

RADICAL CUBE ZERO WEAKLY SYMMETRIC ALGEBRAS AND SUPPORT VARIETIES

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ABSTRACT. One of our main results is a classification all the weakly symmetric radical cube zero finite dimensional algebras over an algebraically closed field having a theory of support via the Hochschild cohomology ring satisfying Dade's Lemma. Along the way we give a characterization of when a finite dimensional Koszul algebra have such a theory of support in terms of the graded centre of the Koszul dual.

INTRODUCTION

The support variety of a module, if it exists, is a powerful invariant. For group algebras of finite groups, or cocommutative Hopf algebras it is defined in terms of the maximal ideal spectrum of the group cohomology ring. For a more general finite-dimensional algebra Λ , instead of group cohomology one can take a subalgebra of the Hochschild cohomology provided it has suitable finite generation properties (see Section 1 for details). It was shown in [7] that if these hold and Λ is self-injective, then many of the standard results from the theory of support varieties for finite groups generalize to this situation.

The question is now to understand whether or not these finite generation properties hold for given classes of algebras. Since Hochschild cohomology is difficult to calculate explicitly, one would rather not do this and have other ways to detect finite generation.

In this paper we present a method, for Koszul algebras, which gives a criterion in terms of the Koszul dual, to show that the finite generation condition holds. We denote this condition by **(Fg)**. We apply this method to a general weakly symmetric algebra with radical cube zero, and also to quantum exterior algebras.

In [1], D. Benson characterized the rate of growth of resolutions for weakly symmetric algebras with radical cube zero. We use his results, and we show that almost all such algebras which are of tame representation type satisfy the finite generation hypothesis. There are only two exceptional cases, where a deformation parameter q appears and whenever q is not a root of unity. It is clear from [1] that the algebras in Benson's list which have wild type, cannot satisfy the finite generation condition. Using [5] it follows easily that all weakly symmetric (in fact all selfinjective) algebras of finite representation type over an algebraically closed field satisfy the finite generation condition. The precise answer is the following (see Theorem 1.5 for the definition of the quivers involved).

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Theorem. *Let Λ be a finite dimensional symmetric algebra over an algebraically closed field with radical cube zero and radical square non-zero. Then Λ satisfies **(Fg)** if and only if Λ is of finite representation type, Λ is of type $\widetilde{\mathbb{D}}_n$ for $n \geq 4$, $\widetilde{\mathbb{Z}}_n$ for $n > 0$, $\widetilde{D}\mathbb{Z}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, or Λ is of type $\widetilde{\mathbb{Z}}_0$ or $\widetilde{\mathbb{A}}_n$ when q is a root of unity.*

In addition we show that a quantum exterior algebra satisfies **(Fg)** if and only if all deformation parameters are roots of unity. This is generalized in [2]. Furthermore, this result, together with [7] also generalizes almost all of Theorem 2.2 in [6].

As observed in [3], even “nice” selfinjective algebras does not satisfy **(Fg)**, in spite of sharing many of the same representation structural properties with algebras having **(Fg)**. Hence, whether or not a finite dimensional algebra has the property **(Fg)** is a more subtle question than one first would expect. However, the list of classes of algebras satisfying **(Fg)** is quite extensive: (i) any block of a group ring of a finite group [8, 10, 23], (ii) any block of a finite dimensional cocommutative Hopf algebra [9], (iii) in the commutative setting for a complete intersection [Gu], (iv) any exterior algebra, (v) all finite dimensional selfinjective algebras of finite representation type over an algebraically closed field [5], (vi) quantum complete intersections [2], (vii) all Gorenstein Nakayama algebras (announced by H. Nagase), (viii) any finite dimensional pointed Hopf algebra, having abelian group of group-like elements, under some mild restrictions on the group order [17].

A weakly symmetric algebra is in particular a finite dimensional selfinjective algebra. In [19] it is shown that a finite dimensional Koszul algebra Λ over a field k with degree zero part isomorphic to k , is selfinjective with finite complexity if and only if the Koszul dual $E(\Lambda)$ is an Artin-Schelter regular Koszul algebra. This was extended in [15] to finite dimensional Koszul algebras Λ over a field k with $\Lambda_0 \simeq k^n$ for some positive integer n . Then it is natural to say that a (non-connected) Koszul k -algebra $R = \bigoplus_{i \geq 0} R_i$ is *Artin-Schelter regular algebra of dimension d* , if

- (i) $\dim_k R_i < \infty$ for all $i \geq 0$,
- (ii) $R_0 \simeq k^n$ for some positive integer n ,
- (iii) $\text{gldim } R = d$,
- (iv) the Gelfand-Kirillov dimension of R is finite,
- (v) for all simple graded R -modules S we have

$$\text{Ext}_R^i(S, R) \simeq \begin{cases} (0), & i \neq d, \\ S', & i = d \text{ and some simple graded } R\text{-module } S'. \end{cases}$$

As in [18] it follows that classifying all selfinjective Koszul algebra of finite complexity d and Loewy length $m + 1$ (up to isomorphism) is the same as classifying Artin-Schelter regular Koszul algebras with Gelfand-Kirillov dimension d and global dimension m (up to isomorphism). Moreover, if R is such an Artin-Schelter regular Koszul algebra, then $\text{Ext}_{R^{\text{op}}}^d(\text{Ext}_R^d(S, R), R) \simeq S$ and consequently $\text{Ext}_R^d(S, -)$ is “permutation” of the simple graded R - and R^{op} -modules. The selfinjective algebra Λ being weakly symmetric, corresponds to the property that $R = E(\Lambda)$, the Koszul dual of Λ , satisfies $\text{Ext}_R^d(S, R) \simeq S^{\text{op}}$ for all simple graded R -modules S . Hence, the weakly symmetric algebras with radical cube zero found in [1] of finite complexity classify all the Artin-Schelter regular Koszul algebras of dimension 2, where $\text{Ext}_R^2(-, R)$ is the “identity permutation”. So, in this context, we classify the Artin-Schelter regular Koszul algebras of dimension 2 with $\text{Ext}_R^2(-, R)$ being the

identity permutation and a finitely generated module over their (ordinary) centre. A similar classification is also carried out for Artin-Schelter regular Koszul algebras of dimension 1 (see Proposition 1.4). These algebras correspond to radical square zero algebras, so after having stated this result, we exclude this class of algebras from the discussion of weakly symmetric radical cube zero algebras.

An early version of this paper was called *Finite generation of the Hochschild cohomology ring of some Koszul algebras*. All the results that existed in that early version are included in full in the current paper.

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1. BACKGROUND AND PRELIMINARY RESULTS

Throughout let Λ be a finite dimensional algebra over an algebraically closed field k with Jacobson radical \mathfrak{r} . Cohomological support varieties for finite dimensional modules over Λ using the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ were introduced in [20] and further studied in [7]. It follows that Λ has a good theory of cohomological support varieties via $\mathrm{HH}^*(\Lambda)$ of Λ , if $\mathrm{HH}^*(\Lambda)$ is Noetherian and $\mathrm{Ext}_\Lambda^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated $\mathrm{HH}^*(\Lambda)$ -module. Denote this condition by **(Fg)**. The aim of this paper is to characterize when a weakly symmetric algebra Λ with $\mathfrak{r}^3 = (0)$ satisfies **(Fg)**.

For the algebras Λ we consider, it is well-known that $\Lambda \simeq kQ/I$ for some finite quiver Q and some ideal I in kQ , up to Morita equivalence. Furthermore there is a homomorphism of graded rings

$$\varphi_M: \mathrm{HH}^*(\Lambda) \rightarrow \mathrm{Ext}_\Lambda^*(M, M) = \bigoplus_{i \geq 0} \mathrm{Ext}_\Lambda^i(M, M)$$

for all Λ -modules M , with $\mathrm{Im} \varphi_M \subseteq Z_{\mathrm{gr}}(\mathrm{Ext}_\Lambda^*(M, M))$ (see [20, 24]). Here $Z_{\mathrm{gr}}(\mathrm{Ext}_\Lambda^*(M, M))$ denotes the graded centre of $\mathrm{Ext}_\Lambda^*(M, M)$. The graded centre of a graded ring R is generated by

$$\{z \in R \mid rz = (-1)^{|r||z|} zr, z \text{ homogeneous}, \forall n \geq 0, \forall r \in R_n\},$$

where $|x|$ denotes the degree of a homogeneous element x in R .

Weakly symmetric algebras are selfinjective algebras where all indecomposable projective modules P have the property that $P/\mathfrak{r}P \simeq \mathrm{Soc}(P)$. All selfinjective algebras Λ of finite representation type are shown to be periodic algebras [5], meaning that $\Omega_{\Lambda \otimes_k \Lambda^{\mathrm{op}}}^n(\Lambda) \simeq \Lambda$ for some $n \geq 1$. It is easy to see that all periodic algebras Λ satisfy **(Fg)**. Furthermore, for selfinjective algebras Λ with $\mathfrak{r}^3 = (0)$ and $\mathfrak{r}^2 \neq (0)$ we have the following result.

Theorem 1.1 ([13, 15]). *Let Λ be a selfinjective algebra with $\mathfrak{r}^3 = (0)$ and $\mathfrak{r}^2 \neq (0)$. Then Λ is Koszul if and only if Λ is of infinite representation type.*

Hence in our study of weakly symmetric algebras Λ with $\mathfrak{r}^3 = (0)$, we can concentrate on infinite representation type and consequently Koszul algebras. For Koszul algebras Λ the homomorphism of graded rings from $\mathrm{HH}^*(\Lambda)$ to the Ext-algebra of the simple modules has an even nicer property than for general algebras, as we discuss next.

For a quotient of a path algebra $\Lambda = kQ/I$, given by a quiver Q with relations I over a field k , it was independently observed by Buchweitz and Green-Snashall-Solberg, that when Λ is a Koszul algebra, the image of the natural map from the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ to the Koszul dual $E(\Lambda)$ is equal to the graded

centre $Z_{\text{gr}}(E(\Lambda))$ of $E(\Lambda)$. Here $E(\Lambda) = \bigoplus_{i \geq 0} \text{Ext}_{\Lambda}^i(\Lambda_0, \Lambda_0)$, where Λ_0 is the degree 0 part of Λ . This isomorphism was obtained by Buchweitz as a part of a more general isomorphism between the Hochschild cohomology ring of Λ and the graded Hochschild cohomology ring of the Koszul dual. This isomorphism has since been generalised by Keller [14]. The statement from [4] reads as follows.

Theorem 1.2 ([4]). *Let $\Lambda = kQ/I$ be a Koszul algebra. Then the image of the natural map $\varphi_{\Lambda_0} : \text{HH}^*(\Lambda) \rightarrow E(\Lambda)$ is the graded centre $Z_{\text{gr}}(E(\Lambda))$.*

This enables us to characterize when **(Fg)** holds for a finite dimensional Koszul algebra Λ over an algebraically closed field. Let $\Lambda = kQ/I$ be a path algebra over an algebraically closed field k . In [7] the following conditions are crucial with respect to having a good theory of cohomological support varieties over Λ :

- Fg1**: there is a commutative Noetherian graded subalgebra H of $\text{HH}^*(\Lambda)$ with $H^0 = \text{HH}^0(\Lambda)$,
- Fg2**: $\text{Ext}_{\Lambda}^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated H -module, where \mathfrak{r} denotes the Jacobson radical of Λ .

In [22, Proposition 5.7] it is shown that these conditions are satisfied if and only if the condition **(Fg)** holds for Λ . Suppose that **(Fg)** is satisfied for Λ . Since $Z_{\text{gr}}(E(\Lambda))$ is a $\text{HH}^*(\Lambda)$ -submodule of $E(\Lambda)$, the Hochschild cohomology ring $\text{HH}^*(\Lambda)$ is Noetherian and $E(\Lambda)$ a finitely generated $\text{HH}^*(\Lambda)$ -module, we infer that $Z_{\text{gr}}(E(\Lambda))$ is a finitely generated $\text{HH}^*(\Lambda)$ -module as well. Hence $Z_{\text{gr}}(E(\Lambda))$ is a Noetherian algebra and $E(\Lambda)$ is clearly finitely generated as a $Z_{\text{gr}}(E(\Lambda))$ -module. If Λ is a finite dimensional Koszul algebra, the converse is also true. Suppose that Λ is a finite dimensional Koszul algebra with $Z_{\text{gr}}(E(\Lambda))$ a Noetherian algebra and $E(\Lambda)$ a finitely generated module over $Z_{\text{gr}}(E(\Lambda))$. Then $Z_{\text{gr}}(E(\Lambda))$ contains a commutative Noetherian graded subalgebra \tilde{H} , such that $Z_{\text{gr}}(E(\Lambda))$ is a finitely generated module over \tilde{H} . Let H be an inverse image of \tilde{H} in $\text{HH}^*(\Lambda)$. Then H is (can be chosen to be) commutative Noetherian graded subalgebra of $\text{HH}^*(\Lambda)$. Therefore the conditions **(Fg1)** and **(Fg2)** are satisfied, and consequently **(Fg)** holds true for Λ by [22, Proposition 5.7]. Hence we have the following.

Theorem 1.3. *Let $\Lambda = kQ/I$ be a finite dimensional algebra over an algebraically closed field k , and let $E(\Lambda) = \text{Ext}_{\Lambda}^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$.*

- (a) *If Λ satisfies **(Fg)**, then $Z_{\text{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\text{gr}}(E(\Lambda))$.*
- (b) *When Λ is Koszul, then the converse implication also holds, that is, if $Z_{\text{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\text{gr}}(E(\Lambda))$, then Λ satisfies **(Fg)**.*

Our main focus in this paper is radical cube zero algebras. However, let us illustrate the above result on radical square zero algebras. Let $\Lambda = kQ/J^2$ be a finite dimensional radical square zero algebra over a field k . Hence J is the ideal generated by the arrows in Q . Since all monomial algebras are Koszul [12], Λ is a Koszul algebra and $E(\Lambda) = kQ^{\text{op}}$. If Q does not have any oriented cycles, the global dimension of Λ is finite and there are no interesting theory of support varieties via the Hochschild cohomology ring. So, assume that Q has at least one oriented cycle. Consequently kQ^{op} is an infinite dimensional k -algebra. In addition we have that

$$Z_{\text{gr}}(E(\Lambda)) = \begin{cases} k, & \text{if } Q \text{ is not an oriented cycle } (\tilde{\mathbb{A}}_n) \\ k[T], & \text{otherwise,} \end{cases}$$

The following result is an immediate consequence of the above.

Proposition 1.4. *Let $\Lambda = kQ/J^2$ for some quiver Q with at least one oriented cycle and a field k . Then Λ satisfies **(Fg)** if and only if Λ is a radical square zero Nakayama algebra.*

Let Λ be a weakly symmetric algebra with $\mathfrak{r}^3 = (0)$ and $\mathfrak{r}^2 \neq (0)$. Denote by $\{S_1, \dots, S_n\}$ all the non-isomorphic simple Λ -modules, and let E_Λ be the $n \times n$ -matrix given by $(\dim_k \text{Ext}_\Lambda^1(S_i, S_j))_{i,j}$. These algebras are classified in [1], and among other things the following is proved there.

Theorem 1.5 ([1]). *Let Λ be a finite dimensional indecomposable basic weakly symmetric algebra over an algebraically closed field k with $\mathfrak{r}^3 = (0)$ and $\mathfrak{r}^2 \neq (0)$. Then the matrix E_Λ is a symmetric matrix, and the eigenvalue λ of E_Λ with largest absolute value is positive.*

- (a) *If $\lambda > 2$, then the dimensions of the modules in a minimal projective resolution of any finitely generated Λ -module has exponential growth.*
- (b) *If $\lambda = 2$, then the dimensions of the modules in a minimal projective resolution of any finitely generated Λ -module are either bounded or grow linearly. The matrix E_Λ is the adjacency matrix of a Euclidean diagram $\tilde{\mathbb{A}}_n$ for $n \geq 1$, $\tilde{\mathbb{D}}_n$ for $n \geq 4$, $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$, or*

$$\tilde{\mathbb{Z}}_n: \bigcirc 0 \text{ --- } 1 \text{ --- } \dots \text{ --- } n-1 \text{ --- } n \bigcirc$$

for $n \geq 0$ or

$$\tilde{\mathbb{D}}_n: \begin{array}{c} 0 \\ \diagdown \quad \diagup \\ 2 \text{ --- } 3 \text{ --- } \dots \text{ --- } n-1 \text{ --- } n \bigcirc \\ \diagup \quad \diagdown \\ 1 \end{array}$$

for $n \geq 2$.

- (c) *If $\lambda < 2$, then the dimensions of the modules in a minimal projective resolution of any finitely generated Λ -module is bounded.*

The trichotomy in Theorem 1.5 corresponds to the division in wild, tame and finite representation type as pointed out in [1]. By [7, Theorem 2.5] the complexity of any finitely generated module over an algebra satisfying **(Fg)** is bounded above by the Krull dimension of the Hochschild cohomology ring, hence finite. It follows from this that a weakly symmetric algebra with radical cube zero only can satisfy **(Fg)** in case (b) and (c) in the above theorem. We remarked above that all the algebras in (c) satisfy **(Fg)**, so we only need to consider the algebras given in (b).

The above result gives the quiver of the algebra Λ , but since Λ is supposed to be weakly symmetric with $\mathfrak{r}^3 = (0)$, it is easy to write down the possible relations. In these relations one can introduce scalars from the field. Most of the times the results are independent of these scalars, except in the $\tilde{\mathbb{Z}}_0$ case and the $\tilde{\mathbb{A}}_n$ case, where it suffices to introduce one scalar q in one commutativity relation. The next sections are devoted to discussing these cases.

We end this section with a remark which is a good guide in doing actual computations. Let Λ be a finite dimensional selfinjective Koszul algebra over an algebraically closed field k satisfying **(Fg)**. In finding non-nilpotent generators for $Z_{\text{gr}}(E(\Lambda))$ as an algebra, then [7, Proposition 4.4] and Theorem 1.2 says essentially

that degree of a non-nilpotent generator for $Z_{\text{gr}}(E(\Lambda))$ is a multiple of the Ω -periodicity of some Ω -periodic module. In actual computations this usually gives a good indication where to look for non-nilpotent generator.

2. THE $\tilde{\mathbb{A}}_n$ -CASE

In this section we characterize when a weakly symmetric algebra over a field k with radical cube zero of type $\tilde{\mathbb{A}}_n$ satisfies **(Fg)**. For a computation of the Hochschild cohomology ring of a more general class of algebras containing the algebras we consider in this section consult [21].

Let Q be the quiver given by

$$0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{\bar{a}_0} \end{array} 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{\bar{a}_1} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{\bar{a}_2} \end{array} \cdots \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{\bar{a}_{n-2}} \end{array} n-1 \begin{array}{c} \xrightarrow{a_{n-1}} \\ \xleftarrow{\bar{a}_{n-1}} \end{array} n \begin{array}{c} \xrightarrow{a_n} \\ \xleftarrow{\bar{a}_n} \end{array} 0$$

where the extreme vertices are identified as the notation suggests. Let k be a field, and let I' be the ideal in kQ generated by the elements $\{a_i a_{i+1}\}_{i=0}^n$, $\{\bar{a}_{i-1} \bar{a}_i\}_{i=0}^n$, and $\{a_i \bar{a}_i + q_i \bar{a}_{i-1} a_{i-1}\}_{i=0}^n$, for some nonzero elements q_i in k . Here we compute the indices modulo $n+1$. By changing basis the algebra $\Lambda = kQ/I'$ can be represented by the same quiver, but an ideal I generated by the elements $\{a_i a_{i+1}\}_{i=0}^n$, $\{\bar{a}_{i-1} \bar{a}_i\}_{i=0}^n$, and $\{a_i \bar{a}_i + \bar{a}_{i-1} a_{i-1}\}_{i=1}^n \cup \{a_0 \bar{a}_0 + q \bar{a}_n a_n\}$ for some nonzero element q in k . That is, $\Lambda \simeq kQ/I$.

Next we apply Theorem 1.3 to characterize when Λ satisfies **(Fg)**.

Proposition 2.1. *Let Q and I in kQ be as above for a field k . Then $\Lambda = kQ/I$ satisfies **(Fg)** if and only if q is a root of unity.*

Proof. The graded centre of $E(\Lambda)$ and $E(\Lambda)^{\text{op}}$ are the same and the criterion for checking **(Fg)** can equivalently be performed using $E(\Lambda)^{\text{op}}$. The algebra $E(\Lambda)^{\text{op}}$ is given by $kQ/\langle \{a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}\}_{i=1}^n, q a_0 \bar{a}_0 - \bar{a}_n a_n \rangle$. Consider the length left-lexicographic ordering of the paths in kQ by letting the vertices be less than any arrow, and order the arrows like $\bar{a}_0 < \cdots < \bar{a}_n < a_0 < \cdots < a_n$. Then $\{a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}\}_{i=1}^n, a_0 \bar{a}_0 - q \bar{a}_n a_n$ is a Gröbner basis for $E(\Lambda)^{\text{op}}$. In addition $E(\Lambda)$ is bigraded via the degrees in a -arrows and \bar{a} -arrows. As a consequence the elements in $Z_{\text{gr}}(E(\Lambda))$ are homogeneous in bigraded degree.

Let

$$x = \sum_{i=0}^n a_i a_{i+1} \cdots a_n a_0 \cdots a_{i-1}$$

and

$$y = \sum_{i=0}^n \bar{a}_{i-1} \bar{a}_{i-2} \cdots \bar{a}_0 \bar{a}_n \cdots \bar{a}_i$$

in $E(\Lambda)$. Any path in kQ viewed as an element in $E(\Lambda)$ can be written uniquely as $\bar{A} A y^r x^s$ for some natural numbers r and s , and some path \bar{A} in the arrows $\{\bar{a}_{i-1}\}$ and some path A in the arrows $\{a_i\}$, each of which has length at most n .

We have that $a_i x = x a_i$, $\bar{a}_{i-1} y = y \bar{a}_{i-1}$, $\bar{a}_{i-1} x = q x \bar{a}_{i-1}$ and $a_i y = q^{-1} y a_i$ for all i . Hence, if q is a d -th root of unity, then $\{x^d, y^d\}$ is in $Z_{\text{gr}}(E(\Lambda))$. It is immediate that $E(\Lambda)$ is a finitely generated module over $Z_{\text{gr}}(E(\Lambda))$ when q is a root of unity.

Suppose that q is not a root of unity. Assume that z is an element in $Z_{\text{gr}}(E(\Lambda))$ of homogeneous bi-degree (r, s) with $r + s \geq 1$. Then $z = z_0 + z_1 + \cdots + z_n$ with z_i in $e_i E(\Lambda) e_i$ of bi-degree (r, s) for $i = 0, 1, \dots, n$. If $z_i \neq 0$, then $z_i a_i = a_i z_{i+1}$

and $\bar{a}_{i-1}z_i = z_{i-1}\bar{a}_{i-1}$ are non-zero, hence both z_{i-1} and z_{i+1} are non-zero. It follows that each z_i is non-zero for all i . By considering some power of z we can assume that each $z_i = \alpha_i e_i y^{r'} x^{s'} e_i$ for some positive integers r' and s' and α_i in $k \setminus \{0\}$. Using the same arguments as above it follows that $q^{r'(n+1)} = 1$. This is a contradiction. Hence, when q is not a root of unity, $Z_{\text{gr}}(E(\Lambda))$ is k . Furthermore **(Fg)** is not satisfied, and we have shown that Λ satisfies **(Fg)** if and only if q is a root of unity. \square

3. THE \tilde{Z}_n -CASE

This section is devoted to proving that the weakly symmetric algebras over a field k with radical cube zero of type \tilde{Z}_n all satisfy **(Fg)** for $n > 0$, again using Theorem 1.3. The case $n = 0$ is discussed in Section 9.

The quiver Q of the algebras of type \tilde{Z}_n are given by

$$b \circlearrowleft 0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n \circlearrowright c$$

$$\xleftarrow{\bar{a}_0} \xleftarrow{\bar{a}_1} \xleftarrow{\bar{a}_2} \cdots \xleftarrow{\bar{a}_{n-2}} \xleftarrow{\bar{a}_{n-1}}$$

where we impose the following relations when $n > 0$

$$\{b^2 + a_0\bar{a}_0, ba_0, \bar{a}_0b, \{a_i a_{i+1}\}_{i=0}^{n-2}, \{\bar{a}_i \bar{a}_{i-1}\}_{i=1}^{n-1}, \\ \{a_i \bar{a}_i + \bar{a}_{i-1} a_{i-1}\}_{i=1}^{n-1}, a_{n-1}c, c\bar{a}_{n-1}, c^2 + q\bar{a}_{n-1}a_{n-1}\}.$$

Let I denote the ideal generated by this set of relations.

One could deform this algebra by introducing non-zero coefficients in all the ‘‘commutativity’’ relations above. However, with a suitable basis change, we can remove all these commutativity coefficients and only have one remaining, for instance the q as chosen above. In addition, if k contains a square root of q , then we can replace q by 1.

Proposition 3.1. *Let $\Lambda = kQ/I$, where Q and I are as above. Then Λ satisfies **(Fg)**.*

Proof. We have that

$$E(\Lambda)^{\text{op}} = kQ/(b^2 - a_0\bar{a}_0, \{\bar{a}_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}\}_{i=1}^{n-1}, qc^2 - \bar{a}_{n-1}a_{n-1}).$$

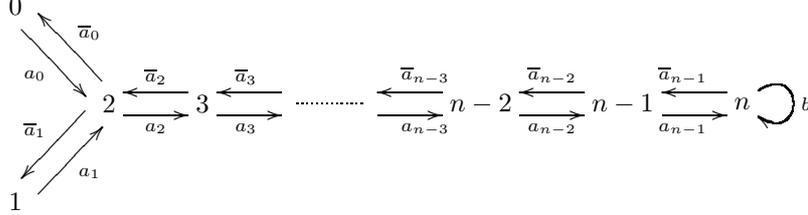
Let $x = \bar{a}_{n-1}a_{n-1} + \sum_{i=0}^{n-1} a_i \bar{a}_i$. Then direct calculations show that x is in $Z_{\text{gr}}(E(\Lambda))$. For d in $\{b, c\}$, denote by $d[s]$ the shortest closed path in Q , starting in vertex s involving d . Then let $y = \sum_{i=0}^n (b[i]c[i] + c[i]b[i])$. Straightforward computations show that y is in $Z_{\text{gr}}(E(\Lambda))$.

We use the above to show that $E(\Lambda)$ is a finitely generated module over $Z_{\text{gr}}(E(\Lambda))$ (or actually the subalgebra generated by x and y). Any path in Q , starting and ending in the same vertex and involving only the arrows a_i and \bar{a}_i , is as an element in $E(\Lambda)$ a power of x times the appropriate idempotent. Then any path p in Q , as an element in $E(\Lambda)$, can be written as $x^m p'$ for some path p' , where $p' = X^l r_X$ for X in $\{b[i]c[i], c[i]b[i]\}_{i=0}^n$ and r_X is a proper subpath of X from the left. Hence, any element in $E(\Lambda)$ is a linear combination of elements of the form $\{x^m (b[i]c[i])^l r_{B_i}, x^m (c[i]b[i])^l r_{C_i}\}_{i=0}^n$. We directly verify that $b[i]^2$ and $c[i]^2$ are in the span of $x^m e_i$ for all i . Then by induction it is easy to see that $(b[i]c[i])^l$ and $(c[i]b[i])^l$ are in $k\langle x, y \rangle (b[i]c[i], c[i]b[i], e_i)$ for all l . It follows from this that $E(\Lambda)$ is a finitely generated $Z_{\text{gr}}(E(\Lambda))$ -module, and hence Λ satisfies **(Fg)**. \square

4. THE \widetilde{DZ}_n -CASE

This section is devoted to showing that a symmetric finite dimensional algebra Λ over a field k of type \widetilde{DZ}_n satisfies **(Fg)**.

Let Q be the quiver given by



Assume that $n > 2$. Let I be the ideal in kQ generated by the elements

$$\begin{aligned} & \{a_0\bar{a}_1, a_0a_2, a_1\bar{a}_0, \bar{a}_2\bar{a}_1, \bar{a}_0a_0 - \bar{a}_1a_1, \bar{a}_2\bar{a}_0, \bar{a}_1a_1 - a_2\bar{a}_2, \\ & \{a_i a_{i+1}\}_{i=1}^{n-2}, \{\bar{a}_i \bar{a}_{i-1}\}_{i=2}^{n-1}, \{\bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i\}_{i=3}^{n-1}, \\ & a_{n-1}b, \bar{a}_{n-1}a_{n-1} + qb^2, b\bar{a}_{n-1} \} \end{aligned}$$

for some q in $k \setminus \{0\}$. When $n = 2$, the ideal I we factor out is slightly different. This ideal is implicitly given in the end of the proof of the next result.

Before proving these algebras satisfies **(Fg)**, we discuss deformations of the algebra Λ . One could deform this algebra by introducing non-zero coefficients in all the commutativity relations above. However with a suitable basis change, we can remove all these commutativity coefficients and only have one remaining, for instance the q as chosen above. Furthermore, if k has the square root of q , then we can replace q by 1.

Proposition 4.1. *Let Q , I and Λ be as above. Then Λ satisfies **(Fg)**.*

Proof. Suppose $n \geq 3$. Again we apply Theorem 1.3. The opposite $E(\Lambda)^{\text{op}}$ of the Koszul dual of Λ is given by kQ modulo the relations generated by

$$\{a_0\bar{a}_0, a_1\bar{a}_1, \bar{a}_0a_0 + \bar{a}_1a_1 + a_2\bar{a}_2, \{\bar{a}_{i-1}a_{i-1} - a_i\bar{a}_i\}_{i=3}^{n-1}, q\bar{a}_{n-1}a_{n-1} - b^2\}.$$

Let $\alpha = \bar{a}_0a_0$, $\beta = \bar{a}_1a_1$ and $\gamma = a_2\bar{a}_2$. Note that $\alpha\beta + \beta\alpha = \gamma^2$ and $\alpha\bar{a}_0 = a_0\alpha = \beta\bar{a}_1 = a_1\beta = 0$. Let $x = \sum_{i=0}^n x_i$ with

$$x_i = \begin{cases} (\bar{a}_{n-1}a_{n-1})^2, & \text{for } i = n, \\ (a_i\bar{a}_i)^2, & \text{for } 2 \leq i \leq n-1, \\ a_0\beta\bar{a}_0, & \text{for } i = 0, \\ a_1\alpha\bar{a}_1, & \text{for } i = 1. \end{cases}$$

Then direct computations show that x is in $Z_{\text{gr}}(E(\Lambda))$.

For η in $\{\alpha, \beta, b\}$ denote by $\eta[s]$ the shortest path in Q starting in vertex s involving η , whenever this makes sense. In particular, $\alpha[s] = \bar{a}_{s-1}\bar{a}_{s-2} \cdots \bar{a}_2\alpha a_2 a_3 \cdots a_{s-1}$ for $3 \leq s \leq n$, and $\alpha[2] = \alpha$. We leave it to the reader to write out the similar formulae for the other cases.

Let $y' = \sum_{i=0}^n y'_i$ with

$$y'_i = \begin{cases} b[0], & i = 0, \\ b[1], & i = 1, \\ b[i]\alpha[i] - \alpha[i]b[i], & 2 \leq i \leq n \text{ and } i \text{ even}, \\ b[i]\beta[i] - \alpha[i]b[i], & 2 \leq i \leq n \text{ and } i \text{ odd}. \end{cases}$$

We want to show that $y = (y')^2$ is in $Z_{\text{gr}}(E(\Lambda))$. In doing so the following equalities are useful to have,

$$(\dagger) \quad \alpha\gamma = -\alpha\beta = \gamma\beta$$

$$(\ddagger) \quad \beta\gamma = -\beta\alpha = \gamma\alpha$$

and

$$b[i]\alpha[i] - \alpha[i]b[i] = -(b[i]\beta[i] - \beta[i]b[i])$$

for all $2 \leq i \leq n$. Note that the last property is equivalent to having $b[i]\gamma[i] = \gamma[i]b[i]$ for $2 \leq i \leq n$. The most cumbersome calculations involve vertex n , and here it is useful to note that $\alpha[n]\gamma[n]$ is equal to $\gamma[n]\alpha[n]$ if n is odd, while it is equal to $\gamma[n]\beta[n]$ for n even. Pointing out that $\alpha[n]^2 = 0$ and $\beta[n]\alpha[n] = 0$ when n is even and odd, respectively, we leave it to the reader to show that y is in $Z_{\text{gr}}(E(\Lambda))$.

Next we want to show that $E(\Lambda)$ is a finitely generated module over the subalgebra Z_2 generated by $\{x, y\}$. Let $E_2 = e_2 E(\Lambda)^{\text{op}} e_2$. Then E_2 is generated as a k -algebra by $\{\alpha, \gamma, b[2]\}$. Let μ be any non-zero monomial in $\{\alpha, \gamma, b[2]\}$. We have that $\alpha\gamma = -\gamma^2 - \gamma\alpha$ and $b[2]\gamma = \gamma b[2]$, so we can suppose that all the γ 's in μ can be moved to the left (since $\gamma^2 = x_2$). Furthermore, we have that $b[2]^2 = q\gamma^{2n-3}$ and $\alpha^2 = 0$, so that we can write $\mu = \gamma^t \mu_1 \mu_2 \cdots \mu_r$ with μ_i in $\{\alpha, b[2]\}$ for all i with $\mu_i \neq \mu_{i+1}$. We have that $x_2 = \gamma^2$ and $y_2 = (b[2]\alpha)^2 + (\alpha b[2])^2 + q\alpha\gamma^{2(n-1)}$. Let \mathcal{A} be the set of monomials in $\{\alpha, \gamma, b[2]\}$ with at most three factors. It is then easy to see that the factor $E_2/Z_2\mathcal{A}$ is zero, and hence E_2 is a finitely generated Z_2 -module. Any oriented cycle in Q not going through the vertex 2, can be written as a power of x times a finite set of cycles. It follows from this that **(Fg)** is satisfied for Λ , when $n \geq 2$.

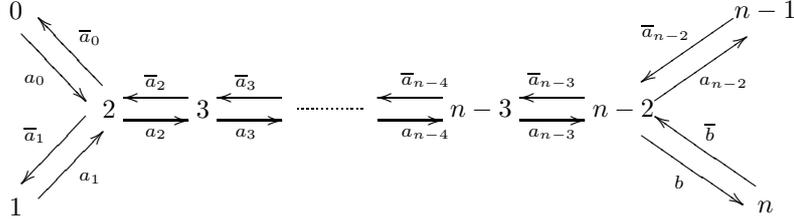
Let $n = 2$. Then $E(\Lambda)^{\text{op}}$ is given by kQ modulo the relations $\{a_0\bar{a}_0, a_1\bar{a}_1, \bar{a}_0a_0 + b^2 + \bar{a}_1a_1\}$. Let $\alpha = \bar{a}_0a_0$, $\beta = \bar{a}_1a_1$ and $\gamma = b^2$. Let $x_0 = \gamma[0]$, $x_1 = \gamma[1]$, and $x_2 = \gamma^2 = -(\alpha\gamma + \gamma\alpha) = -(\beta\gamma + \gamma\beta)$. Let $y_0 = b\alpha b[0]$, $y_1 = b\beta b[1]$ and $y_2 = (b\alpha - \alpha b)^2 = (b\beta - \beta b)^2$. Then it is easy to see that $x = x_0 + x_1 + x_2$ and $y = y_0 + y_1 + y_2$ are in $Z_{\text{gr}}(E(\Lambda))$. Using that $x_2 = -\alpha b^2 - b^2\alpha$ and $\alpha y_2 = \alpha b\alpha b\alpha$, it is immediate that Λ satisfies **(Fg)** also in this case. \square

This example also provide us with additional information on Betti numbers of periodic modules. Considering the Λ -module M with radical layers $\binom{n}{n}$ it is easy to see that M is Ω -periodic with period $2n - 1$, and all the projective modules in an initial periodic minimal projective resolution are indecomposable except projective number $n - 1$, which is a direct sum of two indecomposable projective modules. This gives an example of a Ω -periodic module with non-constant Betti numbers.

5. THE $\tilde{\mathbb{D}}_n$ -CASE

This section is devoted to proving that the weakly symmetric algebras over a field k with radical cube zero of type $\tilde{\mathbb{D}}_n$ all satisfy **(Fg)**.

Let Q be the quiver given by



Assume that $n > 4$. Let I be the ideal in kQ generated by the elements

$$\begin{aligned} & \{a_0\bar{a}_1, a_0a_2, a_1\bar{a}_0, \bar{a}_2\bar{a}_1, \bar{a}_0a_0 - \bar{a}_1a_1, \bar{a}_2\bar{a}_0, \bar{a}_1a_1 - a_2\bar{a}_2, \\ & \{a_i a_{i+1}\}_{i=1}^{n-3}, \{\bar{a}_i \bar{a}_{i-1}\}_{i=2}^{n-2}, \{\bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i\}_{i=3}^{n-3}, \\ & \bar{a}_{n-2}b, \bar{b}a_{n-2}, a_{n-3}b, a_{n-2}\bar{a}_{n-2} - \bar{b}\bar{b}, \bar{b}\bar{b} - \bar{a}_{n-3}a_{n-3}\}. \end{aligned}$$

When $n = 4$ then I is generated by

$$\begin{aligned} & \{a_0\bar{a}_1, a_0a_2, a_0b, a_1\bar{a}_0, a_1a_2, a_1b, \bar{a}_2\bar{a}_0, \bar{a}_2\bar{a}_1, \bar{a}_2b, \\ & \bar{b}\bar{a}_0, \bar{b}\bar{a}_1, \bar{b}a_2, \bar{a}_0a_0 - \bar{a}_1a_1, \bar{a}_1a_1 - \bar{b}\bar{b}, \bar{b}\bar{b} - a_2\bar{a}_2\}. \end{aligned}$$

Similarly as before, deformations via coefficients in the commutativity relations can be removed via an appropriate basis change. Given this we can show the following.

Proposition 5.1. *Let Q , I and Λ be as above. Then Λ satisfies **(Fg)**.*

Proof. We apply again Theorem 1.3. The opposite algebra $E = E(\Lambda)^{\text{op}}$ of the Koszul dual of Λ is given by kQ modulo the relations generated by

$$\begin{aligned} & \{a_0\bar{a}_0, a_1\bar{a}_1, \bar{a}_0a_0 + \bar{a}_1a_1 + a_2\bar{a}_2, \{\bar{a}_{i-1}a_{i-1} - a_i\bar{a}_i\}_{i=3}^{n-3}, \\ & \bar{a}_{n-2}a_{n-2}, \bar{b}\bar{b}, a_{n-2}\bar{a}_{n-2} + \bar{b}\bar{b} + \bar{a}_{n-3}a_{n-3}\} \end{aligned}$$

when $n > 4$. For $n = 4$ the relations are given by

$$\{a_0\bar{a}_0, a_1\bar{a}_1, \bar{a}_2a_2, \bar{b}\bar{b}, \bar{a}_0a_0 + \bar{a}_1a_1 + a_2\bar{a}_2 + \bar{b}\bar{b}\}.$$

Let $\alpha = \bar{a}_0a_0$, $\beta = \bar{a}_1a_1$ and $\gamma = a_2\bar{a}_2$. Furthermore we write $\delta = a_{n-2}\bar{a}_{n-2}$, $\omega = \bar{b}\bar{b}$ and $\eta = \bar{a}_{n-3}a_{n-3}$. Note that $\alpha\beta + \beta\alpha = \gamma^2$ and $\alpha\bar{a}_0 = a_0\alpha = \beta\bar{a}_1 = a_1\beta = 0$ and as well $\delta\bar{a}_{n-2} = a_{n-2}\delta = \omega\bar{b} = b\omega = 0$.

Let $x = \sum_i x_i$, where

$$\begin{aligned} x_0 &= a_0\beta\bar{a}_0, & x_{n-2} &= \eta^2, \\ x_1 &= a_1\alpha\bar{a}_1, & x_{n-1} &= \bar{a}_{n-2}\omega a_{n-2}, \\ x_i &= (a_i\bar{a}_i)^2, \text{ for } 2 \leq i < n-2, & x_n &= \bar{b}\delta\bar{b}. \end{aligned}$$

Then direct computations show that x is in $Z_{\text{gr}}(E(\Lambda))$. This is also true when $n = 4$. Note that in this case we have $\alpha + \beta + \delta + \omega = 0$.

When $n = 4$ we find another element of degree 4 in the centre of E , namely $w = \sum_{i=0}^4 w_i$, where

$$\begin{aligned} w_0 &= a_0\delta\bar{a}_0, & w_2 &= (\alpha + \delta)^2 = \alpha\delta + \delta\alpha, & w_4 &= \bar{b}\beta\bar{b}, \\ w_1 &= a_1\omega\bar{a}_1, & w_3 &= \bar{a}_3\alpha a_3, \end{aligned}$$

We assume now that $n > 4$. Suppose ρ is one of $\alpha, \beta, \delta, \omega$. As before, we write $\rho[s]$ for the shortest closed path starting at s which involves α .

Let $y = \sum_{i=0}^n y_i$, where the y_i are defined as follows. For $2 \leq r \leq n-2$,

$$y_r = \begin{cases} \alpha[r]\delta[r] + \delta[r]\alpha[r], & r, n \text{ even,} \\ \alpha[r]\delta[r] + \omega[r]\beta[r], & r \text{ odd, } n \text{ even,} \\ \alpha[r]\delta[r] - \omega[r]\alpha[r], & r \text{ even, } n \text{ odd,} \\ \beta[r]\omega[r] - \omega[r]\alpha[r], & r, n \text{ odd.} \end{cases}$$

Furthermore,

$$y_0 = \delta[0], \quad y_1 = \omega[1], \quad y_{n-1} = \alpha[n-1], \quad y_n = \beta[n].$$

We want to show that y is in the centre of $E(\Lambda)$. One way to prove this is to first establish the following identities. For $2 \leq r \leq n-2$,

$$(1) \quad \alpha[r](\delta[r] + \omega[r]) = \begin{cases} (\delta[r] + \omega[r])\alpha[r], & n-r-1 \text{ even,} \\ (\delta[r] + \omega[r])\beta[r], & n-r-1 \text{ odd.} \end{cases}$$

Moreover

$$(2) \quad \delta[r](\alpha[r] + \beta[r]) = \begin{cases} (\alpha[r] + \beta[r])\delta[r], & r-1 \text{ even,} \\ (\alpha[r] + \beta[r])\omega[r], & r-1 \text{ odd.} \end{cases}$$

Similar formulae hold by interchanging α and β in (1), and by interchanging δ and ω , in (2). Using these formulae one gets several identities for the y_r . Assume $2 \leq r \leq n-2$. Then

$$y_r = \begin{cases} \beta[r]\omega[r] + \omega[r]\beta[r], & n, r \text{ even,} \\ \delta[r]\alpha[r] + \beta[r]\omega[r], & r \text{ odd, } n \text{ even,} \\ \delta[r]\alpha[r] - \beta[r]\delta[r] = \omega[r]\beta[r] - \alpha[r]\omega[r], & n, r \text{ odd,} \\ \delta[r]\alpha[r] - \alpha[r]\omega[r] = \omega[r]\beta[r] - \beta[r]\delta[r], & r \text{ even, } n \text{ odd.} \end{cases}$$

These show in particular that the anti-homomorphism induced by $a \rightarrow \bar{a}$ and $\bar{a} \rightarrow a$ of E^+ fixes each y_r . This means that one only has to check that y commutes with the arrows a_r , then it automatically commutes with \bar{a}_r . Furthermore, one checks that $\alpha[r]\delta[r]a_r = a_r\beta[r+1]\omega[r+1]$, and similarly $\beta[r]\omega[r]a_r = a_r\alpha[r+1]\delta[r+1]$, for $2 \leq r < n-2$. Using all these details, it is not difficult to check that y commutes with all a_i , and with b .

Next, we want to show that $E(\Lambda)$ is a finitely generated module over the subalgebra generated by $\{x, y\}$. As in the previous section, it suffices to show that the local algebra $E_2 = e_2E(\Lambda)e_2$ is finitely generated as a module over the subalgebra Z generated by $\{x_2, y_2\}$.

Recall $x_2 = \gamma^2$; and we take

$$y_2 = \begin{cases} \alpha\delta[2] + \delta[2]\alpha, & n \text{ even} \\ \delta[2]\alpha - \alpha\omega[2], & n \text{ odd.} \end{cases}$$

Note that this is also correct when $n = 4$, so we can deal with arbitrary $n \geq 4$ at the same time.

The algebra E_2 is generated by $\{\alpha, \gamma, \delta[2]\}$, note that $\delta[2] + \omega[2] = -\gamma^{n-3}$. We have further identities, namely

$$\gamma\alpha = -\gamma^2 - \alpha\gamma, \quad \delta[2]\gamma = \gamma\omega[2].$$

One checks that if n is even, $\delta[2]^2 = 0$, and that for n odd, $\delta[2]\omega[2] = 0$.

Let \mathcal{A} be the set of monomials in $\{\alpha, \delta[2], \gamma\}$ with at most three factors. We want to show that $E_2/Z\mathcal{A}$ is zero, hence that E_2 is finitely generated over Z .

Assume $\mu \in E_2$ is a non-zero monomial in $\alpha, \delta[2]$ and γ . We can move all even powers of γ to the left, note that these lie in Z . Furthermore, any factor of α in μ can be moved to the left, using $\gamma\alpha = -\alpha\gamma + z$ for $z \in Z$, and also using that for n even, $\delta[2]\alpha = -\alpha\delta[2] + y_2$ and for n odd, $\delta[2]\alpha = y_2 + \alpha\omega[2] = y_2 - \alpha\delta[2] - \alpha z$ where $z = \gamma^{n-3} \in Z$. Hence we may assume none except possibly the first factor of μ is equal to α . Next, consider submonomials of length three in $\delta[2], \gamma$ of μ where successive factors are different. If it is of the form $\delta[2]\gamma\delta[2] = \delta[2]\omega[2]\gamma$ then is zero if n is odd, and if n is even, it is equal to $-\delta[2]^2 - \delta[2]\gamma^{n-3} = z\delta[2]\gamma$ with $z \in Z$. Otherwise, it is of the form $\gamma\delta[2]\gamma = \gamma^2\omega[2] = z\delta[2] + z'\gamma^j$ with $z, z' \in Z$ and $j = 0$ or 1 . Using these one shows by induction on the length of μ that μ belongs to $Z\mathcal{A}$. \square

6. THE $\widetilde{\mathbb{E}}_6$ -CASE

This section is devoted to showing that the weakly symmetric algebras over a field k with radical cube zero of type $\widetilde{\mathbb{E}}_6$ satisfy **(Fg)**.

Let Q be the quiver

$$\begin{array}{ccccccc}
 & & & 4 & & & \\
 & & & \downarrow \bar{a}_3 & \uparrow a_3 & & \\
 & & & 3 & & & \\
 & & & \downarrow \bar{a}_2 & \uparrow a_2 & & \\
 & & & 2 & & & \\
 \bar{a}_0 \leftarrow & 1 & \leftarrow \bar{a}_1 & 2 & \leftarrow \bar{a}_4 & 5 & \leftarrow \bar{a}_5 \\
 \xrightarrow{a_0} & & \xrightarrow{a_1} & & \xrightarrow{a_4} & & \xrightarrow{a_5} & 6
 \end{array}$$

with relations

$$\begin{aligned}
 & \{ \{a_i a_{i+1}\}_{i=0}^4, \{ \bar{a}_i \bar{a}_{i-1} \}_{i=1}^5, \{ \bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i \}_{i=1,3,5}, a_1 a_4, \\
 & \bar{a}_2 a_4, \bar{a}_4 a_2, \bar{a}_4 a_1, \bar{a}_1 a_1 - a_2 \bar{a}_2, a_2 \bar{a}_2 - a_4 \bar{a}_4 \}.
 \end{aligned}$$

Let $\Lambda = kQ/I$, where I is the ideal generated by the relations given above for a field k . As before, we could deform the algebra by introducing non-zero scalars in the commutativity relations, but by a suitable basis change all the scalars can be removed. Then we have the following.

Proposition 6.1. *Let Q , I and Λ be as above. Then Λ satisfies **(Fg)**.*

Proof. The opposite $E(\Lambda)^{\text{op}}$ of the Koszul dual of Λ is given by kQ modulo the relations generated by

$$\{ a_0 \bar{a}_0, \bar{a}_3 a_3, \bar{a}_5 a_5, \{ \bar{a}_{i-1} a_{i-1} - a_i \bar{a}_i \}_{i=1,3,5}, \bar{a}_1 a_1 + a_2 \bar{a}_2 + a_4 \bar{a}_4 \}.$$

Let $\alpha = \bar{a}_1 a_1$, $\beta = a_4 \bar{a}_4$ and $\gamma = a_2 \bar{a}_2$. Let $x_0 = \gamma[0]$, $x_1 = (\alpha\gamma + \gamma\alpha)[1]$, and $x_2 = \alpha^2\gamma + \alpha\gamma\alpha + \gamma\alpha^2$. We want to define x_i for $i = 3, 4, 5, 6$ by symmetry. To do so, we need the following details. Using the relations in $E(\Lambda)$ it follows that

$a_1\alpha^2 = \alpha^2\bar{a}_1 = 0$, and therefore $\alpha^3 = \beta^3 = \gamma^3 = 0$. Applying this we infer that

$$\begin{aligned} (\dagger) \quad \alpha^2\gamma + \alpha\gamma\alpha + \gamma\alpha^2 &= -(\alpha\gamma^2 + \gamma\alpha\gamma + \gamma^2\alpha) \\ \gamma^2\alpha &= -\gamma^2\beta \\ \gamma\alpha\gamma &= -\gamma\beta\gamma \\ \alpha\gamma^2 &= -\beta\gamma^2 \end{aligned}$$

and similar formulae. We define now $x_3 = -(\alpha\gamma + \gamma\alpha)[3]$, $x_4 = -\alpha[4]$, $x_5 = -(\alpha\beta + \beta\alpha)[5]$ and $x_6 = -\alpha[6]$. Utilizing symmetry direct computations then show that $x = \sum_{i=0}^6 x_i$ is in $Z_{\text{gr}}(E(\Lambda))$.

Computing $(\dagger) \cdot \gamma - \gamma \cdot (\dagger)$ we obtain

$$\gamma^2\alpha^2 - \alpha\gamma\alpha\gamma = \alpha^2\gamma^2 - \gamma\alpha\gamma\alpha.$$

Furthermore, since $a_0\alpha^2 = 0 = \alpha^2\bar{a}_0$, we have that

$$\gamma^2\alpha[1] + \alpha\gamma^2[1] = -\gamma\alpha\gamma[1] - \alpha\gamma\alpha[1].$$

Using that $a_0\alpha\gamma\alpha[1] = a_0\alpha\gamma^2[1] = 0 = \alpha\gamma\alpha[1]\bar{a}_0 = \gamma^2\alpha[1]\bar{a}_0$ and $y_2 = \beta^2\gamma^2 - \gamma\beta\gamma\beta$, and letting

$$\begin{aligned} y_2 &= \gamma^2\alpha^2 - \alpha\gamma\alpha\gamma, \\ y_1 &= -\gamma\alpha\gamma[1], \\ y_0 &= \gamma^2[0], \end{aligned}$$

we obtain by symmetry an element $y = \sum_{i=0}^6 y_i$ in $Z_{\text{gr}}(E(\Lambda))$.

Next we show that $E(\Lambda)$ is a finitely generated module over the subalgebra generated by $\{x, y\}$. The algebra $e_2E(\Lambda)e_2$ is generated by $\{\alpha, \gamma\}$ as an algebra. Given a monomial μ in α and γ we can use $-\alpha\gamma^2 = x_2 + \gamma\alpha\gamma + \gamma^2\alpha$ to move the occurrence of γ^2 to the left in μ . Hence, except for a short initial part, we can assume that μ is a word in $\{\alpha, \alpha^2, \gamma\}$. If $\alpha\gamma\alpha^2$ occurs somewhere in μ , the equality $\gamma\alpha^2 = -x_2 - \alpha^2\gamma - \alpha\gamma\alpha$ gives $\alpha\gamma\alpha^2 = -\alpha x_2 - \alpha^2\gamma\alpha$ and the occurrence of α^2 is moved further to the left in creating one new monomial and one expression $-x_2\mu'$, where μ' is a monomial of degree three less than μ . Hence, except for a short initial part, we can assume that μ is a word in $\{\alpha, \gamma\}$. The equality $y_2 + \gamma^2\alpha^2 = \alpha\gamma\alpha\gamma$ implies that $\gamma y_2 = \gamma\alpha\gamma\alpha\gamma$ and $y_2\alpha = \alpha\gamma\alpha\gamma\alpha$. By induction we obtain that any monomial in α and γ can be written as a linear combination of products of powers of $\{x_2, y_2\}$, and a finite set of monomials in α and γ . Hence $e_2E(\Lambda)e_2$ is a finitely generated module over the algebra generated by $\{x_2, y_2\}$. As before it follows from this that **(Fg)** holds for Λ . \square

7. THE $\widetilde{\mathbb{E}}_7$ -CASE

This section is devoted to proving that the weakly symmetric algebras over a field k with radical cube zero of type $\widetilde{\mathbb{E}}_7$ satisfy **(Fg)**.

Let Q be the quiver

$$\begin{array}{ccccccc} & & & & 4 & & \\ & & & & \uparrow & & \\ & & & & \bar{a}_3 & & \\ & & & & \downarrow & & \\ & & & & \bar{a}_4 & & \\ & & & & \uparrow & & \\ & & & & \bar{a}_5 & & \\ & & & & \bar{a}_6 & & \\ 0 & \xleftarrow{\bar{a}_0} & 1 & \xleftarrow{\bar{a}_1} & 2 & \xleftarrow{\bar{a}_2} & 3 & \xleftarrow{\bar{a}_3} & 4 & \xleftarrow{\bar{a}_4} & 5 & \xleftarrow{\bar{a}_5} & 6 & \xleftarrow{\bar{a}_6} & 7 \\ & \xrightarrow{a_0} & & \xrightarrow{a_1} & & \xrightarrow{a_2} & & \xrightarrow{a_3} & & \xrightarrow{a_4} & & \xrightarrow{a_5} & & \xrightarrow{a_6} & \end{array}$$

with relations

$$\{\{a_i a_{i+1}\}_{i=0}^5, \{\bar{a}_i \bar{a}_{i-1}\}_{i=1}^6, \{\bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i\}_{i=1,2,5,6}, a_2 a_4, \\ \bar{a}_3 a_4, \bar{a}_4 a_3, \bar{a}_4 \bar{a}_2, \bar{a}_2 a_2 - a_3 \bar{a}_3, a_3 \bar{a}_3 - a_4 \bar{a}_4\}.$$

Let $\Lambda = kQ/I$, where I is the ideal generated by the relations given above for a field k . As for the $\widetilde{\mathbb{E}}_6$ -case, deforming the algebra by introducing non-zero scalars in the commutativity relations does not change the algebra up to isomorphism. Then we have the following.

Proposition 7.1. *Let Q , I and Λ be as above. Then Λ satisfies (Fg).*

Proof. The opposite $E(\Lambda)^{\text{op}}$ of the Koszul dual of Λ is given by kQ modulo the relations generated by

$$\{a_0 \bar{a}_0, \bar{a}_3 a_3, \bar{a}_6 a_6, \{\bar{a}_{i-1} a_{i-1} - a_i \bar{a}_i\}_{i=1,2,5,6}, \bar{a}_2 a_2 + a_3 \bar{a}_3 + a_4 \bar{a}_4\}.$$

Let $\alpha = \bar{a}_2 a_2$, $\beta = a_4 \bar{a}_4$ and $\gamma = a_3 \bar{a}_3$. One easily shows that $\alpha^4 = \beta^4 = \gamma^2 = 0$. Furthermore using that $\alpha + \beta + \gamma = 0$, $\beta\gamma = -\alpha\gamma$ and $\gamma\beta = -\gamma\alpha$, we get

$$\begin{aligned} (\dagger) \quad \beta^3 \gamma + \beta^2 \gamma \beta + \beta \gamma \beta^2 + \gamma \beta^3 &= -[(\beta\gamma)^2 + \gamma\beta^2\gamma + (\gamma\beta)^2] \\ &= -(\beta\gamma + \gamma\beta)^2 = -(\alpha\gamma + \gamma\alpha)^2 \\ &= -[(\alpha\gamma)^2 + \gamma\alpha^2\gamma + (\gamma\alpha)^2] \\ &= \alpha^3\gamma + \alpha^2\gamma\alpha + \alpha\gamma\alpha^2 + \gamma\alpha^3. \end{aligned}$$

Let $x = \sum_{i=0}^7 x_i$ be the element of degree 8 defined as follows

$$\begin{aligned} x_0 &= \gamma[0], & x_7 &= \gamma[7], \\ x_1 &= (\alpha\gamma + \gamma\alpha)[1], & x_6 &= (\beta\gamma + \gamma\beta)[6], \\ x_2 &= (\alpha^2\gamma + \alpha\gamma\alpha + \gamma\alpha^2)[2], & x_5 &= (\beta^2\gamma + \beta\gamma\beta + \gamma\beta^2)[5], \\ x_3 &= \alpha^3\gamma + \alpha^2\gamma\alpha + \alpha\gamma\alpha^2 + \gamma\alpha^3, & x_4 &= -(\alpha\gamma\alpha)[4]. \end{aligned}$$

Using (\dagger) , symmetry in α and β and performing straightforward computations, we infer that x is in $Z_{\text{gr}}(E(\Lambda))$.

Define $y = \sum_{i=0}^7 y_i$ as the following degree 12 element in $E(\Lambda)$ with

$$\begin{aligned} y_0 &= -\gamma\alpha\gamma[0], & y_7 &= -\gamma\beta\gamma[7], \\ y_1 &= \gamma\alpha^2\gamma[1], & y_6 &= \gamma\beta^2\gamma[6], \\ y_2 &= (-\alpha^2\gamma\alpha^2 - \alpha^2\gamma\alpha\gamma + \gamma\alpha^2\gamma\alpha)[2], & y_5 &= (-\beta^2\gamma\beta^2 - \beta^2\gamma\beta\gamma + \gamma\beta^2\gamma\beta)[5], \\ y_3 &= \alpha^2\gamma\alpha^2\gamma + \gamma\alpha^2\gamma\alpha^2, & y_4 &= \alpha^2\gamma\alpha^2[4]. \end{aligned}$$

Using the last equality in (\dagger) and $a_0 \bar{a}_0 = 0$, it follows that that y_0 and y_1 “commute” with a_0 and \bar{a}_0 . Premultiply the last equality in (\dagger) with α^2 gives

$$\alpha^3\gamma\alpha^2 + \alpha^2\gamma\alpha^3 = -[\alpha^2(\alpha\gamma)^2 + \alpha^2(\gamma\alpha)^2 + \alpha^2\gamma\alpha^2\gamma].$$

In computing $a_2 y_3 - y_2 a_2$ we obtain

$$a_2 y_3 - y_2 a_2 = a_2[\alpha^2\gamma\alpha^2\gamma + \alpha^2\gamma\alpha\gamma\alpha + \alpha^2\gamma\alpha^3].$$

Furthermore, substitute for $\alpha^2\gamma\alpha^3$ using the above expression, we can then cancel four terms and are left with

$$a_2 y_3 - y_2 a_2 = a_2[-\alpha^3\gamma\alpha^2 - \alpha^2(\alpha\gamma)^2] = 0,$$

The structure of the proof is as before, first we exhibit two elements x and y in $Z_{\text{gr}}(E(\Lambda))$. Then we show that $E(\Lambda)$ is a finitely generated module over the subalgebra generated by $\{x, y\}$, through analyzing the local algebra E_2 . The algebra E_2 is generated by $\{\alpha, \gamma\}$, and they satisfy $\alpha^3 = 0$ and $\gamma^2 = 0$. Furthermore we have $\beta^6 = 0$. This is equivalent with

$$\begin{aligned} (\dagger) \quad 0 &= (\gamma\alpha^2\gamma\alpha^2 + \alpha^2\gamma\alpha^2\gamma) + (\alpha\gamma\alpha\gamma\alpha^2 + \alpha^2\gamma\alpha\gamma\alpha) \\ &\quad + (\gamma\alpha^2\gamma\alpha\gamma + \gamma\alpha\gamma\alpha^2\gamma) + \alpha\gamma\alpha^2\gamma\alpha + (\gamma\alpha)^3 + (\alpha\gamma)^3. \end{aligned}$$

This will be used often. Furthermore, we have, from expanding $-\beta^3 = (\alpha + \gamma)^3$ that

$$(\ddagger) \quad -\beta^3 + \gamma\beta\gamma = -\beta^3 - \gamma\alpha\gamma = (\gamma\alpha^2 + \alpha\gamma\alpha + \alpha^2\gamma).$$

Denote this element by ρ . Note that ρ commutes with α .

In giving elements in $Z_{\text{gr}}(E(\Lambda))$, the following identity is helpful. Let $\zeta = \rho^2$, then ζ has degree 12, and it lies in the centre of E_2 . To this end, first note that it can be written in different ways.

$$\begin{aligned} (\Delta) \quad \zeta &= \gamma\alpha^2\gamma\alpha^2 + \alpha^2\gamma\alpha^2\gamma + \alpha^2\gamma\alpha\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha\gamma\alpha^2 \\ &= -(\gamma\alpha^2\gamma\alpha\gamma + \gamma\alpha\gamma\alpha^2\gamma + (\gamma\alpha)^3 + (\alpha\gamma)^3) \\ &= (\gamma\alpha^2 + \alpha\gamma\alpha + \alpha^2\gamma)^2. \end{aligned}$$

By the second identity in (Δ) we have $\zeta\gamma = -(\gamma\alpha)^3\gamma = \gamma\zeta$; and since ρ commutes with α , so does ζ . Hence ζ is in the centre of E_2 .

An element of degree 12 in $Z_{\text{gr}}(E(\Lambda))$. Define $x = \sum_{i=0}^8 x_i$, where

$$\begin{aligned} x_0 &= \gamma\alpha^2\gamma[0], & x_3 &= -(\alpha\gamma)^2\alpha[3], & x_6 &= \sum_{i=0}^2 \beta^{2-i}\gamma\beta^i[6], \\ x_1 &= (\alpha\gamma\alpha^2\gamma + \gamma\alpha^2\gamma + (\alpha\gamma)^2\alpha)[1], & x_4 &= \sum_{i=0}^4 \beta^{4-i}\gamma\beta^i[4], & x_7 &= (\beta\gamma + \gamma\beta)[7], \\ x_2 &= \zeta, & x_5 &= \sum_{i=0}^3 \beta^{3-i}\gamma\beta^i[5], & x_8 &= \gamma[8]. \end{aligned}$$

We claim that x is in the centre of $E(\Lambda)$. On the branch of the quiver starting with vertex 0 one uses the first expression for ζ given in (Δ) . On the branch with vertex 3 one uses the second expression for ζ . For the long branch, we need ζ in terms of β and γ . Namely, we have

$$\zeta = (-\beta^3 + \gamma\beta\gamma)^2 = (\beta^5\gamma + \beta^4\gamma\beta + \beta^3\gamma\beta^2 + \cdots + \gamma\beta^5).$$

To see this, write the RHS as

$$\beta^3[\beta^2\gamma + \beta\gamma\beta + \gamma\beta^2] + [\beta^2\gamma + \beta\gamma\beta + \gamma\beta^2]\beta^3.$$

Expanding $\alpha^3 = 0$ gives $\beta^2\gamma + \beta\gamma\beta + \gamma\beta^2 = -\beta^3 - \gamma\beta\gamma$, substitute this and use $\beta^6 = 0$.

An element of degree 20 in $Z_{\text{gr}}(E(\Lambda))$. Next we find an element y of degree 20 in $Z_{\text{gr}}(E(\Lambda))$. Define

$$\omega = \alpha^2\gamma\alpha\gamma + \alpha\gamma\alpha\gamma\alpha + \gamma\alpha\gamma\alpha^2.$$

This element commutes with α . Furthermore, ω^2 commutes with γ , using the following which is easy to check.

$$(\square) \quad \gamma\omega + \omega\gamma = -\zeta.$$

Define $y = \sum_{i=0}^8 y_i$, where

$$\begin{aligned} y_0 &= \gamma\alpha\gamma\alpha^2\gamma\alpha\gamma[0], & y_5 &= \sum_{i=0}^3 \beta^{3-i}\alpha^2\gamma\alpha^2\beta^i[5], \\ y_1 &= (\alpha(\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma) + (\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma)\alpha + (\alpha\gamma)^4\alpha)[1], & y_6 &= \sum_{i=0}^2 \beta^{2-i}\alpha^2\gamma\alpha^2\beta^i[6], \\ y_2 &= \omega^2, & y_7 &= \sum_{i=0}^1 \beta^{1-i}\alpha^2\gamma\alpha^2\beta^i[7], \\ y_3 &= (\alpha^2\gamma\alpha\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha\gamma\alpha^2)[3], & y_8 &= \alpha^2\gamma\alpha^2[8]. \\ y_4 &= \sum_{i=0}^4 \beta^{4-i}\alpha^2\gamma\alpha^2\beta^i + 2(\beta^4\gamma\beta^4)[4], \end{aligned}$$

We claim that y is an element in the centre of $E(\Lambda)$. It is straightforward to check that on each branch, away from the branch vertex, we have $a_i y = y a_i$ and $\bar{a}_i y = y \bar{a}_i$, and similarly $a_1 y = y a_1$ and $\bar{a}_1 y = y \bar{a}_1$. The remaining identity at the long branch will follow directly if we show

$$(A) \quad \omega^2 = \sum_{i=1}^5 \beta^{5-i}\alpha^2\gamma\alpha^2\beta^i + 2[\beta^5\gamma\beta^4 + \beta^4\gamma\beta^5].$$

Furthermore, the remaining identity at vertex 3 will follow directly from

$$(B) \quad \omega^2 = \gamma[\alpha^2\gamma\alpha\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha\gamma\alpha^2] \\ + [(\alpha^2\gamma\alpha\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha^2\gamma\alpha + \alpha\gamma\alpha^2\gamma\alpha\gamma\alpha^2)\gamma].$$

We use the following identity

$$(\diamond) \quad (\alpha\gamma)^3\alpha^2\gamma\alpha + \alpha\gamma\alpha^2(\gamma\alpha)^3 + \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha = 0,$$

which is obtained from (\dagger) by premultiplying with $\alpha\gamma$ and postmultiplying with $\gamma\alpha$.

We start with proving (A). First we calculate

$$(A)\text{-I} = \sum_{i=1}^5 \beta^{5-i}\alpha^2\gamma\alpha^2\beta^i.$$

This can be written as $\beta^3 m + m\beta^3$, where

$$\begin{aligned} m &= \alpha\gamma\alpha^2\gamma\alpha^2 + \gamma\alpha^2\gamma\alpha^2\gamma + \alpha^2\gamma\alpha^2\gamma\alpha \\ &= \alpha\zeta + \gamma\alpha^2\gamma\alpha^2\gamma - \alpha^2\gamma\alpha\gamma\alpha^2. \end{aligned}$$

Recall from (\ddagger) that $\beta^3 = -\rho - \gamma\alpha\gamma$, where $\rho^2 = \zeta$. Substituting this gives

$$\begin{aligned} \beta^3 m + m\beta^3 &= (-\rho m - \gamma\alpha\gamma m - m\gamma\alpha\gamma - m\rho) \\ &= -\rho m - (\gamma\alpha)^2\zeta + (\gamma\alpha\gamma\alpha^2)^2 - (\alpha\gamma)^2\zeta + (\alpha^2\gamma\alpha\gamma)^2 - m\rho. \end{aligned}$$

Note that the two squares occur in ω^2 . Next

$$\begin{aligned} -\rho m &= -\rho\alpha\zeta - \rho\gamma\alpha^2\gamma\alpha^2\gamma + \rho\alpha^2\gamma\alpha\gamma\alpha^2 \\ &= (\alpha\gamma)^4\alpha^2 + \alpha^2(\gamma\alpha)^4 - \rho\gamma\alpha^2\gamma\alpha^2\gamma + \rho\alpha^2\gamma\alpha\gamma\alpha^2 \end{aligned}$$

(and reversing each term gives an identity for $-m\rho$). Note that the first two terms occur in ω^2 . In the expression for $\beta^3 m + m\beta^3 - \omega^2$ we can cancel two of the terms immediately. Namely, we obtain that

$$\rho\alpha^2\gamma\alpha\gamma\alpha^2 + \alpha^2\gamma\alpha\gamma\alpha^2\rho = 0,$$

by first substituting $\rho\alpha^2 = \alpha^2\gamma\alpha^2 = \alpha^2\rho$ and then pre- and post multiply (\ddagger) with α^2 .

(A)-II. We get from this that $\beta^3 m + m\beta^3 - \omega^2$ is equal to

$$(\alpha\gamma\alpha\gamma\alpha)^2 + (\alpha\gamma)^4\alpha^2 + \alpha^2(\gamma\alpha)^4 - (\gamma\alpha)^2\zeta - (\alpha\gamma)^2\zeta - \rho(\gamma\alpha^2\gamma\alpha^2\gamma) - (\gamma\alpha^2\gamma\alpha^2\gamma)\rho.$$

(A)-III. By definition and an obvious substitution

$$\begin{aligned} \beta^5\gamma\beta^4 + \beta^4\gamma\beta^5 &= \beta^4[\beta\gamma + \gamma\beta]\beta^4 &= \beta^4(\alpha^2 - \beta^2)\beta^4 &= \beta^4\alpha^2\beta^4 \\ &= \beta^3\gamma\alpha^2\gamma\beta^3 &= (\rho + \gamma\alpha\gamma)\gamma\alpha^2\gamma(\gamma\alpha\gamma + \rho) &= \rho(\gamma\alpha^2\gamma)\rho. \end{aligned}$$

So we must show that

$$(*) \quad (\text{A})\text{-II} = -2\rho\gamma\alpha^2\gamma\rho.$$

Using that $\gamma\alpha^2\gamma = \gamma(\rho - \alpha\gamma\alpha) = (\rho - \alpha\gamma\alpha)\gamma$ and $\rho^2 = \zeta$, we infer that $\rho(\gamma\alpha^2\gamma\alpha\gamma\alpha) - \zeta(\gamma\alpha)^2 = -\rho(\alpha\gamma)^3\alpha$ and $\alpha\gamma\alpha\gamma\alpha^2\gamma\rho - (\alpha\gamma)^2\zeta = -(\alpha\gamma)^3\alpha\rho$. Using the same identity as above we get that

$$\alpha^2(\alpha\gamma)^4 - \rho(\alpha\gamma)^3\alpha = -\alpha\gamma\alpha^2(\gamma\alpha)^3.$$

The identity obtained from this by reversing the order in each monomial holds similarly. We add the appropriate equations and cancel, and we get

$$\alpha^2(\gamma\alpha)^4 - \zeta(\gamma\alpha)^2 = -\alpha\gamma\alpha^2(\gamma\alpha)^3 - \rho(\gamma\alpha^2\gamma\alpha\gamma\alpha).$$

The identity obtained by reversing the order in each term also holds. We substitute these into (A)-II and obtain that it is equal to

$$\begin{aligned} & -(\alpha\gamma\alpha\gamma\alpha)^2 - \alpha\gamma\alpha^2(\gamma\alpha)^3 - (\alpha\gamma)^3\alpha^2\gamma\alpha \\ & - \rho(\gamma\alpha^2\gamma\alpha\gamma\alpha) - (\alpha\gamma\alpha\gamma\alpha^2\gamma)\rho - \rho(\gamma\alpha^2\gamma\alpha^2\gamma) - (\gamma\alpha^2\gamma\alpha^2\gamma)\rho. \end{aligned}$$

The sum of the first three terms is zero, by (\diamond) . Now we note that

$$(*) \quad \rho\gamma\alpha^2\gamma\rho = \rho\gamma\alpha^2\gamma\alpha^2\gamma + \rho\gamma\alpha^2\gamma\alpha\gamma\alpha,$$

so we can replace two terms in (A)-II by $-\rho\gamma\alpha^2\gamma\rho$. The identity obtained from $(*)$ by reversing the order also holds, and we can therefore replace the other two terms in (A)-II, which involve ρ by $-\rho\gamma\alpha^2\gamma\rho$. So in total we get that (A)-II is equal to $-2\rho\gamma\alpha^2\gamma\rho$ as required. This proves (A).

We prove now (B), that is

$$\begin{aligned} (\diamond) \quad & \gamma\alpha\gamma\alpha^2\gamma\alpha^2\gamma\alpha + \gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha^2 + \alpha\gamma\alpha^2\gamma\alpha^2\gamma\alpha\gamma \\ & + \alpha\gamma\alpha^2\gamma\alpha\gamma\alpha^2\gamma - \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha - (\alpha\gamma)^4\alpha^2 - \alpha^2(\gamma\alpha)^4 = 0. \end{aligned}$$

Take relation (\dagger) , and pre- and post-multiply it with $\alpha\gamma$, this gives

$$(B)\text{-I} \quad \alpha\gamma\alpha^2\gamma\alpha^2\gamma\alpha\gamma + \alpha\gamma\alpha^2\gamma\alpha\gamma\alpha^2\gamma = -\alpha\gamma\alpha\gamma\alpha^2\gamma\alpha^2\gamma - (\alpha\gamma)^5,$$

and we can replace terms 3 and 4 in (\diamond) by (B)-I. Symmetrically we can replace terms 1 and 2 in (\diamond) by

$$(B)\text{-I}^* \quad -\gamma\alpha^2\gamma\alpha^2\gamma\alpha\gamma\alpha - (\gamma\alpha)^5.$$

Next we claim that (B)-I is equal to

$$(B)\text{-II} \quad (\alpha\gamma)^3\alpha^2\gamma\alpha + (\alpha\gamma)^4\alpha^2 + \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha.$$

Namely

$$\begin{aligned}
 \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha^2\gamma &= \alpha\gamma\alpha\gamma\alpha^2(\rho - \alpha\gamma\alpha) \\
 &= (\alpha\gamma)^2(\rho - \alpha\gamma\alpha)\rho - \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha \\
 &= \alpha\gamma)^2\zeta - (\alpha\gamma)^3\alpha\rho - \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha \\
 &= -(\alpha\gamma)^5 - (\alpha\gamma)^3\alpha^2\gamma\alpha - (\alpha\gamma)^3\alpha\gamma\alpha^2 - \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha.
 \end{aligned}$$

Symmetrically we can replace (B)-I* by

$$(B)\text{-II}^*. \quad \alpha\gamma\alpha^2(\gamma\alpha)^3 + \alpha^2(\gamma\alpha)^4 + \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha$$

We substitute (B)-II and (B)-II* into (\diamond) and cancel, this leaves us to show that

$$(\alpha\gamma)^3\alpha^2\gamma\alpha + \alpha\gamma\alpha^2(\gamma\alpha)^3 + \alpha\gamma\alpha\gamma\alpha^2\gamma\alpha\gamma\alpha = 0.$$

We get this directly by premultiplying the identity (\dagger) with $\alpha\gamma$ and postmultiplying with $\gamma\alpha$. This proves the identity (B), and hence that y is in $Z_{\text{gr}}(E(\Lambda))$.

Finite generation. We want to show that E_2 is finitely generated over the subalgebra generated by the central elements ζ and ω^2 . Let ϕ be a monomial in $\{\alpha, \gamma\}$. We define its length $l(\phi)$, to be the total number of factors $\{\alpha, \gamma\}$. As a first goal towards finite generation, we want to express ϕ as a polynomial in ζ, ω^2 , where the coefficients are polynomials in $\{\alpha, \gamma\}$, and so that we have only finitely many coefficients. To this end, we have already seen that

- (i) ω and ρ commute with α ,
- (ii) $\omega\gamma + \gamma\omega = -\zeta$,
- (iii) $\omega\rho + \rho\omega = -3\zeta\alpha^2$,
- (iv) $\gamma\rho = \rho\gamma + (\gamma\alpha)^2 - (\alpha\gamma)^2$.

Furthermore, we have that $(\gamma\alpha)^{3k}\gamma = (-1)^k\zeta^k\gamma$ for $k \geq 1$, since $\zeta\gamma = -(\gamma\alpha)^3\gamma$. And, $(\alpha^2\gamma)^{2k+1} = \rho^{2k}\alpha^2\gamma = \zeta^k\alpha^2\gamma$ for $k \geq 1$, since $\alpha^2\gamma = \omega - \alpha\gamma\alpha - \gamma\alpha^2$. These observations are a main step towards our first goal. At the next step we get expressions, which have ω as a factor. Although this is not central, we can move factors of ω to the left, as a consequence of the next lemma.

Lemma. *Assume $\phi = p\omega q$ where p, q are monomials in α, γ . Then*

$$\phi = -\omega pq + \zeta \sum \phi_i,$$

where ϕ_i are monomials in α, γ of length $l(\phi_i) < l(p) + l(q)$.

We leave the proof to the reader, only pointing out that induction and the identity $\gamma\omega = -(\omega - \zeta)$ are used. The final crucial step is the following.

Lemma. *Assume ϕ is a monomial in $\{\alpha, \beta\}$. Then ϕ is a linear combination of elements of the form*

$$\zeta^a\omega^b\phi_1,$$

where ϕ_1 is a monomial in $\{\alpha, \gamma\}$ of the form

$$(*) \quad \alpha^i(\gamma\alpha)^r(\gamma\alpha^2)^s\gamma\alpha^j$$

with $r \leq 2$ and $s \leq 2$ and $i, j \leq 2$.

Proof. It suffices to express ϕ as a combination of elements $\zeta^a \omega^b \phi_1$ with ϕ_1 as in (*), but for arbitrary r and s . Then the coefficients can be reduced according to the first Lemma. Suppose therefore that ϕ is not of the form as in (*). Then we can write

$$\phi = p(\alpha^2 \gamma \alpha \gamma)q,$$

where p and q are monomials in $\{\alpha, \gamma\}$. This is equal to

$$(**) \quad p[\omega - \alpha \gamma \alpha \gamma \alpha - \gamma \alpha \gamma \alpha^2]q = p\omega q - p\alpha \gamma \alpha \gamma \alpha q - p\gamma \alpha \gamma \alpha^2 q$$

and $p\omega q = \omega p q + \zeta \sum \phi_i$ with $l(\phi_i) < l(p) + l(q)$. For $p q$ we use induction. The second term in (**) has one factor α^2 less, and in the third term of (**), α^2 occurs further to the right. So if we start with the rightmost α^2 in ϕ and iterate the above substitution, then we can move α^2 completely to the right. Hence we get terms with coefficients either of shorter length, or with fewer α^2 and where to the right only submonomials $\dots \gamma \alpha \gamma \dots$ occur, but where the length does not increase. The claim follows now by induction. \square

The claim that Λ satisfies **(Fg)** now follows immediately. \square

9. QUANTUM EXTERIOR ALGEBRAS

This final section is devoted to characterizing when the quantum exterior k -algebra

$$\Lambda = k\langle x_1, x_2, \dots, x_n \rangle / (\{x_i x_j + q_{ij} x_j x_i\}_{i < j}, \{x_i^2\}_{i=1}^n, q_{ij} \in k^*)$$

satisfies **(Fg)** for a field k . This completes the discussion of the $\widetilde{\mathbb{Z}}_d$ -case, in that we treat the case $d = 0$.

It is well-known that Λ is a Koszul algebra and that the Koszul dual $E(\Lambda)$ is given by $E(\Lambda) = k\langle x_1, x_2, \dots, x_n \rangle / (\{x_i x_j - q_{ij} x_j x_i\}_{i < j}, q_{ij} \in k^*)$. To answer our question we apply Theorem 1.3, so our first task is to study the graded centre of $E(\Lambda)$.

Any element in $E(\Lambda)$ can be written uniquely as a linear combination of the elements $\{x_1^{t_1} x_2^{t_2} \dots x_n^{t_n} \mid t_j \geq 0\}$. Given that $E(\Lambda)$ is generated in degree 1 as an algebra over k , to compute the graded centre of $E(\Lambda)$ it is necessary and sufficient to check graded commutation with the variables $\{x_i\}_{i=1}^n$. Multiplication with the variables $\{x_i\}_{i=1}^n$ from the left and the right on an element $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$ as above is given by:

$$x_1^{t_1} x_2^{t_2} \dots x_n^{t_n} x_i = \begin{cases} x_1^{t_1} x_2^{t_2} \dots x_n^{t_n+1}, & i = n, \\ (\prod_{j=n}^{i+1} q_{ij}^{-t_j}) x_1^{t_1} x_2^{t_2} \dots x_{i-1}^{t_{i-1}} x_i^{t_i+1} x_{i+1}^{t_{i+1}} \dots x_n^{t_n}, & i < n, \end{cases}$$

and

$$x_i x_1^{t_1} x_2^{t_2} \dots x_n^{t_n} = \begin{cases} x_1^{t_1+1} x_2^{t_2} \dots x_n^{t_n+1}, & i = 1, \\ (\prod_{j=1}^{i-1} q_{ji}^{-t_j}) x_1^{t_1} x_2^{t_2} \dots x_{i-1}^{t_{i-1}} x_i^{t_i+1} x_{i+1}^{t_{i+1}} \dots x_n^{t_n}, & i > 1. \end{cases}$$

Proposition 9.1. *The quantum exterior k -algebra*

$$\Lambda = k\langle x_1, x_2, \dots, x_n \rangle / (\{x_i x_j + q_{ij} x_j x_i\}_{i < j}, \{x_i^2\}_{i=1}^n, q_{ij} \in k^*)$$

*satisfies **(Fg)** if and only if all the elements $\{q_{ij}\}_{i < j}$ are roots of unity.*

Proof. First we characterise when x_i^p is in $Z_{\text{gr}}(E(\Lambda))$, where $E(\Lambda)$ is as above. It follows immediately from the formulae above that

$$x_i^p x_j = \begin{cases} x_i^p x_j, & i < j, \\ x_i^{p+1}, & i = j, \\ q_{ji}^{-p} x_j x_i^p, & i > j, \end{cases}$$

and

$$x_j x_i^p = \begin{cases} q_{ij}^{-p} x_i^p x_j, & i < j, \\ x_i^{p+1}, & i = j, \\ x_j x_i^p, & i > j. \end{cases}$$

From these formulae we obtain that x_i^p is in $Z_{\text{gr}}(E(\Lambda))$ if and only if

- (i) $1 = (-1)^p q_{ij}^{-p}$ for $j > i$,
- (ii) $1 = (-1)^p$ for $j = i$,
- (iii) $q_{ji}^{-p} = (-1)^p$ for $j < i$.

For $\text{char } k \neq 2$, this is clearly equivalent to (A) p is even and (B) $\{q_{ij}\}_{j>i}$ are all p -th roots of unity. For $\text{char } k = 2$, then this is equivalent to just (B).

Suppose now that not all the elements $\{q_{ij}\}_{j>i}$ are roots of unity for some i in $\{1, 2, \dots, n-1\}$. Hence x_i^p is not in $Z_{\text{gr}}(E(\Lambda))$ for any $p \geq 1$. On the other hand x_i^l is in $E(\Lambda)$ for any $l \geq 1$. But since $E(\Lambda)$ is a domain and $Z_{\text{gr}}(E(\Lambda))$ (and $E(\Lambda)$) is multi-graded by \mathbb{N}^n , the only way the elements x_i^l can be generated by the graded centre $Z_{\text{gr}}(E(\Lambda))$ is that it is one of the generators. Hence $E(\Lambda)$ is infinitely generated as a module over $Z_{\text{gr}}(E(\Lambda))$. Consequently **(Fg)** is not satisfied for $\Lambda = E(E(\Lambda))$.

Suppose now that all the elements $\{q_{ij}\}_{j>i}$ are roots of unity for all i in $\{1, 2, \dots, n-1\}$. Suppose that each q_{ij} is a root of unity of degree d_{ij} . Let N be the least common even multiple of all the d_{ij} 's. From the calculations above we infer that x_i^N is in $Z_{\text{gr}}(E(\Lambda))$ for all i in $\{1, 2, \dots, n\}$. Then the set $\{x_1^{t_1} x_2^{t_2} \dots x_n^{t_n} \mid t_i < N, \forall i \in \{1, 2, \dots, n\}\}$ is a generating set of $E(\Lambda)$ as a module over the graded centre $Z_{\text{gr}}(E(\Lambda))$. Hence $E(\Lambda)$ is a finitely generated module over $Z_{\text{gr}}(E(\Lambda))$, and therefore **(Fg)** is satisfied for $\Lambda = E(E(\Lambda))$. \square

This result is generalized by Bergh and Oppermann in [2, Theorem 5.5].

Remark. The Nakayama automorphism ν of Λ is given by $\nu(x_i) = (-1)^{n-1} \frac{\prod_{j=i+1}^n q_{ij}}{\prod_{j=1}^{i-1} q_{ji}} x_i$ for $i = 1, 2, \dots, n$. For $n = 3$, we can choose $q_{12} = q$, $q_{13} = q^{-1}$ and $q_{23} = q_{12}$, where q is not a root of unity. Then the Nakayama automorphism ν is the identity, while **(Fg)** is not satisfied. Hence the property **(Fg)** is not linked to the Nakayama automorphism being of finite order, or even the identity.

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