that, in general, $\mathcal{K}_T(\Lambda)$ is a subcategory of $\mathcal{FR}(\mathcal{X}_T \cap \mathcal{K}_T(\Lambda))$. Let M be in $\mathcal{K}_T(\Lambda)$. Then M_i is in $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ for $i \geq 0$ by definition. Then we have an exact sequence of graded modules $0 \to X^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0$. By definition, $X^1[-1]$ is contained in $\mathcal{K}_T(\Lambda)$. Thus X^1_n is in $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ for all $n \geq 1$. It now follows by induction that M is in $\mathcal{FR}(\mathcal{X}_T \cap \mathcal{K}_T(\Lambda))$.

Now assume that $\mathcal{Y} = \mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is resolving. We show that $\mathcal{FR}(\mathcal{Y})$ is a subcategory of $\mathcal{K}_T(\Lambda)$ to complete the proof. To do this, we use that $\mathcal{K}_T(\Lambda)$ is the largest subcategory $\mathcal{X}$ of $\mathrm{gr}_0 \Lambda$ whose objects M have the following properties:

1. $\mathcal{E}(M)$ is generated in degree 0.
2. M_0 is in $\mathcal{X}$
3. $M^1[-1] = \ker(\Lambda \otimes_{\Lambda_0} M_0 \to M)[-1]$ is in $\mathcal{X}$.
4. The submodule of $\Omega_\Lambda(M)$ generated in degree 0 is in $\mathcal{X}$.
5. $\mathrm{rej}_T(M)[-1]$ is in $\mathcal{X}$.
6. $\mathrm{corej}_T(M)$ is in $\mathcal{X}$.
7. $\mathrm{Im}(M \to M_T)$ is in $\mathcal{X}$.

We show that $\mathcal{FR}(\mathcal{Y})$ satisfies all the above properties. Let M be in $\mathcal{FR}(\mathcal{Y})$. Similarly as in Proposition 1.11 (a) in Chapter II we can show that $\mathcal{E}(M)$ is generated in degree 0. Directly from the definition of $\mathcal{FR}(\mathcal{Y})$ and since $\mathcal{Y}$ is resolving, it is clear that property 3 holds. We have that Λ_0 is in $\mathcal{FR}(\mathcal{Y})$. Hence by the $\otimes$-Tor-conditions $\mathcal{Y}$ is a subcategory of $\mathcal{FR}(\mathcal{Y})$, and therefore properties 2, 4, 6, and 7 hold. It remains to show that $\mathrm{rej}_T(M)[-1]$ is in $\mathcal{FR}(\mathcal{Y})$. But we have a short exact sequence

$$0 \to M^1 \to \Lambda_{\geq 1} \otimes_{\Lambda_0} M_0 \to \mathrm{rej}_T(M) \to 0,$$

where $\mathrm{rej}_T(M)_i = M_i$ is in $\mathcal{Y}$ for all $i \geq 1$ and the modules M^1 and $\Lambda_{\geq 1} \otimes_{\Lambda_0} M_0$ are in $\mathcal{FR}(\mathcal{Y})$. To see that $\Lambda_{\geq 1} \otimes_{\Lambda_0} M_0$ is in $\mathcal{FR}(\mathcal{Y})$, observe that $\Lambda_{\geq 1} \otimes_{\Lambda_0} M_0 = (M_0)^1$. As in Proposition 1.13 in Chapter II we can prove that $\mathrm{rej}_T(M)$ is in $\mathcal{FR}(\mathcal{Y})$. This shows that $\mathcal{FR}(\mathcal{Y})$ is contained in $\mathcal{K}_T(\Lambda)$, and consequently we have proved that $\mathcal{FR}(\mathcal{Y}) = \mathcal{K}_T(\Lambda)$. ☐

The above result motivates the following question.

QUESTION 5.3. If Λ is a generalized T-Koszul algebra, is $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ resolving? If $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is not resolving, is $\mathcal{K}_T(\Lambda)$ of the form $\mathcal{FR}(\mathcal{Y})$ for a reasonable subcategory $\mathcal{Y}$ of $\mathrm{mod}(\Lambda_0)$, perhaps resolving?

The difference between an algebra being a generalized Koszul algebra with respect to T and a Koszul algebra with respect to T is not well understood. We have shown that if Λ is T-Koszul, then Λ is generalized T-Koszul. Also it follows from this that if Λ is strongly T-Koszul, then Λ is generalized strongly T-Koszul. However, we do not know if being strongly T-Koszul is equivalent to being generalized strongly T-Koszul. We have the following information in this direction.

PROPOSITION 5.4. *If Λ is a generalized strongly T-Koszul for a Wakamatsu cotilting module T, then $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is resolving. In particular, $\mathcal{K}_T(\Lambda) = \mathcal{FR}(\mathcal{X}_T \cap \mathcal{K}_T(\Lambda))$.*

PROOF. Assume that Λ is generalized strongly T-Koszul for a Wakamatsu cotilting module T. We want to show that $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is resolving. By definition Λ is in $\mathcal{K}_T(\Lambda)$, and therefore Λ_0 is in $\mathcal{K}_T(\Lambda)$. We have that Λ_0 is in $\mathcal{X}_T$, so that $\mathcal{P}(\Lambda_0)$ is contained in $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$.

By assumption $\mathcal{K}_T(\Lambda) = \mathcal{G}_T(\Lambda)$. Since the degree zero part of any module in $\mathcal{G}_T(\Lambda)$ is in $\mathcal{X}_T$, any exact sequence in $\mathcal{G}_T(\Lambda)$ is F^T-exact. Then by Proposition 1.12 (b), the category $\mathcal{G}_T(\Lambda)$ is closed under any extensions. The category $\mathcal{X}_T$ is also closed under extensions, so that we infer that $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is closed under extensions.

Finally we need to show that $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is closed under kernels of epimorphisms. Let $\eta\colon 0 \to A \to B \to C \to 0$ be exact with B and C in $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$. Since $\mathcal{X}_T$ is resolving, A is in $\mathcal{X}_T$. Since Λ satisfies the Tor-condition with respect to $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ and C is $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$, the sequence $0 \to \Lambda \otimes_{\Lambda_0} A \to \Lambda \otimes_{\Lambda_0} B \to \Lambda \otimes_{\Lambda_0} C \to 0$ is exact. The modules $\Lambda \otimes_{\Lambda_0} B$ and $\Lambda \otimes_{\Lambda_0} C$ are fat modules. Using the long exact sequence induced by η when applying the functor $\operatorname{Hom}_\Lambda(\ ,T)$, it follows that $\mathcal{E}(\Lambda \otimes_{\Lambda_0} A)$ is concentrated in degree 0. Hence $\Lambda \otimes_{\Lambda_0} A$ is in $\mathcal{G}_T(\Lambda) = \mathcal{K}_T(\Lambda)$. Therefore $(\Lambda \otimes_{\Lambda_0} A)_0 = A$ is in $\mathcal{K}_T(\Lambda)$. This shows that A is in $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$, and consequently that $\mathcal{X}_T \cap \mathcal{K}_T(\Lambda)$ is resolving.

The last claim is a direct consequence of Proposition 5.2. $\square$

In view of the above proposition the following question is natural.

QUESTION 5.5. If Λ is a generalized T-Koszul algebra for a Wakamatsu cotilting module T, find necessary and sufficient conditions so that $\mathcal{K}_T(\Lambda) \cap \mathcal{X}_T = \mathcal{X}_T$, that is, find necessary and sufficient conditions for Λ to be a Koszul algebra with respect to T.

We can also ask similar questions to Question 5.1 in this general case. More precisely,

QUESTION 5.6. Let Λ be a generalized T-Koszul algebra. Is there a largest subcategory of $\operatorname{gr}_0(\Lambda)$ containing the T-Koszul modules such that $\mathcal{E}$ restricts to a duality? Does $\mathcal{E}$ restrict to a duality on the subcategory of $\mathcal{G}_T(\Lambda)$?

We have the following related question.

QUESTION 5.7. Suppose that Λ is a generalized T-Koszul algebra with respect to a Wakamatsu cotilting Λ_0-module T. Is there a finite set of criteria to determine if a module M in $\operatorname{gr}_0(\Lambda)$ is in $\mathcal{K}_T(\Lambda)$?

Proposition 3.4 motivates the following question.

QUESTION 5.8. Suppose that Λ is a graded algebra generated in degrees 0 and 1, with Λ_0 an artin algebra and Λ_1 a finitely generated as a left Λ_0-module. Let T be a Wakamatsu cotilting module. Suppose that $\mathcal{E}'\mathcal{E}(\Lambda)$ is isomorphic to Λ as graded algebras. Under what additional hypothesis, if any, can we conclude that Λ is a (generalized) Koszul algebra with respect to T?

Our next question can be asked of either T-Koszul or generalized T-Koszul algebras.

QUESTION 5.9. Suppose Λ is either a T-Koszul or generalized T-Koszul algebra for a Wakamatsu cotilting Λ_0-module T. Is there a right Λ_0-module T', for example $D(T)$, such that Λ^{op} is either a T'-Koszul or generalized T'-Koszul algebra?

We have seen many examples of strongly T-Koszul algebras. The next question is related to this class.

QUESTION 5.10. Let T be a Wakamatsu cotilting Λ_0-module. Under what additional conditions, if any, does Λ being strongly T-Koszul imply that Γ is strongly T-Koszul?

Another question relates to tensor products.

QUESTION 5.11. Suppose that A and B are (generalized) T-Koszul k-algebras, for Wakamatsu cotilting modules U and V over A_0 and B_0 respectively, where k is a field. Is $A \otimes_k B$ a (generalized) T-Koszul algebra for $U \otimes_k V$?

We see that generalized T-Koszul algebras display similar features to Koszul algebras. To further highlight this similarity we include a generalization of another result from the Koszul case.

PROPOSITION 5.12. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degrees 0 and 1, where Λ_0 is an artin algebra. Let T be a finitely generated Λ_0-module with T in $\operatorname{fgsyz}\Lambda$. If M is in $\mathcal{G}_T(\Lambda)$, then $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T[j]) = (0)$ for $j \neq i$ and $i > 0$.*

PROOF. By definition $\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Ext}^i_\Lambda(M,T) = \operatorname{Ext}^{i+1}_\Lambda(M,T)$ for all $i \geq 0$. Since M is in $\operatorname{fgsyz}\Lambda$ and generated in degree 0, we have that $\operatorname{Ext}^i_\Lambda(M,T) = \oplus_{j\geq 0}\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T[j])$. Similarly $\operatorname{Ext}^1_\Lambda(T,T) = \oplus_{j\geq 0}\operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T[j])$. We have that $\operatorname{Ext}^1_{\operatorname{gr}\Lambda}(T,T[j]) = (0)$ for $j \neq 1$. Introducing this decomposition into the above product it follows by induction that $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T[j]) = (0)$ for $j \neq i$ and $i > 0$. □

This naturally raises the question.

QUESTION 5.13. When are the modules M in $\mathcal{G}_T(\Lambda)$ characterized by the property that $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T[j]) = (0)$ for $j \neq i$ and $i > 0$?

One situation where this can be answered to the affirmative is given in the next result.

PROPOSITION 5.14. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a left finite graded ring generated in degree 0 and 1 with Λ_0 an artin algebra. Assume that $(\Lambda_i)_{\Lambda_0}$ is projective, and let $T = D(\Lambda^{\mathrm{op}})$. Suppose M is in $\operatorname{gr}_0(\Lambda)$ such that $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(M,T[j]) = (0)$ for $j \neq i$ and $i > 0$. Then M is in $\mathcal{G}_T(\Lambda)$.*

PROOF. Let M be in $\mathrm{gr}_0(\Lambda)$ such that $\mathrm{Ext}^i_{\mathrm{gr}\,\Lambda}(M, T[j]) = (0)$ for $j \neq i$ and $i > 0$. We have the standard exact sequence

$$0 \to M^1 \to \Lambda \otimes_{\Lambda_0} M_0 \to M \to 0.$$

Applying the functor $\mathrm{Hom}_{\mathrm{gr}\,\Lambda}(\ , T[j])$ to this sequence induces the exact sequence

$$0 \to {}_{\mathrm{gr}\,\Lambda}(M, T[j]) \to {}_{\mathrm{gr}\,\Lambda}(\Lambda \otimes_{\Lambda_0} M_0, T[j]) \to {}_{\mathrm{gr}\,\Lambda}(M^1, T[j]) \to \mathrm{Ext}^1_{\mathrm{gr}\,\Lambda}(M, T[j]) \to 0,$$

since $\mathrm{Ext}^1_{\mathrm{gr}\,\Lambda}(\Lambda \otimes_{\Lambda_0} M_0, T[j]) = (0)$ for all $j \geq 0$. The two first are zero for $j \neq 0$ and both are isomorphic $\mathrm{Hom}_{\Lambda_0}(M_0, T)$ for $j = 0$. Hence we get that $\mathrm{Hom}_{\mathrm{gr}\,\Lambda}(M^1, T[j]) \simeq \mathrm{Ext}^1_{\mathrm{gr}\,\Lambda}(M, T[j])$ for all $j \geq 0$. It follows from the assumptions that M^1 is generated in degree 1. Using that $\mathrm{Ext}^2_{\mathrm{gr}\,\Lambda}(M, T[j]) = (0)$ for all $j \neq 2$ and so on, we have that M has a linear fat resolution. Hence M is in $\mathcal{G}_T(\Lambda)$. □

List of Notation

Bibliography

[1] M. Auslander, *Artin algebras I*, Comm. Algebra 1 (1974) 177–268.

[2] M. Auslander, R.-O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Colloque en l'honneur de Pierre Samuel (Orsay 1987), Mém. Soc. Math. France, (N.S.) No. 38 (1989) 5-37.

[3] M. Auslander, I. Reiten, *Applications of homologically finite subcategories*, Adv. in Math., 86, No. 1 (1991) 111–152.

[4] M. Auslander, I. Reiten, S. O. Smalø, Representation theory of artin algebras, Cambridge studies in advanced mathematics, 36, Cambridge University Press.

[5] M. Auslander, S. O. Smalø, *Preprojective modules over artin algebras*, J. Algebra 66, (1980) No. 1, 61–122.

[6] M. Auslander, Ø. Solberg, *Representation theory and relative homology* I*: Relative homology and homologically finite subcategories*, Comm. Algebra, 21 (9) (1993) 2995–3031.

[7] M. Auslander, Ø. Solberg, *Relative homology and representation theory* III*: Cotilting modules and Wedderburn correspondence*, Comm. Algebra, 21 (9) (1993) 3081–3097.

[8] H. Cartan, S. Eilenberg, Homological algebra, Oxford University Press, Princeton University Press.

[9] A. Belinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., Vol. 9, No. 2 (1996) 473–527.

[10] E. L. Green, R. Martinéz-Villa, *Koszul and Yoneda algebras*, Representation theory of algebras (Cocoyoc, 1994), 247–297, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.

[11] E. L. Green, R. Martinéz-Villa, *Koszul and Yoneda algebras* II, Algebras and modules, II (Geiranger, 1996), 227–244, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[12] D. Happel, C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), No. 2, 399–443.

[13] Y. Miyashita, *Generalized tilting modules and applications to module theory*, Math. J. Okayama Univ. 34 (1992), 75–98 (1994).

[14] C. Nâstâsescu, F. van Oystaeyen, Graded ring theory, North-Holland Mathematical Library, 28. North-Holland Publishing Co., Amsterdam-New York, 1982. ix+340 pp. ISBN: 0-444-86489-X

[15] C. M. Ringel, *The preprojective algebra of a quiver*, Algebras and modules, II (Geiranger, 1996), 467–480, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[16] J. J. Rotman, An introduction to homological algebra, Academic Press, 1979.

[17] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra 134 (1990), No. 2, 298–325.

[18] D. Woodcock, *Cohen-Macaulay complexes and Koszul rings*, J. London Math. Soc. (2) 57 (1998), No. 2, 398–410.

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