THE REPRESENTATION-FINITE SYMMETRIC ALGEBRAS WITH LIFTABLE SIMPLE MODULES

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1. INTRODUCTION

In [8] G. J. Janusz showed that a p -block algebra Λ of a finite group G of finite representation type over a splitting field F with characteristic $p > 0$ is uniserial if and only if every simple Λ -module M can uniquely be lifted to a simple $\hat{\Lambda} \otimes_R S$ -module, where R is a complete discrete rank one valuation ring with maximal ideal πR , residue class field $F = R/\pi R$, and quotient field S with characteristic zero, and where $\hat{\Lambda}$ is an RG -block such that $\Lambda \cong \hat{\Lambda} \otimes_R F$. If Λ has at least two non-isomorphic simple modules, then using standard results of the theory of blocks with cyclic defect groups [5], p. 302, and J. A. Green's work [6] it can easily be seen that this liftability condition is equivalent to the requirements that $\text{Ext}_\Lambda^2(M, M) = (0) = \text{Ext}_\Lambda^1(M, M)$ for every simple Λ -module M .

These Ext-conditions also imply the unique liftability of the simple modules M of an arbitrary finite-dimensional symmetric algebra Λ over a commutative field K in the sense of Auslander, Ding and the second author [1]. It is also known that $\text{Ext}_\Lambda^2(M, M) = (0)$ implies $\text{Ext}_\Lambda^1(M, M) = (0)$. In order to generalize Janusz' theorem to the theory of finite dimensional symmetric algebras Λ over an algebraically closed field K it therefore suffices to classify all such connected algebras Λ of finite representation type satisfying $\text{Ext}_\Lambda^2(M, M) = (0)$ for all simple Λ -modules M . In this article we give such a classification.

In order to state it the following definitions and notations are needed. Let $J = J(\Lambda)$ denote the radical of Λ and let $\text{soc}(X)$ denote the socle of a module X . If P_M denotes the projective cover of the simple Λ -module M , then $H_M = P_M J / \text{soc}(P_M)$ is called the *heart* of P_M . The K -algebra Λ is said to have multiplicity $c(\Lambda) = 1$ if $P_M J$ is a multiplicity-free Λ -module for every simple Λ -module M . Now we can state our main result.

Theorem 1.1. *Let Λ be a connected, representation-finite, symmetric algebra over the algebraically closed field K with at least two non-isomorphic simple Λ -modules. Then the following conditions are equivalent:*

- (1) $\text{Ext}_\Lambda^2(M, M) = (0)$ for every simple Λ -module M .
- (2) *Either Λ is uniserial or $c(\Lambda) = 1$ and the hearts H_M of all indecomposable projective Λ -modules P_M are indecomposable Λ -modules.*

This result is proved in section 3. The non uniserial algebras Λ with multiplicity $c(\Lambda) = 1$ satisfying condition (2) are classified by Theorem 4.1 in section 4. This classification is based on the classification of all representation-finite connected basic symmetric algebras of Dynkin type D_n by Bretscher, Läser and Riedtmann [4], and Roggen's description [11] of the quivers and relations of the basic symmetric algebras of Dynkin types E_6 , E_7 and E_8 .

In section 1 the relation between the module structure of the heart H_M of the projective cover P_M of the simple Λ -module M and the condition $\text{Ext}_\Lambda^2(M, M) =$

(0) is studied. These subsidiary results are the main new ingredients of the proof of Theorem 1.1.

Concerning our terminology and notation we refer to the books by Auslander, Reiten, Smalø [2] and Feit [5].

2. HEARTS OF INDECOMPOSABLE PROJECTIVES

This section is devoted to showing that if $\text{Ext}_\Lambda^2(M, M) = (0)$ for a simple module M over a weakly symmetric artin algebra Λ , then the heart H_M of the projective cover P_M of M is indecomposable. This is a special case of a more general result for artin algebras stating that for an indecomposable projective module P with simple socle, then if $\text{Ext}_\Lambda^2(P/PJ, \text{soc}(P)) = (0)$, the heart $PJ/\text{soc}(P)$ is zero or indecomposable. For weakly symmetric finite dimensional algebras over an algebraically closed field we prove a converse statement.

We first discuss the general case for artin algebras.

Proposition 2.1. *Let Λ be an artin algebra and suppose that P is an indecomposable projective Λ -module with simple socle. If $\text{Ext}_\Lambda^2(P/PJ, \text{soc}(P)) = (0)$ and $PJ/\text{soc}(P)$ is nonzero, then the heart $PJ/\text{soc}(P)$ is indecomposable.*

Proof. Assume that $P/PJ = M$ and $\text{soc}(P) = M'$ for some simple Λ -modules M and M' . We have that $\text{Ext}_\Lambda^2(M, M') \simeq \text{Ext}_\Lambda^1(PJ, M')$. Since $\text{soc}(PJ) = M'$, we have that $I(PJ) = I(M')$, where $I(X)$ denotes the injective hull of the Λ -module X . Moreover, we have the exact sequence

$$0 \rightarrow (PJ, M') \rightarrow (PJ, I(M')) \rightarrow (PJ, I(M')/M') \rightarrow \text{Ext}_\Lambda^2(M, M') \rightarrow 0.$$

Suppose that $PJ/M' = X \oplus Y$, where both X and Y are nonzero. Then consider the following composition of maps

$$\alpha: PJ \rightarrow PJ/M' = X \oplus Y \rightarrow X \rightarrow I(M')/M',$$

where all the maps are the canonical projections or inclusions. We want to show that the homomorphism α does not factor through the homomorphism $\pi: I(M') \rightarrow I(M')/M'$ and therefore represents a nonzero element in $\text{Ext}_\Lambda^2(M, M')$.

Suppose that there is a map $\beta: PJ \rightarrow I(M')$ such that $\alpha = \pi \circ \beta$. Let x be in PJ such that $x + M'$ is in $X \setminus XJ$. There exists an element r_x in J such that $x \cdot r_x$ is a nonzero element in $\text{soc}(PJ) = M'$. Moreover, $\alpha(x) = x + M' = \pi \circ \beta(x)$, so that $\beta(x) = x + m$ for some m in M' . Therefore $\beta(xr_x) = \beta(x)r_x = (x + m)r_x = xr_x \neq 0$. Hence, $\beta(s) \neq 0$ for any element s in $\text{soc}(PJ) = M'$.

Let y be in PJ such that $y + M'$ is in $Y \setminus YJ$. There exists an element r_y in J such that $y \cdot r_y$ is a nonzero element in $\text{soc}(PJ) = M'$. Hence $\beta(yr_y) \neq 0$ and $\beta(y) \neq 0$. If $\beta(y)$ is in M' , then $0 = \beta(y)r_y = \beta(yr_y)$. This is impossible, so that $\pi \circ \beta(y) \neq 0$. Since $\alpha(y) = \pi \circ \beta(y) = 0$, we have a contradiction. Therefore the map α defined as above does not factor through the homomorphism $\pi: I(M') \rightarrow I(M')/M'$. From the above exact sequence follows that $\text{Ext}_\Lambda^2(M, M') \neq (0)$. This completes the proof. $\square$

For weakly symmetric artin algebras all indecomposable projectives have simple socles. In this case we have the following corollary which we apply in section 3.

Corollary 2.2. *Let Λ be a weakly symmetric artin algebra and M a nonprojective simple Λ -module. If $\text{Ext}_\Lambda^2(M, M) = (0)$, the heart H_M is indecomposable. In particular, the length of P_M is at least 3.*

Proof. If H_M were zero, then we would have an exact sequence of the form $0 \rightarrow M \rightarrow P_M \rightarrow M \rightarrow 0$. Therefore $\text{Ext}_\Lambda^1(M, M) \cong \text{Ext}_\Lambda^2(M, M) = 0$, a contradiction to M not being projective. Thus H_M is nonzero. The claim follows from the previous lemma, and the length of P_M is at least 3. $\square$

In view of the above corollary it is natural to ask if $\text{Ext}_\Lambda^1(M, M) = (0)$ and H_M being indecomposable for a simple module M over a weakly symmetric artin algebra Λ implies that $\text{Ext}_\Lambda^2(M, M) = (0)$. This is not the case as the following example shows.

Let Q be the following quiver

and K a field. Let I be the ideal in the path algebra KQ generated by the elements $\{\alpha_{i-1}\beta_i - \beta_{i+1}\alpha_i, \alpha_{i+2}\alpha_{i+1}\alpha_i, \beta_{i+2}\beta_{i+1}\beta_i\}_{i=1}^3$, where the indices are calculated modulo 3. Then it is easy to see that the algebra $\Lambda = KQ/I$ is weakly symmetric, all the hearts of the indecomposable projective Λ -modules are indecomposable and $\text{Ext}_\Lambda^1(M, M) = (0)$ for all simple Λ -modules M . As $\text{Ext}_\Lambda^2(M, M) \neq (0)$ for all simple Λ -modules M , we see that the converse of Corollary 2.2 is by no means true.

A partial converse of Corollary 2.2 is the following.

Proposition 2.3. *Let Λ be a weakly symmetric finite dimensional algebra over a field K . Let M be a simple Λ -module such that $\text{End}_\Lambda(H_M) = K$ and $\text{Ext}_\Lambda^1(M, M) = (0)$. Then $\text{Ext}_\Lambda^2(M, M) = (0)$.*

Proof. Let M be a simple Λ -module such that $\text{End}_\Lambda(H_M) = K$ and $\text{Ext}_\Lambda^1(M, M) = (0)$. Since Λ is selfinjective, we have that

$$\text{Ext}_\Lambda^2(M, M) \simeq \text{Ext}_\Lambda^1(P_M J, M) \simeq \underline{\text{Hom}}_\Lambda(P_M J, P_M/M).$$

We want to show that $\underline{\text{Hom}}_\Lambda(P_M J, P_M/M) = (0)$. Let $\alpha: P_M J \rightarrow P_M/M$ be a Λ -homomorphism. Since $\text{Ext}_\Lambda^1(M, M) = (0)$, the simple module M is neither a composition factor of $P_M J/P_M J^2$ nor of $\text{soc}(P_M/M)$. As $\alpha(P_M J)/\alpha(P_M J)J$ is a factor module of $P_M J/P_M J^2$, it follows that $\alpha(P_M J) \subset P_M J/M$. Since $\alpha(\text{soc}(P_M J)) \subset \text{soc}(P_M)/M$, we have that $\text{soc}(P_M J) = M$ is contained in $\ker \alpha$. Therefore every homomorphism $\alpha: P_M J \rightarrow P_M/M$ induces an endomorphism $\bar{\alpha}$ of H_M . It follows that $\varphi: \text{End}_\Lambda(H_M) \rightarrow \underline{\text{Hom}}_\Lambda(P_M J, P_M/M)$ given by $\varphi(\beta) = \nu \circ \beta \circ \pi$ for β in $\text{End}_\Lambda(H_M)$ is an isomorphism, where $\pi: P_M J \rightarrow H_M$ and $\nu: H_M \rightarrow P_M/M$ are the natural homomorphisms. Since $\nu \circ \pi: P_M J \rightarrow P_M/M$ factors through the natural projection $P_M \rightarrow P_M/M$ and $\text{End}_\Lambda(H_M) = K$, it follows that any homomorphism $\alpha: P_M J \rightarrow P_M/M$ factors through P_M and therefore $\text{Ext}_\Lambda^2(M, M) = (0)$. $\square$

A situation where the heart H_M of P_M for a simple module M satisfies the hypothesis of the previous result is the following. We apply this in section 3.

Corollary 2.4. *Let Λ be a weakly symmetric finite dimensional algebra over an algebraically closed field K . Let M be a simple Λ -module such that $P_M J$ is a multiplicity free Λ -module and the heart H_M of P_M is indecomposable. Then $\text{Ext}_\Lambda^2(M, M) = (0)$.*

Proof. As H_M is indecomposable $E = \text{End}_\Lambda(H_M)$ is a finite dimensional local K -algebra. Hence $E/J(E) \simeq K$. Suppose there is a nonzero element α in $J(E)$. Then α is nilpotent. Hence both $\ker \alpha$ and $\alpha(H_M)$ are nonzero proper submodules of H_M containing together all composition factors of H_M because $\alpha(H_M) \simeq H/\ker \alpha$. As H_M is multiplicity free $\alpha(H_M) \cap \ker \alpha = (0)$. Hence $H_M = \alpha(H_M) \oplus \ker \alpha$, a contradiction to H_M being indecomposable. Thus $\text{End}_\Lambda(H_M) = K$ and the claim follows from the previous result. $\square$

The condition of $\text{Ext}_\Lambda^2(M, M)$ being zero also has come up in work on liftability of modules in [1]. We recall the notion of a lifting of a module from [1].

Let $\varphi: \Gamma \rightarrow \Lambda$ be a homomorphism of rings and let M be a finitely generated Γ -module. Then finitely generated Γ -module L is called a *lifting* of M to Γ if $M \simeq L \otimes_\Gamma \Lambda$ and $\text{Tor}_i^\Gamma(L, \Lambda) = (0)$ for all $i \geq 1$. The Λ -module M is said to be *liftable* to Γ when such a Γ -module L exists.

It is shown in Proposition 1.7 and Proposition 2.6 of [1] that if $\varphi: \Gamma \rightarrow \Lambda$ with Γ and Λ suitably given, then for a Λ -module X the condition $\text{Ext}_\Lambda^2(X, X) = (0)$ implies that X is liftable to Γ . Furthermore, if a Λ -module X is liftable to Γ with $\text{Ext}_\Lambda^1(X, X) = (0)$, then the lifting of X to Γ is unique.

In the following let Λ be an artin algebra. Next we show that if $\text{Ext}_\Lambda^2(M, M) = (0)$ for a simple Λ -module M , then $\text{Ext}_\Lambda^1(M, M) = (0)$. Hence the sufficient condition $\text{Ext}_\Lambda^2(M, M) = (0)$ for liftability of a simple Λ -module M implies uniquely liftability.

Let M be a simple Λ -module. Denote by $\tau_X(Y)$ the trace of X in Y for two Λ -modules X and Y , that is,

$$\tau_X(Y) = \sum_{\varphi \in \text{Hom}_\Lambda(X, Y)} \varphi(X).$$

Let $\{M_1, M_2, \dots, M_{n-1}, M_n = M\}$ be a complete set of nonisomorphic simple Λ -modules and let $Q = \bigoplus_{i=1}^{n-1} P_{M_i}$. For a Λ -module A let $A(M) = A/\tau_Q(A)$. Then it is clear that the only simple occurring as a composition factor in $A(M)$ is M and that $A(M) = (0)$ if and only if M does not occur in the top, A/AJ , of A . Moreover, $Q \simeq e\Lambda$ for some idempotent e and $\tau_Q(A) = A(\Lambda e\Lambda)$ for a Λ -module A . Therefore we have that $A(M) = A \otimes_\Lambda \Lambda/\Lambda e\Lambda$ for a Λ -module A and in particular it gives rise to a right exact functor.

The corollary to the following result is known. For the convenience of the reader we include a proof.

Proposition 2.5. *Suppose M is a simple Λ -module and X is a Λ -module such that $X(M)$ is a nonprojective $\Lambda(M)$ -module. Then $\text{Ext}_\Lambda^1(X, M) \neq (0)$.*

Proof. Let M be a simple Λ -module and assume that $X(M)$ is a nonprojective $\Lambda(M)$ -module. Now $\text{Ext}_\Lambda^1(X, M) \neq (0)$ if and only if $\text{Hom}_\Lambda(\Omega(X)/\Omega(X)J, M) \neq (0)$, where Ω denotes the Heller operator.

We have the following exact commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega(X) & \longrightarrow & P_X & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & (\Omega(X))(M) & \longrightarrow & P_X(M) & \longrightarrow & X(M) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The map $\varphi: P_X(M) \rightarrow X(M)$ is a projective cover. Since $X(M)$ is a nonprojective $\Lambda(M)$ -module, $\ker \varphi$ is a nonzero $\Lambda(M)$ -module. Therefore $(\Omega(X))(M)$ must be

nonzero. Hence M occurs in the top of $\Omega(X)$. The claim follows from this by the above observation. $\square$

Corollary 2.6. *Let M be a simple Λ -module. Then $\text{Ext}_\Lambda^2(M, M) = (0)$ implies that $\text{Ext}_\Lambda^1(M, M) = (0)$.*

Proof. Assume that $\text{Ext}_\Lambda^1(M, M) \neq (0)$. Then we know that M occurs as a composition factor in the top $\Omega(M) = P_M J$. It is easy to see that $0 \rightarrow (P_M J)(M) \rightarrow P_M(M) \rightarrow M \rightarrow 0$ is exact. Therefore $(P_M J)(M)$ is a nonprojective Λ -module and $\text{Ext}_\Lambda^1(P_M J, M) \simeq \text{Ext}_\Lambda^2(M, M) \neq (0)$. This completes the proof of the corollary. $\square$

3. PROOF OF THE MAIN RESULT

In this section we assume that all algebras Λ are finite dimensional connected symmetric algebras over an algebraically closed field K .

Let $\{e_i\}_{i=1}^n$ be a complete set of primitive orthogonal idempotents for Λ . Recall that Λ is called *regular* if for each pair of i and j there exists x and y in $e_i \Lambda e_j$ such that $e_i \Lambda e_j = e_i \Lambda e_i x$ and $e_i \Lambda e_j = y e_j \Lambda e_j$. Moreover, the *multiplicity* of Λ is defined by $c(\Lambda) = \max\{\dim_K e_i \Lambda e_i\}_{i=1}^n - 1$.

Proof of Theorem 1.1. Let Λ be a connected, representation-finite, symmetric K -algebra with at least two non-isomorphic simple modules such that $\text{Ext}_\Lambda^2(M, M) = (0)$ for every simple Λ -module M . Since all these properties of Λ are preserved under Morita equivalence we may assume that Λ is a basic K -algebra.

If Λ is not regular, then by [13, Satz 1.8] the multiplicity of Λ is at least 2. Hence by [13, Satz 1] Λ is isomorphic to one of the algebras which are denoted in [13] by $B(T, S, \epsilon)$, where T is a Brauer tree, S an extremal point on T and the scalar ϵ is 0 or 1. Since Λ has at least two simple modules, it follows from the description of the quivers and relations of the algebras $B(T, S, \epsilon)$ that there is a simple Λ -module M such that M occurs in the top of $P_M J$. From Corollary 2.6 follows that $\text{Ext}_\Lambda^2(M, M) \neq (0)$, a contradiction.

Hence we may assume that Λ is regular. If the multiplicity of Λ is at least 2, then by [13, Satz 1] and [12, Corollary a] we have that Λ is biserial. By Corollary 2.2 the hearts H_M of all indecomposable projective Λ -modules P_M are indecomposable. Therefore Λ is uniserial.

Thus we may assume that Λ is regular with multiplicity $c(\Lambda) = 1$. Therefore assertion (2) of Theorem 1.1 is a consequence of assertion (1).

The proof of the converse is split into two parts. At first we assume that Λ is uniserial. Since Λ has at least two non-isomorphic simple modules, it follows from Corollary 2.12 of [2], p. 117, that all simple Λ -modules are contained in one orbit of the square Ω^2 of the Heller operator Ω . In particular, $\text{Ext}_\Lambda^2(M, M) = \text{Hom}_\Lambda(\Omega^2 M, M) = (0)$ for every simple Λ -module.

Now we assume that Λ has multiplicity $c(\Lambda) = 1$ and all hearts H_M of the indecomposable projective Λ -modules are indecomposable. Then $P_M J$ is a multiplicity-free Λ -module for each simple Λ -module M by Satz 1.1 of [9] and Theorem 2 of [10] because Λ is of finite representation type. Hence $\text{Ext}_\Lambda^2(M, M) = (0)$ for all simple Λ -modules M by Corollary 2.4. This completes the proof. $\square$

with n relations generated by the elements $\gamma_2\gamma_1 - \delta_2\delta_1$, $\gamma_1\alpha^{n-3}\delta_2$, $\delta_1\alpha^{n-3}\gamma_2$ and $\alpha^{n-2-i}\gamma_2\gamma_1\alpha^i$ for $i = 1, \dots, n-3$, or it is

where $n_1 + n_2 + n_3 = n \geq 5$, $n_i \geq 1$, and where the relations are generated by the elements $\gamma_3\gamma_2 - \alpha_1^{n_1}$, $\gamma_1\gamma_3 - \alpha_2^{n_2}$, $\gamma_2\gamma_1 - \alpha_3^{n_3}$, $\alpha_2\alpha_3$, $\alpha_1\alpha_2$, $\alpha_3\alpha_1$, $\alpha_j^{n_j-i}\gamma_j\alpha_j^{i+1}$ for $i = 0, 1, \dots, n_j - 1$ and $j = 1, 2, 3$.

(2) There are 9 non-isomorphic such algebras Λ of Dynkin type E_6 . Their quivers and relations are given in the following table.

Number	Dimension	Quiver and relations
10	30	14 24 34 45 46 51 53 61 62
13	30	12 14 23 25 36 31 45 51 62
14	30	12 13 24 34 45 46 41 53 62
17	32	12 15 23 24 36 41 56 61
18	34	12 14 23 25 31 45 56 61
19	34	12 15 23 34 36 41 56 61
20	34	12 13 25 34 46 56 61
21	36	12 23 25 34 36 41 56 61
22	38	12 23 34 36 45 51 61

(3) There are 41 non-isomorphic such algebras Λ of Dynkin type E_7 . Their quivers and relations are given in the following table.

Number	Dimension	Quiver and relations
36s	36	12 14 16 23 25 31 45 47 51 67 71
39	36	12 14 23 25 31 45 46 57 51 61 72
49s	36	12 14 23 25 36 31 45 51 67 62 73
53	38	12 15 23 26 34 41 56 67 61 75
57	38	12 15 23 24 36 47 41 56 61 72
58	38	12 15 23 26 34 41 56 57 61 71
59	38	12 14 16 23 27 35 45 51 67 71
61	40	12 15 23 34 36 41 56 57 61 71
63	40	12 15 23 26 34 47 41 56 61 72
67s	40	12 15 23 24 36 41 56 67 61 72
68s	40	12 15 23 34 36 47 41 56 61 73
70	40	12 16 23 25 34 41 57 67 71
72	42	12 16 23 24 35 37 45 51 67 71

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73s	42	12 16 23 25 34 47 51 67 71
75	42	12 15 23 26 34 41 56 67 71
77s	42	12 23 25 34 36 41 56 67 61 73
78s	44	12 15 23 26 34 37 41 56 67 71
79	44	12 16 23 34 35 47 51 67 71
81	44	12 16 23 34 37 45 51 67 71
82s	44	12 23 26 34 45 51 67 71
83	46	12 23 25 34 46 56 57 61 71
84	46	12 16 23 34 45 47 51 67 71
85	48	12 23 25 34 36 41 56 67 71
86s	50	12 23 34 37 45 56 61 71
87s	50	12 23 34 36 45 47 51 67 71
88s	52	12 23 34 45 47 56 61 71

(4) There are 139 non-isomorphic such algebras Λ of Dynkin type E_8 . Their quivers and relations are given in the following table.

<i>Number</i>	<i>Dimension</i>	<i>Quiver and relations</i>
142	42	12 14 23 25 37 31 45 46 58 51 61 72 84
143	42	12 14 16 23 25 31 45 47 58 51 67 71 82
149	42	12 14 23 25 37 31 45 46 51 61 78 72 83
150s	42	12 14 23 25 27 38 31 45 46 51 61 78 82
153s	42	12 14 23 25 36 31 45 58 51 67 62 73 84
158	44	12 15 23 26 34 41 56 57 61 78 71 85
159	44	12 15 23 26 34 41 56 67 61 78 72 86
164	44	12 15 17 23 26 34 41 56 58 61 78 81
167	44	12 15 23 26 34 41 56 57 68 61 71 82
176	44	12 15 23 24 36 41 56 57 68 61 71 85
185s	44	12 15 23 26 34 37 41 56 61 78 72 83
190	44	12 15 23 24 36 48 41 56 57 61 71 82
193	44	12 15 23 24 36 47 41 56 61 78 72 84
194	44	12 14 23 35 45 46 47 51 68 61 78 84
200	44	12 14 16 23 35 45 47 58 51 67 71 83
205s	44	12 15 23 34 41 56 67 61 78 85
208	44	12 14 23 35 45 46 51 67 61 78 74 86
209s	44	12 14 16 23 27 35 45 48 51 67 71 81
210	46	12 15 23 34 36 41 56 57 61 78 71 85
212	46	12 15 23 34 36 41 56 57 68 61 71 85
221	46	12 15 23 34 36 47 41 56 61 78 73 84
227	46	12 15 23 34 36 48 41 56 57 61 71 83
228	46	12 15 17 23 34 36 41 56 58 61 78 81
230	46	12 15 23 27 34 38 41 56 67 71 82
234	46	12 15 23 24 36 41 56 57 61 78 81
241s	46	12 16 23 25 34 38 41 57 67 71 82
244	48	12 16 23 27 34 45 51 67 68 71 81
245s	48	12 16 23 25 34 41 58 67 78 81
248	48	12 15 23 27 34 41 56 67 78 71 85
249	48	12 15 23 34 46 56 57 61 78 71 85
250	48	12 15 23 26 34 41 56 67 68 71 82
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254	48	12 15 23 34 36 41 56 57 68 61 71 82
258	48	12 15 23 34 37 41 56 58 67 71 81
259	48	12 16 23 24 35 37 45 51 67 68 71 81
260	48	12 15 23 26 34 41 56 67 68 71 85
261	48	12 16 23 24 35 37 45 58 51 67 71 83
264	48	12 16 23 34 35 47 48 51 67 71 84
265	48	12 14 23 25 36 45 47 56 61 78 71 84
268	48	12 14 16 23 35 45 47 51 67 78 81
272	48	12 16 23 25 34 47 51 67 68 71 81
275s	50	12 15 23 26 34 41 56 67 78 71 82
277	50	12 16 23 24 35 38 45 51 67 78 81
278	50	12 16 23 34 35 47 58 51 67 71 83
279	50	12 16 23 34 35 47 51 67 68 71 81
282	50	12 15 23 24 36 48 41 56 67 71 82
285	50	12 17 23 26 34 45 51 68 78 81
286s	50	12 15 23 26 34 48 41 56 67 71 82
287	50	12 15 23 26 34 37 41 56 67 68 71 82
288s	50	12 16 23 34 37 45 51 67 78 71 83
289	50	12 17 23 34 35 46 56 68 61 71 84
293	52	12 16 23 34 38 45 51 67 78 81
295	52	12 23 26 34 37 45 51 67 78 71 83
296	52	12 15 23 24 36 41 56 57 68 78 81
298	52	12 16 23 27 34 45 51 67 78 81
301	52	12 17 23 25 34 46 56 58 61 78 81
303	52	12 16 23 34 35 48 51 67 78 81
305	52	12 16 23 34 45 47 51 67 68 71 81
306	52	12 16 23 27 34 45 58 51 67 71 82
307	54	12 15 23 27 34 38 41 56 67 78 81
309	54	12 15 23 34 36 47 56 58 67 71 81
312	54	12 23 25 34 36 41 56 67 68 71 83
315s	54	12 17 23 26 34 45 58 61 78 81
316	56	12 16 23 27 34 45 48 51 67 78 81
318	56	12 15 23 34 46 56 58 67 71 81
319	56	12 14 23 25 36 45 48 56 67 71 81
320s	56	12 23 26 34 38 45 51 67 78 81
321	56	12 17 23 34 36 45 58 61 78 81
323s	56	12 23 27 34 45 56 61 78 81
324	58	12 23 26 34 45 57 67 68 71 81
325	58	12 15 23 24 36 41 56 67 78 81
327	60	12 23 27 34 45 48 56 61 78 81
328	60	12 14 23 25 31 45 56 67 78 81
329s	60	12 23 26 34 37 45 48 51 67 78 81
330	62	12 23 25 34 36 41 56 67 78 81
331	64	12 23 34 37 45 56 58 61 78 81
332s	64	12 23 34 38 45 56 67 71 81
333s	66	12 23 34 45 47 56 58 61 78 81
334s	68	12 23 34 45 56 58 67 71 81

Proof. Let Λ be a representation-finite connected basic symmetric K -algebra with multiplicity $c(\Lambda) = 1$, where K is an algebraically closed field. As Λ is not semi-simple it has at least two non-isomorphic simple modules. If Λ were non-regular, then by [13, Satz 1.8] its multiplicity would be 3, a contradiction to $c(\Lambda) = 1$. Hence Λ is regular. By [7, § 4.1] Λ is a trivial extension of a tilted algebra of Dynkin class A_n, D_n, E_6, E_7 or E_8 . As Λ is not uniserial, it is not of Dynkin class A_n . All the trivial extensions of the tilted algebras are standard [11, Proposition 2.3]. Therefore it remains to classify all standard representation-finite connected basic symmetric K -algebras of Dynkin class D_n, E_6, E_7 and E_8 . Each case is now dealt with separately.

(1) Suppose that Λ is of Dynkin class D_n . All standard representation-finite connected basic symmetric K -algebras of Dynkin class D_n are classified in [4]. They are either two- or three-cornered algebras.

In [4] the two-cornered algebras of Dynkin class D_n are given in the following way. An arrowed Brauer-quiver Q is a finite connected quiver satisfying the conditions:

- (a) each arrow of Q belongs to a simple oriented cycle,
- (b) each vertex belongs exactly to two cycles, and
- (c) two cycles meet in at most one vertex. The arrows belonging to one cycle are all labeled by α or β in such a way that two α -cycles (or β -cycles) never meet.

Let P be an arrowed Brauer-quiver with $n-2$ vertices and a distinguished arrow $\alpha: x \rightarrow y$. Let a_z (b_z) be the length of the α -cycle A_z (β -cycle B_z) of P passing through the vertex z . Then denote by $Q(P)$ the quiver obtained from P replacing the distinguished arrow $\alpha: x \rightarrow y$ by

Let $I(P)$ be the ideal in $KQ(P)$ generated by the elements

- (a) $\alpha\beta, \beta\alpha, \gamma_1\beta, \beta\gamma_2, \delta_1\beta, \beta\delta_2, \delta_1\alpha^{a_x-1}\gamma_2, \gamma_1\alpha^{a_x-1}\delta_2, \gamma_2\gamma_1 - \delta_2\delta_1$
- (b) $\alpha^{a_z} - \beta^{b_z}$ for each vertex z of P lying outside A_x .
- (c) $\alpha^{a_z^+}\gamma_2\gamma_1\alpha^{a_z^-} - \beta^{b_z}$ for each vertex z of A_x , where a_z^+ (a_z^-) denote the length of the α -path of $Q(P)$ from z to x (y to z).

Then the algebra $\Lambda = KQ(P)/I(P)$ is a two-cornered algebra and by [4] all two-cornered algebras are given in this way.

Let Λ be a two-cornered algebra. If there is an α -cycle meeting a β -cycle both of length at least 2, then the relation in (b) is a relation starting and ending in one vertex z . By the main theorem [3] there exists a simple Λ -module M with $\text{Ext}_\Lambda^2(M, M) \neq (0)$. Since there are no loops in the ordinary quiver of Λ , the quiver $Q(P)$ can only consist of one cycle of length at least 2. The claim now follows immediately from the description of the two-cornered algebras.

The three-cornered algebras have the following description. Let P be a Brauer-quiver with a distinguished β -loop, that is, a looped Brauer-quiver. Denote by P^+ the quiver obtained from P by splitting the distinguished vertex x of P into an α -sink x^- and an α -source x^+ , the arrow $\alpha: y \rightarrow x$ ($\alpha: x \rightarrow z$) of P is replaced by an arrow $\alpha: y \rightarrow x^-$ ($\alpha: x^+ \rightarrow z$) if $y \neq x \neq z$, if P has only one vertex, the α -loop $\alpha: x \rightarrow x$ is replaced by an arrow $\alpha: x^+ \rightarrow x^-$ and finally in all cases the distinguished loop $\beta: x \rightarrow x$ is replaced by an arrow $\gamma: x^- \rightarrow x^+$.

Let P_1, P_2 and P_3 be three looped Brauer-quivers with distinguished vertices x_1, x_2 and x_3 . Denote by $Q(P_1P_2P_3)$ the quiver obtained from P_1, P_2 and P_3 by

identifying the vertex x_2^+ with x_3^- , the vertex x_3^+ with x_1^- , the vertex x_1^+ with x_2^- . Let $I(P_1P_2P_3)$ be the ideal in $KQ(P_1P_2P_3)$ generated by the elements

- (a) $\alpha_i\beta_i, \beta_i\alpha_i, \alpha_2\alpha_3, \alpha_1\alpha_2, \alpha_3\alpha_1, \gamma_3\gamma_2 - \alpha_1^{a_{x_1}}, \gamma_1\gamma_3 - \alpha_2^{a_{x_2}}, \gamma_2\gamma_1 - \alpha_3^{a_{x_3}}$, for $i = 1, 2, 3$
- (b) $\alpha^{a_z} - \beta^{b_z}$ for each i and each vertex z of P_i lying outside the α -cycle A_i of P_i which contains x_i .
- (c) $\alpha_i^{a^-} \gamma_i \alpha_i^{a^+} - \beta^{b_z}$ for each i if $z \in A_i \setminus \{x_i\}$, where a^- (a^+) is the length of the α -path of P_i^+ from z to x_i^- (from x_i^+ to z)

If the total number of vertices in P_1, P_2 and P_3 is at least 5, then the path algebra $\Lambda = KQ(P_1P_2P_3)/I(P_1P_2P_3)$ is a three-cornered algebra and all three-cornered algebras are given in this way.

Let Λ be a three-cornered algebra. If the quiver P_i has an α -cycle meeting a β -cycle both of length at least 2 in a vertex z of P_i lying outside the α -cycle A_i of P_i containing x_i , then the relation in (b) is a relation starting and ending in the vertex z .

If there is a β -cycle of length at least 2 meeting the α -cycle A_i in a vertex z lying on A_i and different from x_i in P_i , then the relation in (c) is a relation starting and ending in the vertex z .

Again by using [3] there exists a simple Λ -module M with $\text{Ext}_{\Lambda}^2(M, M) \neq (0)$ in both cases described above. Hence each P_i has at most one cycle of length at least 2. The claim now follows immediately from this and the description of the three-cornered algebras.

(2) Using Roggon's complete list [11] of all basic representation-finite symmetric K -algebras of Λ of Dynkin type E_6 it suffices here to exhibit for each such algebra Λ not listed in the assertion at least one indecomposable projective Λ -module $P(i)$, $i \in \{1, 2, \dots, 6\}$ whose heart $H(i)$ is decomposable. This is done in the following table.

$$\begin{aligned} H(1): & 2, 5, 6, 7, 9, 11, 15 \\ H(2): & 1, 8 \\ H(4): & 4, 16 \\ H(5): & 3, 12 \end{aligned}$$

(3) Using Roggon's complete list [11] of all basic representation-finite symmetric K -algebras of Λ of Dynkin type E_7 it suffices here to exhibit for each such algebra Λ not listed in the assertion at least one indecomposable projective Λ -module $P(i)$, $i \in \{1, 2, \dots, 7\}$ whose heart $H(i)$ is decomposable. This is done in the following table.

$$\begin{aligned} H(1): & 7, 8, 10s, 11, 12, 13, 24s, 26s, 27s, 30, 37s, 38, 40, 43, 46, 51, 55s, \\ & 60, 65, 66, 71, 74s \\ H(2): & 3, 20, 23s, 31, 35, 48 \\ H(3): & 5, 18, 21, 28, 33, 45s, 50, 54, 62 \\ H(4): & 9s, 15, 22, 41, 56 \\ H(5): & 1s, 4s, 6, 16s, 25, 44, 47s, 52, 69, 76, 80s \\ H(6): & 2s, 14, 17s, 19s, 29, 32, 34s, 42, 64s \end{aligned}$$

(4) Using Roggon's complete list [11] of all basic representation-finite symmetric K -algebras of Λ of Dynkin type E_8 it suffices here to exhibit for each such algebra Λ not listed in the assertion at least one indecomposable projective Λ -module $P(i)$, $i \in \{1, 2, \dots, 8\}$ whose heart $H(i)$ is decomposable. This is done in the following table.

$$\begin{aligned} H(1): & 001, 002, 003, 004, 005, 006s, 007, 008, 009, 010, 011s, 012, 014, \\ & 015, 016, 017, 018, 019, 020, 021, 022, 023, 024, 025, 027s, 028, \\ & 029, 030, 031s, 032, 033s, 034, 035, 036, 037, 038s, 039, 040, 042, \end{aligned}$$

043, 044, 046s, 049, 050, 051, 053, 054, 055, 058, 059, 061, 063,
064s, 065s, 067s, 068s, 069, 070, 071, 073, 077s, 081, 083, 084,
086, 088, 095, 096, 099, 102s, 103, 105, 106, 108, 109, 113s, 114,
117, 120, 126, 129s, 132, 133, 135, 136, 139, 144, 145, 151, 154s,
155, 157s, 160, 161, 162, 163, 168, 169, 171, 172s, 174, 179, 186s,
187, 188, 192, 198s, 203s, 213s, 214s, 215, 218s, 223, 224, 231,
232, 233, 235, 240, 252, 253, 255s, 262, 266, 267, 270, 273s, 284,
290, 292, 299, 302s, 304, 308s, 314, 317, 322
 $H(2)$: 052, 056, 057, 074, 089, 090, 101, 104, 112, 147, 183, 191, 263, 269
 $H(3)$: 013s, 026s, 041, 048, 060s, 072, 079s, 082, 093, 094, 116, 125, 128,
130, 134s, 137, 138s, 140, 165, 189, 236, 247, 257
 $H(4)$: 045, 047, 066, 080, 097, 098s, 110s, 111, 115, 118, 119, 124, 131,
141, 148s, 170, 173, 175, 202, 206s, 211, 220, 237, 246s, 280, 283,
311
 $H(5)$: 062s, 076, 078, 085s, 100, 121, 123, 146, 156, 177, 181, 182, 196s,
207, 219, 222, 225, 229s, 238, 242, 271, 276, 300, 310s
 $H(6)$: 087, 092, 107, 122, 127, 152s, 180, 197, 201, 204, 217, 226, 243,
256, 274, 291, 326s
 $H(7)$: 075, 091, 166, 178, 184, 195, 199s, 216, 239s, 251, 281, 294s, 297,
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□

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REFERENCES

- [1] M. Auslander, S. Ding, Ø. Solberg, *Liftings and weak liftings of modules*, J. Algebra, **156**, (1993) 273–317.
- [2] M. Auslander, I. Reiten, S.O. Smalø, *Representation theory of Artin algebras*, Cambridge Univ. Press, Cambridge, 1995.
- [3] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. (2), **28** (1983) 461–469.
- [4] O. Bretscher, C. Läser, C. Riedtmann, *Selfinjective and simply connected algebras*, Manuscripta Math. **36** (1981) 253–307.
- [5] W. Feit, *The representation theory of finite groups*, North-Holland, Amsterdam, 1982.
- [6] J.A. Green, *A lifting theorem for modular representations*, Proc. Roy. Soc. London, Ser. A. (1959) 135–142.
- [7] D. Hughes, J. Waschbüsch, *Trivial extensions of tilted algebras*, Proc. London Math. Soc. **46** (1983) 347–364.
- [8] G.J. Janusz, *Indecomposable modules for finite groups*, Annals of Math. **89** (1969), 209–241.
- [9] H. Kupisch, *Symmetrische Algebren mit endlich vielen unzerlegbaren Darstellungen I*, J. reine u. angew. Math., **219** (1965), 1–25.
- [10] H. Kupisch, *Quasi-Frobenius algebras of finite representation type*, Springer Lecture Notes in Math. **488** (1975), 184–200.
- [11] B. Roggon, *Selfinjective and iterated tilted algebras of type E_6 , E_7 , E_8* , Preprint, Math. Fak. Univ. Bielefeld (1994).
- [12] A. Skowronski, J. Waschbüsch, *Representation-finite biserial algebras*, J. reine u. angew. Math. **345** (1983), 172–181.
- [13] J. Waschbüsch, *Symmetrische Algebren vom endlichen Modultyp*, J. reine u. angew. Math., **341** (1981), 78–98.

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