SUPPORT VARIETIES FOR SELFINJECTIVE ALGEBRAS

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Abstract. Support varieties for any finite dimensional algebra over a field were introduced in [21] using graded subalgebras of the Hochschild cohomology. We mainly study these varieties for selfinjective algebras under appropriate finite generation hypotheses. Then many of the standard results from the theory of support varieties for finite groups generalize to this situation. In particular, the complexity of the module equals the dimension of its corresponding variety, all closed homogeneous varieties occur as the variety of some module, the variety of an indecomposable module is connected, periodic modules are lines and for symmetric algebras a generalization of Webb's theorem is true. As a corollary we show that Webb's theorem generalizes to finite dimensional cocommutative Hopf algebras.

Introduction

Let k be a field of characteristic p and G a finite group. In 1971, Quillen [19] gave a description of the cohomology ring $H^*(G,k)$ modulo nilpotent elements as an inverse limit of cohomology algebras of elementary abelian p-subgroups of G. This has led to the work of Benson, Carlson and others on the theory of varieties for kG-modules, and in general to deep structural information about modular representations of finite groups [4], [6]. Subsequently, analogous results have been obtained for p-Lie algebras (by Friedlander and Parshall [12], Jantzen [17], and others), and also for Steenrod algebras arising in algebraic topology (by Palmieri [18]).

The support variety of a kG-module is a powerful invariant. This is defined in terms of the maximal ideal spectrum of the group cohomology $H^*(G,k)$, a finitely generated (almost) commutative graded ring. It acts on $\mathrm{Ext}^*(M,M)$ for any finitely generated kG-module M and the support variety of the module is the variety associated to the annihilator ideal of this action. This construction is based on the Hopf algebra structure which is generally not available. In [21] an analogous construction for an arbitrary finite dimensional algebra Λ was developed where instead of group cohomology, one takes the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$.

In this paper, we mainly study these varieties for selfinjective algebras. We prove that under appropriate finite generation hypotheses many of the properties known for the group algebra situation have analogues. Note that if the algebra is decomposable, then the Hochschild cohomology ring also decomposes in a natural way using the same complete set of central idempotents as in the decomposition of the algebra. In particular, we can without loss of generality assume that our algebras are indecomposable.

2000 *Mathematics Subject Classification.* Primary: 16E40, 16G10, 16P10, 16P20; Secondary: 16G70.

Throughout this paper Λ always denotes an indecomposable finite dimensional algebra over an algebraically closed field k, with Jacobson radical $\mathfrak{r}$. Recall from [21] that the variety of a finitely generated left Λ-module M relative to a (Noetherian) graded subalgebra H of $\mathrm{HH}^*(\Lambda)$ is given by

$$V_H(M) = \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H \operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r}) \subseteq \mathfrak{m}\},$$

where $\operatorname{MaxSpec} H$ is the maximal ideal spectrum of H and $\operatorname{Ann}_H X$ is the annihilator of an H-module X. In the theory of support varieties for group rings of finite groups ([4, 5, 6]), for more general finite dimensional cocommutative Hopf algebras ([13]), and for complete intersections ([1, 2]), the property of having a Noetherian ring H of cohomological operators over which the extension groups $\operatorname{Ext}^*(M,N)$ are finitely generated H-modules for all finitely generated modules M and N, is one of the corner stones for the whole theory. Hence, two assumptions are central: **Fg1**: H is a commutative Noetherian graded subalgebra of $\mathrm{HH}^*(\Lambda)$ with $H^0 = \mathrm{HH}^0(\Lambda)$, and **Fg2**: $\operatorname{Ext}^*_\Lambda(M,N)$ is a finitely generated H-module for all finitely generated left Λ-modules M and N (see section 1).

In the first section we analyse the consequences of these assumptions, and show that the algebra Λ must be Gorenstein (that is, the injective dimensions of Λ as a left and as a right module are finite). Furthermore, the dimension of the variety of a module is given by the complexity of the module, and the variety of a module is trivial if and only if the module has finite projective dimension.

The second section is devoted to characterizing elements in the annihilator of $\operatorname{Ext}^*_\Lambda(M,M)$ as an $\mathrm{HH}^*(\Lambda)$-module. In the process we introduce for each homogeneous element η in $\mathrm{HH}^*(\Lambda)$ a bimodule M_η that we use in the next section with assumptions **Fg1** and **Fg2** to show that any closed homogeneous variety occurs as the variety of some module (Theorem 3.4). To our knowledge the proof we give of this fact also gives an alternative proof of the same result for support varieties for group rings of finite groups. The proof in the group ring case uses rank varieties and restriction to elementary abelian subgroups. We do not yet have an analogue of a rank variety to offer in our more general setting, but there are partial answers in special cases considered by Erdmann and Holloway in [8].

Again with assumptions **Fg1** and **Fg2**, periodic modules for selfinjective algebras are characterized in the fourth section as modules with complexity one, as in the group ring case. Using this we prove a generalization of Webb's theorem to selfinjective algebras where the Nakayama functor is of finite order for any indecomposable module. In particular using [11, 13] Webb's theorem is true for finite dimensional cocommutative Hopf algebras.

The next section is devoted to briefly discussing relationships between representation type and complexity. In the final section we show that the variety of an indecomposable module is connected whenever **Fg1** and **Fg2** hold.

We end this introduction by setting the notation and the overall general assumptions. For any ring R we denote by $\operatorname{mod} R$ the finitely presented left R-modules. Recall that throughout the paper Λ denotes an indecomposable finite dimensional algebra over an algebraically closed field k with Jacobson radical $\mathfrak{r}$. The stable category of $\operatorname{mod}\Lambda$ is denoted by $\underline{\operatorname{mod}}\,\Lambda$, that is, $\operatorname{mod}\Lambda$ modulo the ideal given by the morphisms factoring through projective modules in $\operatorname{mod}\Lambda$. We denote the enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$ by Λ^e, and we view Λ-bimodules as left Λ^e-modules. For a module X in $\operatorname{mod}\Lambda$ the full subcategory $\operatorname{add} X$ is given by all the direct summands of every finite direct sum of copies of X. For a homogeneous ideal $\mathfrak{a}$

in H let $V_H(\mathfrak{a})$ be the subvariety given by the maximal ideals of H containing $\mathfrak{a}$. Let $D = \operatorname{Hom}_k(-,k)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$ be the usual duality. For a Λ-module B denote by B^* the right Λ-module $\operatorname{Hom}_\Lambda(B,\Lambda)$.

Acknowledgements

Holloway would like to thank EPSRC for their support through a Postdoctoral Fellowship while this research was carried out. Snashall and Solberg would like to express their gratitude for the hospitality and support from the Mathematical Institute, University of Oxford. Taillefer kindly acknowledges financial support from HEFCE through a Career Development Fellowship.

1. Finite generation

Given two Λ-modules M and N in $\operatorname{mod}\Lambda$ the direct sum $\operatorname{Ext}^*_\Lambda(M,N) = \oplus_{i\geq 0}\operatorname{Ext}^i_\Lambda(M,N)$ is a left and a right module over the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ of Λ, where the left and the right actions are related in a graded commutative way (see [21, Theorem 1.1]). Furthermore, it is known that in general $\operatorname{HH}^*(\Lambda)$ is not a finitely generated algebra over k and that $\operatorname{Ext}^*_\Lambda(M,N)$ is not a finitely generated module over $\operatorname{HH}^*(\Lambda)$. This section is devoted to investigating consequences of finite generation of $\operatorname{Ext}^*_\Lambda(M,N)$ as a module over commutative Noetherian graded subalgebras H of $\operatorname{HH}^*(\Lambda)$ for specific pairs and for all pairs of modules in $\operatorname{mod}\Lambda$. In particular, we show that if $\operatorname{Ext}^*_\Lambda(D(\Lambda_\Lambda),\Lambda/\mathfrak{r})$ and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda)$ are finitely generated modules over such an H, then Λ is a Gorenstein algebra.

As we have pointed out already, even seemingly weak finite generation assumptions imply strong conditions on the algebras we consider. To see how severely finite generation can fail in general, we pause to consider the following example. Let $\Lambda = k\langle\alpha_1,\ldots,\alpha_n\rangle/(\{\alpha_i\alpha_j\}^{n,n}_{i,j=1})$ for some $n\geq 2$. This is a finite dimensional Koszul algebra, and the Koszul dual is given by $E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \simeq k\langle\alpha_1^*,\ldots,\alpha_n^*\rangle$, a free algebra in n indeterminates. In [21] it is shown that the image of the map $-\otimes_\Lambda \Lambda/\mathfrak{r}\colon \operatorname{HH}^*(\Lambda)\to E(\Lambda)$ is contained in the graded centre $Z_{\mathrm{gr}}(E(\Lambda))$ of $E(\Lambda)$. As it is well-known that $Z_{\mathrm{gr}}(E(\Lambda)) = k$, in this case, it follows that $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is an infinitely generated module over any graded subalgebra H of $\operatorname{HH}^*(\Lambda)$.

Finite generation can also fail for selfinjective algebras, as the next example shows. Let $\Lambda = k\langle x,y\rangle/(x^2, xy+qyx, y^2)$, where q is in k and not a root of unity. This is a finite dimensional selfinjective Koszul algebra, and the Koszul dual $E(\Lambda)$ is isomorphic to $k\langle x,y\rangle/(yx-qxy)$. The graded centre $Z_{\mathrm{gr}}(E(\Lambda))$ is k. As above, $E(\Lambda)$ is an infinitely generated module over any graded subalgebra H of $\operatorname{HH}^*(\Lambda)$.

Now we introduce the first of two finite generation assumptions that we keep throughout the paper.

Assumption 1 (Fg1). Fix a graded subalgebra H of $\operatorname{HH}^*(\Lambda)$ such that

(i) H is a commutative Noetherian ring.
(ii) $H^0 = \operatorname{HH}^0(\Lambda) = Z(\Lambda)$.

This assumption, **Fg1**, is assumed throughout the paper unless otherwise explicitly stated. One reason for making this assumption is to obtain an affine variety in which to consider the support varieties of finitely generated modules as introduced in [21]. The choice of H plays a crucial role later with respect to the assumption **Fg2**. We now make an equivalent definition of the variety of a pair of modules

(M,N) in $\operatorname{mod}\Lambda$ to the one given in [21]. Let the variety of (M,N) be given by

$$V_H(M,N) = \operatorname{MaxSpec}(H/A_H(M,N)),$$

where $A_H(M,N)$ is the annihilator of $\operatorname{Ext}^*_\Lambda(M,N)$ as an H-module and this is considered as a subvariety of the homogeneous affine variety $V_H = \operatorname{MaxSpec} H$. Recall that $V_H(M,\Lambda/\mathfrak{r}) = V_H(M,M) = V_H(\Lambda/\mathfrak{r},M)$, and this is defined to be the variety $V_H(M)$ of M. By [21, Proposition 4.6 (b)] $\sqrt{A_H(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})}$ is the ideal $\mathcal{N}_H$ in H generated by all homogeneous nilpotent elements in H. Consequently, $V_H(\Lambda/\mathfrak{r}) = V_H$.

To start analysing the consequences of finite generation, we need the following proposition linking the dimension of the variety of a module to the complexity of the module. Recall that the complexity $c_\Lambda(M)$ of a Λ-module M is given by $c_\Lambda(M) = \min\{b \in \mathbb{N}_0 \mid \exists a \in \mathbb{R} \text{ such that } \dim_k Q_n \leq an^{b-1}, \forall n \geq 0\}$, where $\cdots \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0$ is a minimal projective resolution of M.

Proposition 1.1. *Let M be in $\operatorname{mod}\Lambda$.*

(a) If $\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ is a finitely generated H-module, then

$$\dim V_H(M) = c_\Lambda(M) < \infty.$$

(b) If $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},M)$ is a finitely generated H-module, then

$$\dim V_H(M) = c_\Lambda(D(M)) < \infty.$$

Proof. (a) Suppose that $\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ is finitely generated as an H-module. Then using induction on the Loewy length of a module, $\operatorname{Ext}^*_\Lambda(M,N)$ is a finitely generated H-module for all N in $\operatorname{mod}\Lambda$. In particular, $\operatorname{Ext}^*_\Lambda(M,M)$ is a finitely generated H-module. As the H-module structure on $\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ factors through $\operatorname{Ext}^*_\Lambda(M,M)$ we infer that $\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ is a finitely generated module over $\operatorname{Ext}^*_\Lambda(M,M)$. Using these observations the proof given in [4, Proposition 5.3.5 and Proposition 5.7.2] carries over. In particular, the complexity is finite.

(b) As $V_H(M) = V_H(D(M))$ by [21, Proposition 3.5] the claim follows from (a). □

In the theory of support varieties for group rings and complete intersections ([1, 2]) the finite generation of the extension groups $\operatorname{Ext}^*(M,N)$ as modules over the ring of cohomological operators for all modules M and N are of great importance. However we first analyse finite generation for $\operatorname{Ext}^*_\Lambda(M,N)$ as an H-module for specific pairs of Λ-modules.

Proposition 1.2. *Let M be in $\operatorname{mod}\Lambda$.*

(a) Suppose that $\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ is finitely generated as an H-module. If the variety of M is trivial, then the projective dimension of M is finite.

(b) Suppose that $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},M)$ is finitely generated as an H-module. If the variety of M is trivial, then the injective dimension of M is finite.

(c) Suppose that $\operatorname{Ext}^*_\Lambda(D(\Lambda^{\mathrm{op}}),\Lambda/\mathfrak{r})$ and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda)$ are finitely generated as H-modules. Then Λ is a Gorenstein algebra.

Proof. (a) Since $\dim V_H(M) = c_\Lambda(M) = 0$ by Proposition 1.1, it is immediate that $\operatorname{Ext}^n_\Lambda(M,\Lambda/\mathfrak{r}) = (0)$ for $n \gg 0$. Therefore the projective dimension of M is finite.

The statements in (b) and (c) follow directly from (a). □

This has the following immediate consequence concerning the Generalized Nakayama Conjecture.

Corollary 1.3. *Suppose that* $\operatorname{Ext}^i_\Lambda(M, M \oplus \Lambda) = (0)$ *for all* $i > 0$. *If* $\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ *is finitely generated as an* H*-module, then* M *is projective.*

Any finitely generated module over a finite dimensional algebra can be filtered by a finite filtration of semisimple modules. Next we show that the finite generation of $\operatorname{Ext}^*_\Lambda(M, N)$ for all pairs of modules M and N in $\operatorname{mod}\Lambda$ over a subalgebra H of $\operatorname{HH}^*(\Lambda)$ is equivalent to the finite generation of $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ using the filtration in semisimple modules.

Proposition 1.4. *The following are equivalent.*

(i) $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ *is finitely generated as an* H*-module.*
(ii) $\operatorname{Ext}^*_\Lambda(M, N)$ *is finitely generated as an* H*-module for all pairs of* Λ*-modules* M *and* N *in* $\operatorname{mod}\Lambda$.
(iii) $\operatorname{HH}^*(\Lambda, B)$ *is finitely generated as an* H*-module for all* B *in* $\operatorname{mod}\Lambda^e$.

Proof. Since $\operatorname{Ext}^*_\Lambda(M, N) \simeq \operatorname{HH}^*(\Lambda, \operatorname{Hom}_k(M, N))$, it is clear that (iii) implies (ii) and (ii) implies (i). So it remains to prove that (i) implies (iii).

Assume that $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated H-module. This is equivalent to $\operatorname{HH}^*(\Lambda, \operatorname{Hom}_k(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}))$ being a finitely generated H-module. Since any simple Λ^e-module is isomorphic to $\operatorname{Hom}_k(S, T)$ for some simple Λ-modules S and T and any finitely generated Λ^e-module is filtered in simple Λ^e-modules, $\operatorname{HH}^*(\Lambda, B)$ is a finitely generated H-module for all B in $\operatorname{mod}\Lambda^e$. This completes the proof. □

This motivates the second of our two finite generation assumptions.

Assumption 2 (Fg2). $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated H-module.

Remark. In particular note that these two assumptions **Fg1** and **Fg2** imply that $\operatorname{HH}^*(\Lambda)$ is a finitely generated H-module, and consequently $\operatorname{HH}^*(\Lambda)$ itself is finitely generated as a k-algebra. Similarly, $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated k-algebra.

Combining the previous result with our earlier observations in this section we obtain the following.

Theorem 1.5. *Suppose that* Λ *and* H *satisfy* **Fg1** *and* **Fg2**.

(a) *The algebra* Λ *is Gorenstein.*
(b) *The following are equivalent for a module* M *in* $\operatorname{mod}\Lambda$.
  (i) *The variety of* M *is trivial.*
  (ii) *The projective dimension of* M *is finite.*
  (iii) *The injective dimension of* M *is finite.*
(c) $\dim V_H(M) = c_\Lambda(M)$ *for any module* M *in* $\operatorname{mod}\Lambda$.

2. The annihilator of $\operatorname{Ext}^*_\Lambda(M, M)$

In contrast to the previous section, the results in this section do not need any finite generation assumptions.

From [21] the variety of a module M can be defined to be $V_H(M, M)$ for a graded subalgebra H of $\operatorname{HH}^*(\Lambda)$. This has the advantage that it is given by the annihilator of $\operatorname{Ext}^*_\Lambda(M, M)$, which is the target of the graded ring homomorphism from $\operatorname{HH}^*(\Lambda)$ induced by the functor $- \otimes_\Lambda M$. Next we describe the exact sequence of bimodules whereby the annihilator of $\operatorname{Ext}^*_\Lambda(M,M)$ is characterized. To this end let $\cdots \to P^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to \Lambda \to 0$ be a minimal projective resolution of Λ as a Λ^e-module.

Definition 2.1. *Given a homogeneous element η in $\mathrm{HH}^*(\Lambda)$ of degree n, represented by a map $\eta\colon \Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$, we define the Λ^e-module M_η by the following pushout diagram*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_{\Lambda^e}(\Lambda) \arrow[r] \arrow[d, "\eta"] & P^{n-1} \arrow[r] \arrow[d] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda \arrow[r, "\alpha_\eta"] & M_\eta \arrow[r, "\beta_\eta"] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r] & 0
\end{tikzcd}$$

where we denote by $\mathcal{E}_\eta$ the bottom row short exact sequence.

Note that the isomorphism class of the module M_η is independent of the choice of the representation of η as a map $\Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$. For a finite group G the reader should also observe that M_η corresponds to $\Omega^{-1}_{(kG)^e}(\operatorname{Ind}^{G\times G}_{\Delta G}(L_\zeta))$ for ζ a homogeneous element in the group cohomology ring of G, where $L_\zeta \simeq \operatorname{Ker}(\Omega^n_{kG}(k) \xrightarrow{\zeta} k)$. Furthermore, note that M_η is projective as a left and as a right Λ-module, since the same is true for Λ and $\Omega^i_{\Lambda^e}(\Lambda)$ for all $i \geq 0$. Pushing this analogy further we define L_η to be the module $M_\eta \otimes_\Lambda \Lambda/\mathfrak{r}$.

Using the sequence introduced above, the elements in $A_{\mathrm{HH}^*(\Lambda)}(M,M)$ have the following characterization. We leave the proof to the reader as it is similar to the corresponding result for group rings (see [4, Proposition 5.9.5]).

Proposition 2.2. *Let η be a homogeneous element of degree n in $\mathrm{HH}^*(\Lambda)$, and let M be in $\operatorname{mod}\Lambda$. Then the following are equivalent.*

(i) η *is in* $A_{\mathrm{HH}^*(\Lambda)}(M,M)$.
(ii) $\mathcal{E}_\eta \otimes_\Lambda M$ *is a split short exact sequence.*
(iii) $M_\eta \otimes_\Lambda M \simeq M \oplus \Omega^{n-1}_\Lambda(M) \oplus Q$ *for some projective Λ-module Q.*

Let $\{\eta_1,\ldots,\eta_t\}$ be a finite set of homogeneous elements of $\mathrm{HH}^*(\Lambda)$. Define the map $\alpha_{\eta_1,\ldots,\eta_t}$ to be the map

$$\Lambda \otimes_\Lambda \Lambda \otimes_\Lambda \cdots \otimes_\Lambda \Lambda \xrightarrow{\alpha_{\eta_1}\otimes\alpha_{\eta_2}\otimes\cdots\otimes\alpha_{\eta_t}} M_{\eta_1} \otimes_\Lambda M_{\eta_2} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t}.$$

It is easy to see that this map is a monomorphism and therefore induces an exact sequence $\mathcal{E}_{\eta_1,\ldots,\eta_t}$ given by

$$0 \to \Lambda \xrightarrow{\alpha_{\eta_1,\ldots,\eta_t}} M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \to X_{\eta_1,\ldots,\eta_t} \to 0.$$

This construction enables us to give a criterion for when the ideal generated by homogeneous elements $\{\eta_1,\ldots,\eta_t\}$ is in the annihilator $A_{\mathrm{HH}^*(\Lambda)}(M,M)$.

Theorem 2.3. *Let $\{\eta_1,\ldots,\eta_t\}$ be a finite set of homogeneous elements in $\mathrm{HH}^*(\Lambda)$, and let M be in $\operatorname{mod}\Lambda$.*

(a) The following are equivalent.
  (i) The ideal generated by $\{\eta_1,\ldots,\eta_t\}$ is contained in $A_{\mathrm{HH}^*(\Lambda)}(M,M)$.
  (ii) $\mathcal{E}_{\eta_i} \otimes_\Lambda M$ is a split short exact sequence for all $i = 1,2,\ldots,t$.
  (iii) $\mathcal{E}_{\eta_1,\ldots,\eta_t} \otimes_\Lambda M$ is a split short exact sequence.

(b) If a finite set $\{\eta_1,\ldots,\eta_t\}$ of homogeneous elements in $\mathrm{HH}^*(\Lambda)$ is in $A_{\mathrm{HH}^*(\Lambda)}(M,M)$, then M is a direct summand of $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M$.

Proof. (a) By Proposition 2.2 it remains to prove that (ii) and (iii) are equivalent.

Assume that the exact sequence $\mathcal{E}_{\eta_1,\dots,\eta_t}\otimes_\Lambda M$ splits. Since $\alpha_{\eta_1,\dots,\eta_t}$ can be viewed as a composition of $\alpha_{\eta_2,\dots,\eta_t}$ and α_{η_1}, it follows that $\mathcal{E}_{\eta_1}\otimes_\Lambda M$ splits. Similarly we show that $\mathcal{E}_{\eta_i}\otimes_\Lambda M$ splits for all $i=1,2,\dots,t$.

Assume that the exact sequence $\mathcal{E}_{\eta_i}\otimes_\Lambda M$ splits for all $i=1,2,\dots,t$. Since $\alpha_{\eta_1,\dots,\eta_t}$ can be viewed as the composition of the maps $\mathrm{id}_{M_{\eta_1}}\otimes_\Lambda\cdots\otimes_\Lambda\mathrm{id}_{M_{\eta_{t-1}}}\otimes\alpha_{\eta_t},\dots,\ \mathrm{id}_{M_{\eta_1}}\otimes\alpha_{\eta_2}$ and α_{η_1}, the map $\alpha_{\eta_1,\dots,\eta_t}\otimes\mathrm{id}_M$ is a composition of t split monomorphisms and therefore it is a split monomorphism itself. This proves (a).

(b) This is a direct consequence of (a). $\square$

When Λ is selfinjective we have a further characterization of when homogeneous elements are in $A_{\mathrm{HH}^*(\Lambda)}(M,M)$. This is an immediate consequence of Proposition 2.2 and the fact that Λ is selfinjective.

Proposition 2.4. *Let Λ be a selfinjective algebra. Let η be a homogeneous element of $\mathrm{HH}^*(\Lambda)$ of degree n, and let M be in $\operatorname{mod}\Lambda$. Then η is in $A_{\mathrm{HH}^*(\Lambda)}(M,M)$ if and only if $\eta\otimes_\Lambda\mathrm{id}_M\colon\Omega^n_{\Lambda^e}(\Lambda)\otimes_\Lambda M\to\Lambda\otimes_\Lambda M$ is zero in $\underline{\operatorname{mod}}\,\Lambda$.*

We end with a result of which we saw the first glimpses in Proposition 2.2.

Lemma 2.5. *Let $\eta_1,\eta_2,\dots,\eta_t$ be homogeneous elements in $A_{\mathrm{HH}^*(\Lambda)}(M)$. Then $M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda M$ is in $\operatorname{add}\{\Omega^i_\Lambda(M)\}_{i=0}^N\cup\operatorname{add}\Lambda$ for some integer N.*

Proof. If $t=1$, this is the statement of Proposition 2.2. Suppose that $t>1$. Let $\widetilde{M}=M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_{t-1}}\otimes_\Lambda M$. By Proposition 2.2 we have $M_{\eta_t}\otimes_\Lambda M\simeq M\oplus\Omega_\Lambda^{\deg\eta_t-1}(M)\oplus Q$ for some projective Λ-module Q. Tensoring this isomorphism with $M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_{t-1}}$ we obtain

$$M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda M\simeq \widetilde{M}\oplus M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_{t-1}}\otimes_\Lambda\Omega_\Lambda^{\deg\eta_t-1}(M)\oplus Q'$$

for some projective Λ-module Q'. Since the middle direct summand of the right hand side is isomorphic to $\Omega_\Lambda^{\deg\eta_t-1}(\widetilde{M})$ modulo projectives, the claim now follows by induction. $\square$

3. Modules with given varieties

In this section we return to the setting suggested by the first section and require throughout that Λ satisfies **Fg1** and **Fg2** for some graded subalgebra H of $\mathrm{HH}^*(\Lambda)$.

In general the variety of a module is a closed homogeneous variety. Here we show that any closed homogeneous variety occurs as the variety of some module. The module we construct is not necessarily indecomposable.

To prove our results we make use of the bimodules M_η introduced in the previous section. We start by considering, for a homogeneous element η of positive degree in H, the variety of $M_\eta\otimes_\Lambda M$.

Proposition 3.1. *Let η be a homogeneous element of positive degree in H, and let M be in $\operatorname{mod}\Lambda$.*

(a) $V_H(M_\eta\otimes M)\subseteq V_H(L_\eta)\cap V_H(M)$.

(b) The element η^2 is in $A_H(M_\eta\otimes_\Lambda M,\Lambda/\mathfrak{r})$. In particular, $V_H(L_\eta)$ is contained in $V_H(\langle\eta\rangle)$, and consequently $V_H(M_\eta\otimes_\Lambda M)$ is contained in $V_H(\langle\eta\rangle)\cap V_H(M)$.

(c) Let $\{\eta_1,\dots,\eta_t\}$ be homogeneous elements in $\mathrm{HH}^*(\Lambda)$. Then $V_H(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda M)$ is contained in $V_H(\langle\eta_1,\dots,\eta_t\rangle)\cap V_H(M)$.

Proof. (a) The sequence $\mathcal{E}_\eta \otimes_\Lambda M\colon 0 \to M \to M_\eta \otimes_\Lambda M \to \Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \to 0$ is exact. By [21, Proposition 3.4] it follows that $V_H(M_\eta \otimes_\Lambda M) \subseteq V_H(M)$, since $\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \simeq \Omega^{n-1}_\Lambda(M) \oplus F$ for some projective Λ-module F and the variety is invariant under taking syzygies.

Since the module M has a finite filtration in semisimple modules, $V_H(M_\eta \otimes_\Lambda M)$ is contained in $V_H(L_\eta)$ by iterated use of [21, Proposition 3.4]. Hence the claim follows.

(b) The proof is very similar to the group ring case. Since

$$\operatorname{Ext}^i_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\mathfrak{r})$$

is isomorphic to $\operatorname{Ext}^{i+n-1}_\Lambda(M, \Lambda/\mathfrak{r})$ for $i \geq 1$, the short exact sequence

$$\mathcal{E}_\eta \otimes_\Lambda M\colon 0 \to M \to M_\eta \otimes_\Lambda M \to \Omega^n_{\Lambda^e}(\Lambda) \otimes_\Lambda M \to 0$$

gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\cdot\eta} \operatorname{Ext}^{i+n}_\Lambda(M, \Lambda/\mathfrak{r}) \to \operatorname{Ext}^{i+1}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \\ \operatorname{Ext}^{i+1}_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\cdot\eta} \operatorname{Ext}^{i+n+1}_\Lambda(M, \Lambda/\mathfrak{r}) \to \cdots$$

and

$$0 \to \operatorname{Hom}_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \operatorname{Hom}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \\ \operatorname{Hom}_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\cdot\eta} \operatorname{Ext}^n_\Lambda(M, \Lambda/\mathfrak{r}).$$

This induces the following short exact sequences

$$0 \to \operatorname{Hom}_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \operatorname{Hom}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \xrightarrow{\nu_0} \\ \operatorname{Ker}(\cdot\eta)|_{\operatorname{Hom}_\Lambda(M,\Lambda/\mathfrak{r})} \to 0$$

and

$$0 \to \operatorname{Ext}^{*+n-1}_\Lambda(M, \Lambda/\mathfrak{r})/(\eta \operatorname{Ext}^{*-1}_\Lambda(M, \Lambda/\mathfrak{r})) \xrightarrow{\mu_*} \\ \operatorname{Ext}^*_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \xrightarrow{\nu_*} \operatorname{Ker}(\cdot\eta)|_{\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})} \to 0$$

for the index $*$ ranging over natural numbers greater or equal to 1. Let θ be in $\operatorname{Ext}^i_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})$. Then $\nu_{i+n}(\eta\theta) = \eta\nu_i(\theta) = 0$, hence $\eta\theta$ is in $\operatorname{Ker}\nu_{n+i} = \operatorname{Im}\mu_{n+i}$. Since $\operatorname{Im}\mu_{n+i}$ is annihilated by η, it follows that $\eta^2\theta = 0$. The claim follows from this.

(c) This follows immediately from (b). □

It follows that given homogeneous elements $\{\eta_1, \ldots, \eta_t\}$ in H, the module $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda \Lambda/\mathfrak{r}$ has variety contained in $V_H(\langle \eta_1, \ldots, \eta_t\rangle)$. Note that the previous result is true in general. However to show that the inclusion actually is an equality we make full use of our assumptions **Fg1** and **Fg2**.

We stress that throughout this section we assume the conditions **Fg1** and **Fg2**. Recall in particular from Theorem 1.5 that in this case Λ is a Gorenstein ring. Furthermore, these assumptions are satisfied for any block of a group ring of a finite group (see [10, 22]) and more generally for a finite dimensional cocommutative Hopf algebra (see [13]). In addition they hold true for local finite dimensional algebras which are complete intersections (see [14]).

When Λ is Gorenstein the injective dimensions of Λ as a left and a right module over itself are finite and they are equal, say equal to n. Denote by ${}^\perp\Lambda$ the full subcategory $\{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \Lambda) = (0) \text{ for all } i > 0\}$ of $\operatorname{mod}\Lambda$. Since the variety is invariant under taking syzygies, all the different varieties of modules occur for a module in ${}^\perp\Lambda$.

Given a module M in ${}^\perp\Lambda$, there is a complete resolution of M

$$\mathbb{P}_*\colon \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \to \cdots$$

where $\operatorname{Im} d_0 \simeq M$ and P_i are projective modules for all i. This uses that Λ is a cotilting module; for further details see [3]. Recall that the i-th Tate cohomology group $\widehat{\operatorname{Ext}}_\Lambda^i(M,N)$ is defined as the homology of $\operatorname{Hom}_\Lambda(\mathbb{P}_*,N)$ at stage i, i.e. $\widehat{\operatorname{Ext}}_\Lambda^i(M,N) = \operatorname{Ker} d_{i+1}^*/\operatorname{Im} d_i^*$ for i in $\mathbb{Z}$. We may give $\widehat{\operatorname{Ext}}_\Lambda^*(M,N)$ an H-module structure as follows. Let x be an element in $\widehat{\operatorname{Ext}}_\Lambda^i(M,N)$, and let η be in $\operatorname{HH}^n(\Lambda)$. Choose a non-negative integer m such that $i+2m>0$. We consider x as an element of $\widehat{\operatorname{Ext}}_\Lambda^{i+2m}(\Omega_\Lambda^{-2m}(M),N)$, which is equal to $\operatorname{Ext}_\Lambda^{i+2m}(\Omega_\Lambda^{-2m}(M),N)$. In this way we can use the ordinary H-module structure on extension groups to give the H-module structure on $\widehat{\operatorname{Ext}}_\Lambda^*(M,N)$. The reader may check that this is well-defined.

The Tate cohomology gives rise to the exact sequence

$$(*) \qquad 0 \to \mathcal{P}(M,N) \to \operatorname{Ext}_\Lambda^*(M,N) \to \widehat{\operatorname{Ext}}_\Lambda^*(M,N) \to \widehat{\operatorname{Ext}}_\Lambda^-(M,N) \to 0$$

of H-modules, where $\mathcal{P}(M,N)$ denotes all the homomorphisms from M to N factoring through a projective module, and $\widehat{\operatorname{Ext}}_\Lambda^-(M,N) = \oplus_{i\leq -1}\widehat{\operatorname{Ext}}_\Lambda^i(M,N)$.

Let $\mathfrak{m}_{\mathrm{gr}}$ denote the two-sided ideal in H generated by $\{\operatorname{rad} H^0, H^{\geq 1}\}$, which is the unique maximal graded ideal in H. An H-module X is $\mathfrak{m}_{\mathrm{gr}}$-*torsion* if any element x in X is annihilated by some power of $\mathfrak{m}_{\mathrm{gr}}$. Then the end terms in the above exact sequence $(*)$ are $\mathfrak{m}_{\mathrm{gr}}$-torsion modules. The next result is a crucial observation for this section.

Lemma 3.2. *For any maximal ideal $\mathfrak{p}$ in $\operatorname{MaxSpec} H$ with $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$,*

$$\operatorname{Ext}_\Lambda^*(M,N)_\mathfrak{p} \simeq \widehat{\operatorname{Ext}}_\Lambda^*(M,N)_\mathfrak{p}$$

for all modules M in ${}^\perp\Lambda$ and N in $\operatorname{mod}\Lambda$.

Proof. Localisation at a prime is an exact functor, so that using the sequence $(*)$ it is enough to show any $\mathfrak{m}_{\mathrm{gr}}$-torsion H-module localised at $\mathfrak{p}$ is zero for $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$.

Let X be an $\mathfrak{m}_{\mathrm{gr}}$-torsion H-module, and let $\mathfrak{p}$ be in $\operatorname{MaxSpec} H$ with $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$. Let a be in $\mathfrak{m}_{\mathrm{gr}} \setminus \mathfrak{p}$, and let x be in X. Then $a^t x = 0$ for some $t \geq 0$, so that $\frac{x}{s} = \frac{a^t x}{a^t s} = 0$ for any $\frac{x}{s}$ in $X_\mathfrak{p}$. Hence $X_\mathfrak{p} = (0)$, and this completes the proof. $\square$

Next we show that $V_H(M_\eta \otimes_\Lambda M) = V_H(\langle \eta \rangle) \cap V_H(M)$ for any homogeneous element η of positive degree in H and any Λ-module M in $\operatorname{mod}\Lambda$.

Proposition 3.3. *Let η be a homogeneous element of positive degree in H. Then*

$$V_H(M_\eta \otimes_\Lambda M) = V_H(\langle \eta \rangle) \cap V_H(M).$$

In particular, $V_H(L_\eta) = V_H(\langle \eta \rangle)$.

Proof. If $V_H(\langle \eta \rangle) \cap V_H(M)$ is trivial, there is nothing to prove. So, assume that $V_H(\langle \eta \rangle) \cap V_H(M)$ is non-trivial. Let $n>0$ be the degree of η.

When Λ is Gorenstein the syzygies $\Omega_\Lambda^m(X)$ for a Λ-module X are in ${}^\perp\Lambda$ for m at least the injective dimension of Λ. Since the varieties of $M_\eta \otimes_\Lambda M$ and $M_\eta \otimes_\Lambda \Omega_\Lambda^m(M)$ are the same, for the latter is a syzygy of the former, we can assume that M is in ${}^\perp\Lambda$ and therefore $M_\eta \otimes_\Lambda M$ is also in ${}^\perp\Lambda$.

The element η gives rise to the exact sequence

$$0 \to \Lambda \to M_\eta \to \Omega^{n-1}_{\Lambda^e}(\Lambda) \to 0$$

of Λ^e-modules. Tensoring this sequence with M and applying $\mathrm{Hom}_\Lambda(-, \Lambda/\mathfrak{r})$ induces the long exact sequence

$$\cdots \to \widehat{\mathrm{Ext}}^{i-1}_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\partial} \widehat{\mathrm{Ext}}^{i}_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \\ \widehat{\mathrm{Ext}}^{i}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \widehat{\mathrm{Ext}}^{i}_\Lambda(M, \Lambda/\mathfrak{r}) \to \cdots$$

It is easy to see that ∂ is multiplication by η when $\widehat{\mathrm{Ext}}^{i}_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\mathfrak{r})$ is identified with $\widehat{\mathrm{Ext}}^{n+i}_\Lambda(M, \Lambda/\mathfrak{r})$. This yields the exact sequence

$$\widehat{\theta}\colon 0 \to \widehat{\mathrm{Ext}}^{*+n-1}_\Lambda(M, \Lambda/\mathfrak{r})/\eta\,\widehat{\mathrm{Ext}}^{*-1}_\Lambda(M, \Lambda/\mathfrak{r}) \to \\ \widehat{\mathrm{Ext}}^{*}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to \mathrm{Ker}(\ \cdot\eta)|_{\widehat{\mathrm{Ext}}^{*}_\Lambda(M,\Lambda/\mathfrak{r})} \to 0.$$

Choose a maximal ideal $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$ lying over $\langle \eta, \mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})\rangle$. Suppose that $\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})$ is not contained in $\mathfrak{p}$. Then $\mathrm{Ext}^*_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})_\mathfrak{p} = (0)$ and by Lemma 3.2 the localisation $\widehat{\mathrm{Ext}}^{*}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})_\mathfrak{p} = (0)$. From the exact sequence $\widehat{\theta}$ we infer that $\widehat{\mathrm{Ext}}^{*}_\Lambda(M, \Lambda/\mathfrak{r})_\mathfrak{p} \simeq \eta\,\widehat{\mathrm{Ext}}^{*}_\Lambda(M, \Lambda/\mathfrak{r})_\mathfrak{p}$. Since $\mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})_\mathfrak{p} \simeq \widehat{\mathrm{Ext}}^{*}_\Lambda(M, \Lambda/\mathfrak{r})_\mathfrak{p}$ is a finitely generated $H_\mathfrak{p}$-module and η is in $\mathfrak{p}H_\mathfrak{p}$, the Nakayama Lemma implies that $\widehat{\mathrm{Ext}}^{*}_\Lambda(M, \Lambda/\mathfrak{r})_\mathfrak{p} = (0)$. Using Lemma 3.2 again $\mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})_\mathfrak{p} = (0)$, and since $\mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ is a finitely generated H-module, the annihilator $\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ is not contained in $\mathfrak{p}$. This is a contradiction to the choice of $\mathfrak{p}$, and hence $\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})$ is contained in $\mathfrak{p}$. It follows that $V_H(\langle\eta\rangle) \cap V_H(M) \subseteq V_H(M_\eta \otimes_\Lambda M)$. The opposite inclusion is proved in Proposition 3.1, and this completes the proof of the proposition. □

The corresponding proof for a group ring of a finite group uses rank varieties and reduction to elementary abelian subgroups, so the above proof also gives an alternative proof in that case.

Using the above result it is easy to show that any homogeneous variety occurs as a variety of a module.

Theorem 3.4. *Let $\mathfrak{a}$ be any homogeneous ideal in H. Then there exists a module M in $\mathrm{mod}\,\Lambda$ such that $V_H(M) = V_H(\mathfrak{a})$.*

Proof. Suppose that $\mathfrak{a} = \langle \eta_1, \eta_2, \ldots, \eta_t\rangle$ for some homogeneous elements $\{\eta_1, \eta_2, \ldots, \eta_t\}$ in H. Then $V_H(M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda \Lambda/\mathfrak{r}) = V_H(\langle \eta_1, \eta_2, \ldots, \eta_t\rangle)$ by Proposition 3.3. □

Note that the module constructed for the given closed homogeneous variety need not be indecomposable.

4. Periodic modules

A group ring of a finite group over a field is a symmetric algebra, so that the Auslander-Reiten translate τ is isomorphic to Ω^2. Having a good supply of τ-periodic modules gives information about the shape of the stable Auslander-Reiten quiver in this case. As τ-periodic and Ω-periodic modules coincide here and the

Ω-periodic modules are known to be controlled by the support varieties, the theory of support varieties can be used to (re)prove Webb's theorem (see [23]).

In this section Λ is a selfinjective algebra. We take a closer look at the construction and characterization of periodic modules. In particular we show a generalization of Webb's theorem for a finite dimensional selfinjective algebra with a Nakayama functor which is of finite order for each indecomposable module. By a periodic module we mean throughout an Ω-periodic module.

For group rings a module is periodic if and only if the variety is a line. In fact the proof in our setting is the same as in this case, so that we leave the details to the reader. Recall the following result from [5] (see [4, Proposition 5.10.2]).

Proposition 4.1. *Suppose M is an indecomposable periodic module in $\operatorname{mod}\Lambda$, of period n. Then the set of nilpotent elements in $\operatorname{Ext}^*_\Lambda(M,M)$ forms an ideal, denoted by $\mathcal{N}_M$. Moreover as vector spaces we have*

$$\operatorname{Ext}^*_\Lambda(M,M) \cong \mathcal{N}_M \oplus k[x]$$

where x is a non-nilpotent element of degree n.

As in the group ring case one has the following consequence for selfinjective algebras, observing that the assumptions given are sufficient (see [4, Proposition 5.10.2]).

Proposition 4.2. *Suppose that Λ and H satisfy* **Fg1**. *Assume that M is an indecomposable periodic module in $\operatorname{mod}\Lambda$. If $\operatorname{Ext}^*_\Lambda(M,M)$ is a finitely generated H-module, then the variety of M is a line.*

Again, in an analogous way to the group ring case, one can show that if the variety of a module is a line, then the module is a direct sum of periodic modules and a projective module (see [4, Theorem 5.10.4] and [9]). For this we need to assume that Λ is selfinjective and that Λ and H satisfy **Fg1** and **Fg2**, and we keep these assumptions for the rest of this section.

Theorem 4.3. *If the variety of a module M in $\operatorname{mod}\Lambda$ is a line, or equivalently, if the complexity of M is 1, then M is a direct sum of periodic modules and a projective module.*

Proof. Suppose that the variety of M is a line. Choose a homogeneous element η in H^n such that $V_H(\langle\eta\rangle) \cap V_H(M)$ is trivial. We have an exact sequence

$$0 \to \Lambda \to M_\eta \to \Omega^{n-1}_{\Lambda^e}(\Lambda) \to 0.$$

Tensoring by M gives the exact sequence

$$0 \to M \to M_\eta \otimes_\Lambda M \to \Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \to 0.$$

We know from Proposition 3.3 that

$$V_H(M_\eta \otimes_\Lambda M) = V_H(\langle\eta\rangle) \cap V_H(M).$$

So by Theorem 1.5, the module $M_\eta \otimes_\Lambda M$ has finite projective dimension. Since Λ is selfinjective, the module is in fact projective. This implies that

$$\begin{aligned} M &\simeq \Omega^1_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M) \oplus \text{projective} \\ &\simeq \Omega^1_\Lambda(\Omega^{n-1}_\Lambda(M)) \oplus \text{projective} \\ &\simeq \Omega^n_\Lambda(M) \oplus \text{projective} \end{aligned}$$

and the result follows. $\square$

Following the treatment in the group case, a homogeneous element η in $\mathrm{HH}^*(\Lambda)$ is said to *generate the periodicity* for a periodic module M if $V_H(\langle\eta\rangle) \cap V_H(M)$ is trivial. Thus we have yet another analogous result (see [4, Corollary 5.10.6]).

Proposition 4.4. *If H is generated as a subalgebra of $\mathrm{HH}^*(\Lambda)$ by elements $\eta_1, \eta_2, \ldots, \eta_t$ in degrees n_1, n_2,..., n_t and M is an indecomposable periodic module in $\operatorname{mod}\Lambda$, then the period of M divides one of the n_i.*

The above results can be used to construct periodic module(s) W with a variety contained in the variety of any given non-projective indecomposable module M in $\operatorname{mod}\Lambda$. If M is already periodic, we can choose W equal to M. Suppose M is not periodic, or equivalently $\dim V_H(M) > 1$. Let $\{\eta_1, \eta_2, \ldots, \eta_s\}$ be a set of homogeneous generators for $A_H(M, \Lambda/\mathfrak{r})$. Then there exist homogeneous elements $\{\eta_{s+1}, \ldots, \eta_t\}$ in H such that the height of the ideal $\langle\eta_1, \ldots, \eta_s, \eta_{s+1}, \ldots, \eta_t\rangle$ is $\dim V_H - 1$. Hence the variety of the module $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda \Lambda/\mathfrak{r}$ has dimension one. By the above this is the variety of a direct sum of periodic modules and a projective module, so that we can, for example, choose a non-projective indecomposable direct summand of $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda \Lambda/\mathfrak{r}$ to be W. Next we use this to prove a generalization of Webb's theorem.

In [8] a finite dimensional symmetric algebra Λ is said to have *enough* periodic modules if for any nonzero module M in $\underline{\operatorname{mod}}\,\Lambda$ there is an Ω-periodic module V such that $\underline{\operatorname{Hom}}_\Lambda(V, M) \neq (0)$. Recall that $\underline{\operatorname{Hom}}_\Lambda(C, A) \simeq D\operatorname{Ext}^1_\Lambda(A, \tau^{-1}C)$ for any artin algebra and that $\tau^{-1} \simeq \Omega^{-2}$ for a finite dimensional symmetric algebra. So, for a symmetric algebra, showing that $\underline{\operatorname{Hom}}_\Lambda(V, M)$ is nonzero and showing that $\operatorname{Ext}^1_\Lambda(M, V')$ is nonzero for some periodic modules V and V' are equivalent. Using support varieties we show next that finite dimensional selfinjective algebras (satisfying **Fg1** and **Fg2**) have enough periodic modules.

Theorem 4.5. *Suppose that $\dim H \geq 1$. Let V be a closed homogeneous variety with $\dim V \geq 1$. Then there exists an Ω-periodic Λ-module W such that $\operatorname{Ext}^1_\Lambda(W, M) \neq (0)$ for all indecomposable Λ-modules M in $\operatorname{mod}\Lambda$ with $V_H(M) = V$.*

Proof. Suppose that the closed homogeneous variety V is given by a homogeneous ideal $\mathfrak{b}$. Choose a homogeneous ideal $\mathfrak{a}$ of height $\dim H - 1$ lying over $\mathfrak{b}$. Let $\{\eta_1, \ldots, \eta_t\}$ be homogeneous generators for the ideal $\mathfrak{a}$. Let $W = (M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t})^* \otimes_\Lambda \Lambda/\mathfrak{r}$. Recall that for all $i \geq 0$

$$\operatorname{Ext}^i_\Lambda(B \otimes_\Lambda A, C) \simeq \operatorname{Ext}^i_\Lambda(A, \operatorname{Hom}_\Lambda(B, C)) \simeq \operatorname{Ext}^i_\Lambda(A, B^* \otimes_\Lambda C)$$

for A and C in $\operatorname{mod}\Lambda$ and B in $\operatorname{mod}\Lambda^e$, where B is projective as a left and as a right Λ-module. Since $(M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t})^*$ is projective as a left and as a right Λ-module, we have that

$$V_H(W) = V_H(W, \Lambda/\mathfrak{r}) = V_H(\Lambda/\mathfrak{r}, M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda \Lambda/\mathfrak{r}) = V_H(\mathfrak{a})$$

so that the variety of W is a line and therefore W is a direct sum of periodic modules and a projective module by Theorem 4.3. Now redefine W to be the non-projective part of this direct sum.

Let M be an indecomposable Λ-module with $V_H(M) = V$. Then $V_H(M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M)$ is also equal to $V_H(\mathfrak{a})$, so in particular $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M$ is a non-projective (non-injective) module. Therefore

$$(0) \neq \operatorname{Ext}^1_\Lambda(\Lambda/\mathfrak{r}, M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M) \simeq \operatorname{Ext}^1_\Lambda(W, M).$$

The claim follows from this. $\square$

This gives the promised generalization of Webb's theorem.

Theorem 4.6. *Suppose that the Nakayama functor is of finite order on any indecomposable module in* $\operatorname{mod}\Lambda$. *Then the tree class of a component of the stable Auslander-Reiten quiver of* Λ *is one of the following: a finite Dynkin diagram* ($\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_{6,7,8}$)*, an infinite Dynkin diagram of the type* $\mathbb{A}_\infty$, $\mathbb{D}_\infty$, $\mathbb{A}_\infty^\infty$ *or a Euclidean diagram.*

Proof. The Auslander-Reiten translate for a finite dimensional selfinjective algebra is the composition of the Nakayama functor $\mathcal{N}$ with the second syzygy Ω^2, where these two functors commute. Hence, by assumption an indecomposable module is τ-periodic if and only if the module is Ω-periodic. We use this and the above result to construct subadditive functions on the tree class of the stable components of Λ.

Let $\mathcal{C}$ be a component of the stable Auslander-Reiten quiver Γ_s. In Theorem 3.7 in [21] it is shown that all indecomposable modules in $\mathcal{C}$ have the same variety, say V. If $\dim V = 1$, then all modules in the component are periodic. Then the tree class is a finite Dynkin diagram or $\mathbb{A}_\infty$ by [15].

If $\dim V \geq 2$, then no module in the component $\mathcal{C}$ is periodic, and by the previous result there exists a periodic module W such that $\operatorname{Ext}^1_\Lambda(W, M) \neq (0)$ for all M in $\mathcal{C}$. We can assume without loss of generality that W is τ-periodic of period one, that is, $W \simeq \tau W$. Define $f(M) = \dim_k \operatorname{Ext}^1_\Lambda(W, M)$ for any M in $\mathcal{C}$. Then

$$\begin{aligned} f(\tau M) &= \dim_k \operatorname{Ext}^1_\Lambda(W, \tau M) \\ &= \dim_k \operatorname{Ext}^1_\Lambda(\tau^{-1} W, M) \\ &= \dim_k \operatorname{Ext}^1_\Lambda(W, M) = f(M). \end{aligned}$$

Now this gives rise to a subadditive function on the tree class of $\mathcal{C}$, and it follows from [15] that the tree class of $\mathcal{C}$ is one of diagrams listed above. $\square$

Remark. Note that when the dimension of the variety of a module M in the component $\mathcal{C}$ in the previous proof is at least two, then the function constructed is actually additive. To see this, observe that none of the indecomposable nonprojective direct summands of W can lie in $\mathcal{C}$ as their variety has dimension one. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an almost split sequence in $\mathcal{C}$, then the sequence

$$0 \to \operatorname{Ext}^1_\Lambda(W, A) \xrightarrow{f^*} \operatorname{Ext}^1_\Lambda(W, B) \xrightarrow{g^*} \operatorname{Ext}^1_\Lambda(W, C) \to 0$$

is exact as the kernel of f^* and the cokernel of g^* are isomorphic to $\delta^*(\Omega^i_\Lambda(W))$ for $i = 0$ and $i = 1$, respectively. Here δ^* is given by the exact sequence of the functor

$$0 \to \operatorname{Hom}_\Lambda(-, A) \to \operatorname{Hom}_\Lambda(-, B) \to \operatorname{Hom}_\Lambda(-, C) \to \delta^* \to 0,$$

and $\delta^*(X) \neq (0)$ if and only if C is a direct summand of X.

We thank Rolf Farnsteiner for pointing out the following corollary of Theorem 4.6. The proof rests on the fact that the Nakayama automorphism for a finite dimensional Hopf algebra Λ has finite order dividing $2 \dim_k \Lambda$ (see [11, Lemma 1.5]), and that a cocommutative finite dimensional Hopf algebra satisfies the conditions **Fg1** and **Fg2** ([13]).

Corollary 4.7. *Let* Λ *be a finite dimensional cocommutative Hopf algebra. Then the tree class of a component of the stable Auslander-Reiten quiver of* Λ *is one of the following: a finite Dynkin diagram* ($\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_{6,7,8}$)*, an infinite Dynkin diagram of the type* $\mathbb{A}_\infty$, $\mathbb{D}_\infty$, $\mathbb{A}_\infty^\infty$ *or a Euclidean diagram.*

5. Representation type and complexity

Here we give a brief discussion on relationships between representation type and complexity of modules over a selfinjective algebra Λ.

It was first observed by Heller in [16] that if Λ is of finite representation type, then all the indecomposable non-projective modules are Ω-periodic and therefore all of complexity one. The converse of this statement is not true, since there exist finite dimensional preprojective algebras of wild representation type with Λ being a periodic Λ^e-module (and consequently all indecomposable non-projective modules are periodic and of complexity one).

If Λ is of tame representation type, then it is shown by Rickard in [20] that all indecomposable non-projective modules have complexity at most two. By using the same example as above the converse is also not true here. However a partial converse is known, and we include a proof here for completeness.

Proposition 5.1. *Suppose Λ and H satisfy* **Fg1** *and* **Fg2** *with* $\dim H \geq 2$. *Then Λ is of infinite representation type and Λ has an infinite number of indecomposable periodic modules lying in infinitely many different components of the stable Auslander-Reiten quiver.*

Proof. Suppose that $\dim H = d \geq 2$. Then by the Noether Normalisation Theorem there exists a polynomial ring $k[x_1, \dots, x_d]$ generated by homogeneous elements x_i in H of degrees $n_1, n_2, \dots, n_d$, respectively, over which H is a finitely generated module. Choose natural numbers r and s with $r+s$ minimal such that x_1^r and x_2^s have the same degree. Let $\eta_\alpha = \alpha_1 x_1^r + \alpha_2 x_2^s$ for $\alpha = (\alpha_1, \alpha_2)$ in $\mathbb{P}^1(k)$. Consider the modules $C_\alpha = M_{\eta_\alpha} \otimes_\Lambda M_{x_3} \otimes_\Lambda \cdots \otimes_\Lambda M_{x_d} \otimes_\Lambda \Lambda/\mathfrak{r}$. Then $V_H(C_\alpha) = V_H(\langle \eta_\alpha, x_3, \dots, x_d \rangle)$. It is easy to show that η_α is an irreducible element in H when α is different from $(1,0)$ and $(0,1)$, so that $V_H(C_\alpha)$ is an irreducible variety ($\sqrt{\langle \eta_\alpha, x_3, \dots, x_d \rangle}$ is a prime ideal). If X is any indecomposable non-projective direct summand of C_α, then $V_H(X)$ is a closed subvariety of $V_H(C_\alpha)$. We infer that $V_H(X) = V_H(C_\alpha)$. We can then construct a $\mathbb{P}^1(k)$-family of indecomposable periodic modules $\{X_\alpha\}$ by choosing for each α in $\mathbb{P}^1(k)$ an indecomposable non-projective direct summand X_α of C_α. Since $V_H(X_\alpha) \neq V_H(X_{\alpha'})$ for $\alpha \neq \alpha'$ in $\mathbb{P}^1(k)$, the claim follows as the field is infinite. ☐

6. The variety of an indecomposable module is connected

Throughout this section we assume that Λ satisfies **Fg1** and **Fg2** for some graded subalgebra H of $\mathrm{HH}^*(\Lambda)$. This section is devoted to showing that the variety of an indecomposable module is connected. Here $\mathrm{id}X$ denotes the injective dimension of a module X.

To show the result mentioned above we need preliminary results, and we first give a sufficient condition for the vanishing of all high enough extension groups between two modules.

Proposition 6.1. *Let M and N be two Λ-modules in* $\operatorname{mod}\Lambda$. *Suppose that $V_H(M) \cap V_H(N)$ is trivial. Then $\operatorname{Ext}^i_\Lambda(M, N) = (0)$ for all $i > \mathrm{id}\Lambda_\Lambda$.*

In particular, if Λ is selfinjective, then $\operatorname{Ext}^i_\Lambda(M, N) = (0)$ for all $i \geq 1$.

Proof. Suppose that $A_H(M, M)$ is generated by $\{\eta_1, \dots, \eta_t\}$. By Theorem 2.3 the module M is a direct summand of $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M$. Therefore $\operatorname{Ext}^i_\Lambda(M, N)$

is a direct summand of $\operatorname{Ext}^i_\Lambda(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda M, N)$. We have that

$$\operatorname{Ext}^i_\Lambda(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda M, N)\simeq \operatorname{Ext}^i_\Lambda(M,(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t})^*\otimes_\Lambda N).$$

Moreover

$$\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{r},(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t})^*\otimes_\Lambda N)\simeq \operatorname{Ext}^i_\Lambda(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda \Lambda/\mathfrak{r}, N)$$

so that the variety of $(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t})^*\otimes_\Lambda N$ is contained in the intersection of the varieties of $M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda \Lambda/\mathfrak{r}$ and of N. The variety of $M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t}\otimes_\Lambda \Lambda/\mathfrak{r}$ is contained in that of M, so that by assumption the variety of $(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t})^*\otimes_\Lambda N$ is trivial.

By Theorem 1.5 Λ is Gorenstein and $\operatorname{id}(M_{\eta_1}\otimes_\Lambda\cdots\otimes_\Lambda M_{\eta_t})^*\otimes_\Lambda N\leq \operatorname{id}\Lambda_\Lambda$, so that $\operatorname{Ext}^i_\Lambda(M,N)=(0)$ for all $i>\operatorname{id}\Lambda_\Lambda$.

Since $\operatorname{id}\Lambda_\Lambda=0$ when Λ is selfinjective, the last claim is clear. □

In a similar way to the group case we show that the variety of an indecomposable module is connected. The next result is a crucial step in the proof and the assumptions **Fg1** and **Fg2** are not needed for this.

Lemma 6.2. *Given two homogeneous elements η_1 and η_2 of positive degree in $\operatorname{HH}^*(\Lambda)$, there is an exact sequence*

$$0\to\Omega^n_{\Lambda^e}(M_{\eta_1})\to M_{\eta_2\eta_1}\oplus F\to M_{\eta_2}\to 0$$

of Λ^e-modules, where F is projective and n is the degree of η_2.

Proof. Given the map $\eta_1\colon\Omega^m_{\Lambda^e}(\Lambda)\to\Lambda$ we obtain the exact sequence

$$\Theta\colon 0\to\Omega^1_{\Lambda^e}(M_{\eta_1})\oplus Q'\to\Omega^m_{\Lambda^e}(\Lambda)\oplus Q\xrightarrow{\left(\begin{smallmatrix}\eta_1\\ f\end{smallmatrix}\right)}\Lambda\to 0$$

for some projective Λ^e-modules Q and Q'. Using that $\eta_2\eta_1$ is given by the composition $\eta_2\Omega^n_{\Lambda^e}(\eta_1)$, this gives rise to the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, equal] & M_{\eta_2\eta_1} \arrow[r] \arrow[d] & \Omega^{m+n-1}_{\Lambda^e}(\Lambda) \arrow[r] \arrow[d, "\Omega^{n-1}_{\Lambda^e}(\eta_1)"] & 0\\
0 \arrow[r] & \Lambda \arrow[r] & M_{\eta_2} \arrow[r] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r] & 0
\end{tikzcd}$$

By applying the Horseshoe Lemma to the sequence Θ we know that we can add a projective Λ^e-module F to make the map $\Omega^{m+n-1}_{\Lambda^e}(\Lambda)\oplus F\xrightarrow{\left(\begin{smallmatrix}\Omega^{n-1}_{\Lambda^e}(\eta_1)\\ f'\end{smallmatrix}\right)}\Omega^{n-1}_{\Lambda^e}(\Lambda)$ onto with kernel isomorphic to $\Omega^n_{\Lambda^e}(\Lambda)$. By adding F also to $M_{\eta_2\eta_1}$ and making the necessary adjustments to the maps, we obtain the desired sequence. □

By combining Proposition 2.2, Proposition 6.1 and Lemma 6.2 and making the appropriate modifications to the proof in the group ring case we obtain that the variety of an indecomposable is connected (see [7], [4, Theorem 5.12.1]).

Theorem 6.3. *Let M be in $\operatorname{mod}\Lambda$. If $V_H(M)=V_1\cup V_2$ for some homogeneous non-trivial varieties V_1 and V_2 with $V_1\cap V_2$ trivial, then $M\simeq M_1\oplus M_2$ with $V_H(M_1)=V_1$ and $V_H(M_2)=V_2$.*

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