# GLOBAL CONSERVATIVE MULTIPEAKON SOLUTIONS OF THE CAMASSA–HOLM EQUATION

Helge Holden
*Department of Mathematical sciences,*
*Norwegian University of Science and Technology*
*NO–7491 Trondheim, Norway*
*and*
*Centre of Mathematics for Applications, University of Oslo*
*P.O. Box 1053, Blindern, NO–0316 Oslo, Norway*
*holden@math.ntnu.no*

Xavier Raynaud
*Department of Mathematical science,*
*Norwegian University of Science and Technology*
*NO–7491 Trondheim, Norway*
*raynaud@math.ntnu.no*

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**Abstract.** We show how to construct globally defined multipeakon solutions of the Camassa–Holm equation. The construction includes in particular the case with peakon-antipeakon collisions. The solutions are conservative in the sense that the associated energy is constant for almost all times. Furthermore, we construct a new set of ordinary differential equations that determines the multipeakons globally. The system remains globally well-defined.

*Keywords*: Camassa–Holm equation, multipeakons, conservative solutions.

## 1. Introduction

The Cauchy problem for the Camassa–Holm equation [8,9]

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad u|_{t=0} = u_0, \tag{1.1}$$

has received considerable attention the last decade. With $\kappa$ positive it models, see [19], propagation of unidirectional gravitational waves in a shallow water approximation, with $u$ representing the fluid velocity. The Camassa–Holm equation has a bi-Hamiltonian structure and is completely integrable. It has infinitely many conserved quantities. In particular, for smooth solutions the quantities

$$\int u\,dx, \quad \int (u^2 + u_x^2)\,dx, \quad \int (u^3 + uu_x^2)\,dx \tag{1.2}$$

are all time independent.

In this article we consider the case $\kappa = 0$ on the real line, that is,

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.3}$$

and henceforth we refer to (1.3) as the Camassa–Holm equation.

Solutions of the Camassa–Holm equation may experience wave breaking in the sense that the solution develops singularities in finite time, while keeping the $H^1$ norm finite. Continuation of the solution beyond the time of wave breaking is a challenging problem. It is most easily explained in the context of multipeakons, which are special solutions of the Camassa–Holm equation of the form

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|}, \tag{1.4}$$

where the $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot{q}_i = \sum_{j=1}^{n} p_j e^{-|q_i-q_j|}, \quad \dot{p}_i = \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i-q_j|}.$$

Observe that the solution (1.4) is not smooth even with continuous functions $(p_i(t), q_i(t))$; one possible way to interpret (1.4) as a weak solution of (1.3) is to rewrite Eq. (1.3) as

$$u_t + \big(\frac{1}{2}u^2 + (1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2)\big)_x = 0.$$

Peakons interact in a way similar to that of solitons of the Korteweg–de Vries equation, and wave breaking may appear when at least two of the $q_i$'s coincide. If all the $p_i(0)$ have the same sign, the peakons move in the same direction. Furthermore, in that case the solution experiences no wave breaking, and one has a unique global solution. Higher peakons move faster than the smaller ones, and when a higher peakon overtakes a smaller, there is an exchange of mass, but no wave breaking takes place. Furthermore, the $q_i(t)$ remain distinct, and thus there is no collision. However, if some of $p_i(0)$ have opposite sign, wave breaking or collision may incur, see, e.g., [4,20]. For simplicity, consider the case with $n = 2$ and one peakon $p_1(0) > 0$ (moving to the right) and one antipeakon $p_2(0) < 0$ (moving to the left). In the symmetric case ($p_1(0) = -p_2(0)$ and $q_1(0) = -q_2(0) < 0$) the solution will vanish pointwise at the collision time $t^*$ when $q_1(t^*) = q_2(t^*)$, that is, $u(t^*, x) = 0$ for all $x \in \mathbb{R}$, see Fig. 1. Clearly, at least two scenarios are possible; one is to let $u(t,x)$ vanish identically for $t > t^*$, and the other possibility is to let the peakon and antipeakon "pass through" each other in a way that is consistent with the Camassa–Holm equation. In the first case the energy $\int (u^2 + u_x^2)\, dx$ decreases to zero at $t^*$, while in the second case, the energy remains constant except at $t^*$. Clearly, the well-posedness of the equation is a delicate matter in this case. The first solution could be denoted a dissipative solution, while the second one could be called conservative, which is the class of solutions we study here. Other solutions are also possible. Global dissipative solutions of a more general class of equations were derived by Coclite, Holden, and Karlsen [12,13]. In their approach the solution was obtained by first regularizing the equation by adding a small diffusion term $\epsilon u_{xx}$ to the equation, and subsequently analyzing the vanishing viscosity limit $\epsilon \to 0$.

Global conservative solutions of the Camassa–Holm were recently studied by using a completely new approach, see [5,6,15,18]. In this approach the Camassa–Holm equation is reformulated as a system of ordinary differential equations taking values in a Banach space, see Sect. 2. This allows for the construction of a global and stable solution. To obtain a well-posed initial-value problem it is necessary to introduce the associated energy as an additional variable.

We here study in detail this construction in the context of multipeakons, following [18] where the transformation into new variables can be interpreted as a transformation from Eulerian into Lagrangian coordinates. The explicit nature of multipeakons make them very interesting objects to study in a relation to wave breaking. In particular, the singularity corresponds to a focusing of the energy into a Dirac delta-function.

The general construction in [18] is rather complicated, making the case of multipeakons involved. We show that multipeakons are given as continuous solutions $u$ that on intervals $[y_i(t), y_{i+1}(t)]$ satisfy

$$u - u_{xx} = 0 \text{ with boundary conditions } u(t, y_i(t)) = u_i(t),\ u(t, y_{i+1}(t)) = u_{i+1}(t).$$

The $(y_i, u_i)$ are given by a set of ordinary differential equations, which in addition includes a third variable that measures the energy of the system. The variables $y_i$ and $u_i$ denote the location of the point (for fixed time) where the solution $u$ has a discontinuous spatial derivative (the "peak"), and the value of $u$ at that point, respectively. The system of ordinary differential equations, which is new, remains globally well-defined.

In addition to allowing for a detailed study of the property of solutions near wave breaking, multipeakons are important as building blocks for general solutions. Indeed, if the initial data $u_0$ is in $H^1$ and $m_0 := u_0 - u_0''$ is a positive Radon measure, then it can proved, see [17], that one can construct a sequence of multipeakons that converges in $L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1_{\mathrm{loc}}(\mathbb{R}))$ to the unique global solution of the Camassa–Holm equation. See also [6,15].

The method is illustrated by explicit calculations in the cases $n = 1$ and $n = 2$ (see also [2,3,20]) and by numerical computations in the case $n = 4$ with and without wave breaking.

Furthermore, the methods presented in this paper can be used to derive numerical methods that converge to conservative solutions rather than dissipative solutions. This contrasts finite difference methods that normally converge to dissipative solutions, see [16]. See also [17]. Results will be presented separately.

## 2. Global conservative solutions

The goal of this section is to introduce the results obtained in [18], namely the construction of the continuous semigroup of conservative solutions of the Camassa–Holm equation with a change of variable to Lagrangian coordinates. The equation can be rewritten as the following system

$$u_t + uu_x + P_x = 0, \tag{2.1a}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{2.1b}$$

It is not hard to check that the energy density $u^2 + u_x^2$ fulfills the following transport equation

$$(u^2 + u_x^2)_t + (u(u^2 + u_x^2))_x = (u^3 - 2Pu)_x. \tag{2.2}$$

We denote $y_t(t,\xi) = u(t, y(t,\xi))$ the characteristics and set

$$U(t,\xi) = u(t, y(t,\xi)) \text{ and } H(t,\xi) = \int_{-\infty}^{y(t,\xi)} (u^2 + u_x^2)\,dx,$$

which corresponds to the Lagrangian velocity and the Lagrangian cumulative energy distribution, respectively. We set $\zeta(\xi) = y(\xi) - \xi$. From the definition of the characteristics, it follows that

$$U_t(t,\xi) = u_t(t,y) + y_t(t,\xi)u_x(t,y) = -P_x \circ y\,(t,\xi). \tag{2.3}$$

This last term can be expressed uniquely in term of $U$, $y$, and $H$. From (2.1b), we obtain the following explicit expression for $P$,

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}\big(u^2(t,z) + \frac{1}{2}u_x^2(t,z)\big)\,dz. \tag{2.4}$$

Thus we have

$$P_x \circ y\,(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(y(t,\xi) - z)e^{-|y(t,\xi)-z|}\big(u^2(t,z) + \frac{1}{2}u_x^2(t,z)\big)\,dz,$$

and, after the change of variables $z = y(t,\eta)$,

$$P_x \circ y(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \Big[\operatorname{sgn}(y(t,\xi) - y(t,\eta))e^{-|y(t,\xi)-y(t,\eta)|} \times \left(u^2(t,y(t,\eta)) + \frac{1}{2}u_x^2(t,y(t,\eta))\right) y_\xi(t,\eta)\Big]\,d\eta.$$

Finally, since $H_\xi = (u^2 + u_x^2)\circ y\, y_\xi$,

$$P_x \circ y\,(\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(y(\xi) - y(\eta))\exp(-|y(\xi) - y(\eta)|)\big(U^2 y_\xi + H_\xi\big)(\eta)\,d\eta \tag{2.5}$$

where the $t$ variable has been dropped to simplify the notation. It turns out that $y_\xi(t,\xi) \geq 0$ for all $t$ and almost every $\xi$, see Definition 2.1 and [18, Theorem 2.8]. Thus, $P_x \circ y$ is can be replaced by $Q$ where

$$Q(t,\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi - \eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi) - y(\eta))\big)\big(U^2 y_\xi + H_\xi\big)(\eta)\,d\eta, \tag{2.6}$$

and, slightly abusing the notation, we write

$$P(t,\xi)=\frac{1}{4}\int_{\mathbb{R}}\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta. \tag{2.7}$$

Thus $P_x\circ y$ and $P\circ y$ can be replaced by equivalent expressions given by (2.6) and (2.7) which only depend on our new variables $U$, $H$, and $y$. From (2.2), it follows that

$$H_t=\int_{-\infty}^{y}(u^2+u_x^2)_t\,dx+y_t\circ y(u^2+u_x^2)\circ y=\int_{-\infty}^{y}(u^3-2Pu)_x\,dx=U^3-2PU. \tag{2.8}$$

Finally, from (2.3) and (2.8), we infer that the Camassa–Holm equation is formally equivalent to the following system

$$\begin{cases}\zeta_t=U,\\ U_t=-Q,\\ H_t=U^3-2PU.\end{cases} \tag{2.9}$$

We look at (2.9) as a system of ordinary differential equations in the Banach space

$$E=V\times H^1(\mathbb{R})\times V$$

where $V=\{f\in C_b(\mathbb{R})\mid f_\xi\in L^2(\mathbb{R})\}$. By a contraction argument we establish the short-time existence of solutions ([18, Theorem 2.3]). We have

$$Q_\xi=-\frac{1}{2}H_\xi-\left(\frac{1}{2}U^2-P\right)y_\xi\ \text{ and }\ P_\xi=Qy_\xi, \tag{2.10}$$

and, differentiating (2.9) yields

$$\begin{cases}\zeta_{\xi t}=U_\xi\ (\text{or } y_{\xi t}=U_\xi),\\ U_{\xi t}=\frac{1}{2}H_\xi+\left(\frac{1}{2}U^2-P\right)y_\xi,\\ H_{\xi t}=-2Q\,Uy_\xi+\big(3U^2-2P\big)\,U_\xi.\end{cases} \tag{2.11}$$

The system (2.11) is semilinear with respect to the variables $y_\xi$, $U_\xi$ and $H_\xi$.

Global solutions of (2.9) may not exist for all initial data in $E$. However they exist when the initial data $\bar{X}=(\bar{y},\bar{U},\bar{H})$ belongs to the set $\mathcal{G}$ ([18, Theorem 2.8]) where $\mathcal{G}$ is defined as follows:

**Definition 2.1.** *The set $\mathcal{G}$ is composed of all $(\zeta,U,H)\in E$ such that*

$$(\zeta,U,H)\in\left[W^{1,\infty}(\mathbb{R})\right]^3, \tag{2.12a}$$

$$y_\xi\geq 0, H_\xi\geq 0, y_\xi+H_\xi>0 \text{ almost everywhere, and } \lim_{\xi\to-\infty}H(\xi)=0, \tag{2.12b}$$

$$y_\xi H_\xi=y_\xi^2U^2+U_\xi^2 \text{ almost everywhere}, \tag{2.12c}$$

*where we denote $y(\xi)=\zeta(\xi)+\xi$.*

The proof of the global existence of the solution for initial data in $\mathcal{G}$ ([18, Theorem 2.8]) relies essentially on the fact that the set $\mathcal{G}$ is preserved by the flow, that is, if $X(0) \in \mathcal{G}$, then $X(t) \in \mathcal{G}$ for all time $t$, for any solution $X(t)$ of (2.9) with initial data in $\mathcal{G}$ ([18, Lemma 2.7]). We also have that, for almost every $t$, $y_\xi(t,\xi) > 0$ for almost every $\xi$, which implies that for almost every $t$, $\xi \mapsto y(t,\xi)$ is invertible [18].

To obtain a semigroup of solution for (2.1), we have to consider the space $\mathcal{D}$, which characterizes the solutions in *Eulerian coordinates*:

**Definition 2.2.** *The set $\mathcal{D}$ is composed of all pairs $(u,\mu)$ such that $u$ belongs to $H^1(\mathbb{R})$ and $\mu$ is a positive finite Radon measure whose absolute continuous part, $\mu_{ac}$, satisfies*

$$\mu_{ac} = (u^2 + u_x^2)\,dx. \tag{2.13}$$

The set $\mathcal{D}$ allows the energy density to have a singular part and a positive amount of energy can concentrate on a set of Lebesgue measure zero. In [14], the Camassa–Holm equation is derived as a geodesic equation on the group of diffeomorphism equipped with a right-invariant metric. The right-invariance of the metric can be interpreted as an invariance with respect to relabeling as noted in [1]. This is a property that we also observe in our setting. We denote by $G$ the subgroup of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f - \mathrm{Id} \text{ and } f^{-1} - \mathrm{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}) \tag{2.14}$$

where Id denotes the identity function. The set $G$ can be interpreted as the set of relabeling functions. Let $\mathcal{F}$ be the following subset of $\mathcal{G}$

$$\mathcal{F} = \{X = (y,U,H) \in \mathcal{G} \mid y + H \in G\}.$$

For the sake of simplicity, for any $X = (y,U,H) \in \mathcal{F}$ and any function $f \in G$, we denote $(y \circ f, U \circ f, H \circ f)$ by $X \circ f$. The map $(f,X) \mapsto X \circ f$ defines an action of the group $G$ on $\mathcal{F}$ ([18, Proposition 3.4]), and we denote by $\mathcal{F}/G$ the quotient space of $\mathcal{F}$ with respect to the action of the group $G$. The equivalence relation on $\mathcal{F}$ is defined as follows: For any $X, X' \in \mathcal{F}$, $X$ and $X'$ are equivalent if there exists $f \in G$ such that $X' = X \circ f$, that is, if $X$ and $X'$ are equal up to a relabeling.

As proved in [18, Lemma 3.3], $\mathcal{F}$ is preserved by the flow. Let us denote by $S\colon \mathcal{F} \times \mathbb{R}_+ \to \mathcal{F}$ the continuous semigroup which to any initial data $\bar{X} \in \mathcal{F}$ associates the solution $X(t)$ of the system of differential equation (2.9) at time $t$. The Camassa–Holm equation is invariant with respect to relabeling, that is,

$$S_t(X \circ f) = S_t(X) \circ f \tag{2.15}$$

for any initial data $X \in \mathcal{F}$, any time $t$ and any $f \in \mathcal{F}$. Thus the map $\tilde{S}_t$ from $\mathcal{F}/G$ to $\mathcal{F}/G$ given by $\tilde{S}_t([X]) = [S_t X]$ is well-defined and it generates a continuous semigroup. The topology on $\mathcal{F}/G$ is defined by a complete metric which is derived from the $E$-norm restricted to $\mathcal{F}$.

In order to transport the continuous semigroup obtained in the Lagrangian framework (solutions in $\mathcal{F}/G$) into the Eulerian framework (solutions in $\mathcal{D}$), we want to establish a bijection between $\mathcal{F}/G$ and $\mathcal{D}$. Let us denote by $L\colon \mathcal{D} \to \mathcal{F}/G$

the map transforming Eulerian coordinates into Lagrangian coordinates defined as follows: For any $(u,\mu)$ in $\mathcal{D}$, let

$$y(\xi) = \sup\{y \mid \mu((-\infty,y)) + y < \xi\}\,, \tag{2.16a}$$

$$H(\xi) = \xi - y(\xi), \tag{2.16b}$$

$$U(\xi) = u\circ y(\xi)\,. \tag{2.16c}$$

We define $L(u,\mu) \in \mathcal{F}/G$ to be the equivalence class of $(y,U,H)$. In the other direction, we obtain $\mu$, the energy density in Eulerian coordinates, by pushing forward by $y$ the energy density in Lagrangian coordinates, $H_\xi\,d\xi$. Recall that the push-forward of a measure $\nu$ by a measurable function $f$ is the measure $f_\#\nu$ defined as

$$f_\#\nu(B) = \nu(f^{-1}(B))$$

for all Borel sets $B$. Given any element $[X]$ in $\mathcal{F}/G$, let $(u,\mu)$ be

$$u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \tag{2.17a}$$

$$\mu = y_\#(H_\xi\,d\xi). \tag{2.17b}$$

Then $(u,\mu)$ belongs to $\mathcal{D}$ and is independent of the representative $X = (y,U,H) \in \mathcal{F}$ we choose for $[X]$. We denote by $M\colon \mathcal{F}/G \to \mathcal{D}$ the map which to any $[X]$ in $\mathcal{F}/G$ associates $(u,\mu)$ as given by (2.17). The map $M$ corresponds to the transformation from Lagrangian to Eulerian coordinates. In [18, Theorems 3.8, 3.11], it is proven that the maps $L$ and $M$ are well-defined and that $L^{-1} = M$, see [18, Theorem 3.12].

We define the metric $d_{\mathcal{D}}$ on $\mathcal{D}$ as

$$d_{\mathcal{D}}((u,\mu),(\bar u,\bar\mu)) = d_{\mathcal{F}/G}(L(u,\mu),L(\bar u,\bar\mu)).$$

Since $\mathcal{F}/G$ equipped with $d_{\mathcal{F}/G}$ is a complete metric space, $\mathcal{D}$ equipped with the metric $d_D$ is a complete metric space. For each $t\in\mathbb{R}$, we define the map $T_t$ from $\mathcal{D}$ to $\mathcal{D}$ as

$$T_t = M\tilde S_t L.$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D} & \xleftarrow{\;M\;} & \mathcal{F}/G \\ {\scriptstyle T_t}\uparrow & & \uparrow{\scriptstyle \tilde S_t} \\ \mathcal{D} & \xrightarrow{\;L\;} & \mathcal{F}/G \end{array} \tag{2.18}$$

Finally, we have the following main result from [18].

**Theorem 2.3.** $T\colon \mathcal{D}\times\mathbb{R}_+ \to \mathcal{D}$ *(where* $\mathcal{D}$ *is defined by Definition 2.2) defines a continuous semigroup of solutions of the Camassa–Holm equation, that is, given* $(\bar u,\bar\mu)\in\mathcal{D}$*, if we denote* $t\mapsto (u(t),\mu(t)) = T_t(\bar u,\bar\mu)$ *the corresponding trajectory, then* $u$ *is a weak solution of the Camassa–Holm equation* (2.1)*. Moreover* $\mu$ *is a weak solution of the following transport equation for the energy density*

$$\mu_t + (u\mu)_x = (u^3 - 2Pu)_x. \tag{2.19}$$

*Furthermore, we have that*

$$\mu(t)(\mathbb{R}) = \mu(0)(\mathbb{R}) \text{ for all } t \tag{2.20}$$

*and*

$$\mu(t)(\mathbb{R}) = \mu_{ac}(t)(\mathbb{R}) = \|u(t)\|_{H^1}^2 = \mu(0)(\mathbb{R}) \text{ for almost all } t. \tag{2.21}$$

**Remark 2.4.** We denote the unique solution described in the theorem as a *conservative* weak solution of the Camassa–Holm equation.

## 3. Characterization of multipeakon solutions

Peakons are given by

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|} \tag{3.1}$$

where $p_i, q_i$ satisfy the system of ordinary differential equations

$$\dot{q}_i = \sum_{j=1}^{n} p_j e^{-|q_i-q_j|}, \tag{3.2a}$$

$$\dot{p}_i = \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i-q_j|}. \tag{3.2b}$$

Note that (3.2) is a Hamiltonian system, viz.,

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i},$$

with Hamiltonian

$$H(p,q) = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|}.$$

Clearly, if the $q_i$ remain distinct, the system (3.2) allows for a global smooth solution. By inserting that solution into (3.1) we find that $u$ is a global weak solution of the Camassa–Holm equation. See, e.g., [17] for details. In the case where $p_i(0)$ have the same sign for all $i \in \{1, \dots, n\}$, then the $q_i(t)$ remain distinct, and (3.2) admits a unique global solution, see [7,11,10,17]. In this case, the peakons are traveling in the same direction. However, when two peakons have opposite signs, see, e.g., [5,6], collisions may occur, and if so, the system (3.2) blows up, or, more precisely, some of the $p_i$ blow up.

Our aim is to use the variables $(y, U, H)$ to characterize multipeakons in a way that avoids the problems related to blow up. In particular, we will derive a new system of ordinary differential equations for the multipeakon solutions which is well-posed even when collisions occur.

We consider initial data $\bar{u}$ given by

$$\bar{u}(x) = \sum_{i=1}^{n} p_i e^{-|x-\xi_i|}. \tag{3.3}$$

Without loss of generality, we assume that the $p_i$ are all nonzero, and that the $\xi_i$ are all distinct. From Theorem 2.3 we know that there exists a unique and global weak solution with initial data (3.3), and the aim is to characterize this solution explicitly. The most natural way to define a multipeakon is to say that, given a time $t$, there exist $p_i$ and $\xi_i$ such that $u$ can be expressed in the form given in (3.1). However, the variables $p_i$ are not appropriate since they blow up at collisions. That is why we will prefer the following characterization of multipeakons. Given the position of the peaks $x_i$ and the values $u_i$ of $u$ at the peaks, $u$ is defined on each interval $[x_i, x_{i+1}]$ as the solution of the Dirichlet problem

$$u - u_{xx} = 0,\ u(x_i) = u_i,\ u(x_{i+1}) = u_{i+1}.$$

Clearly, the function (3.1) satisfies this for each fixed time $t$, but we will now show that this property persists for conservative solutions.

A multipeakon is piecewise $C^\infty$ with discontinuous first derivative at the peaks. From (3.2a), we infer that

$$\dot{q}_i = u(q_i)$$

which means that the peaks and therefore the discontinuities follow the characteristics. In this case, the Lagrangian point of view becomes very convenient, as the location of the peaks is known a priori. Let us prove that $\bar{X} = (\bar{y}, \bar{U}, \bar{H})$ given by

$$\bar{y}(\xi) = \xi, \tag{3.4a}$$

$$\bar{U}(\xi) = \bar{u}(\xi), \tag{3.4b}$$

$$\bar{H}(\xi) = \int_{-\infty}^{\xi} (u^2 + u_x^2)\, dx, \tag{3.4c}$$

is a representative of $u$ in Lagrangian coordinates, that is, $[\bar{X}] = L(\bar{u}, (\bar{u}^2 + \bar{u}_x^2)dx)$. First we have to check that $\bar{X} \in \mathcal{F}$. Since $\bar{u}$ is a multipeakon, from (3.3), we have that $\bar{u} \in W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})$. Hence, $\bar{U}$ and $\bar{H}$ both belong to $W^{1,\infty}(\mathbb{R})$ while $\bar{y} - \mathrm{Id}$ is identically zero. Due to the exponential decay of $\bar{u}$ and $\bar{u}_x$ and since $\bar{H}_\xi \in L^\infty(\mathbb{R})$, we have $\bar{H}_\xi \in L^2(\mathbb{R})$. The properties (2.12) are straightforward to check. Furthermore, it is not hard to check that $M([\bar{X}]) = (\bar{u}, (\bar{u}^2 + \bar{u}_x^2)dx)$. Hence, since $L \circ M = \mathrm{Id}$, we get $[\bar{X}] = L(\bar{u}, (\bar{u}^2 + \bar{u}_x^2)dx)$. We set $\mathcal{A} = \mathbb{R} \setminus \{\xi_1, \dots, \xi_n\}$. The functions $\bar{U}$ and $\bar{H}$ belong to $C^2(\mathcal{A})$ (they even belong to $C^\infty(\mathcal{A})$). This property is preserved by the equation, as the next proposition shows.

**Proposition 3.1.** *Given $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) \in \mathcal{F}$ such that $\bar{X} \in [C^2(\mathcal{A})]^3$, the solution $X = (y, U, H)$ of (2.9) with $\bar{X}$ as initial data belongs to $C^1(\mathbb{R}_+, [C^2(\mathcal{A})]^3)$.*

**Proof.** We prove this proposition by repeating the contraction argument of [18, Theorem 2.3], replacing $E$ by

$$\bar{E} = E \cap [C^2(\mathcal{A})]^3.$$

The norm on $\bar{E}$ is given by

$$\|X\|_{\bar{E}} = \|X\|_E + \|y - \mathrm{Id}\|_{W^{2,\infty}(\mathcal{A})} + \|U\|_{W^{2,\infty}(\mathcal{A})} + \|H\|_{W^{2,\infty}(\mathcal{A})}$$

We have to prove that $\mathcal{P} : X \mapsto P$ and $\mathcal{Q} : X \mapsto Q$ are Lipschitz maps from bounded sets of $\bar{E}$ into $H^1(\mathbb{R}) \cap C^2(\mathcal{A})$. Given a bounded set $B = \{X \in \bar{E} \mid \|X\|_{\bar{E}} \leq C_B\}$ where $C_B$ is a positive constant we have, from [18, Lemma 2.1], that

$$\left\|\mathcal{Q}(X) - \mathcal{Q}(\bar{X})\right\|_{L^\infty(\mathbb{R})} \leq C \left\|X - \bar{X}\right\|_E \leq C \left\|X - \bar{X}\right\|_{\bar{E}}$$

for a constant $C$ which only depends on $C_B$. The derivative of $Q$ is given by (2.10). When a map is Lipschitz on bounded sets, we will say that it is B-Lipschitz. It is not hard to prove that the products of two B-Lipschitz maps from $\bar{E}$ into $C(\mathcal{A})$ is also a B-Lipschitz map from $\bar{E}$ into $C(\mathcal{A})$. Hence, from (2.10), $\mathcal{Q}$ is B-Lipschitz from $\bar{E}$ into $C^1(\mathcal{A})$. In the same way, we obtain the same result for $\mathcal{P}$. We can compute the derivative of $P_\xi$ and $Q_\xi$ on $\mathcal{A}$, and we obtain

$$Q_{\xi\xi} = -\frac{1}{2}H_{\xi\xi} - (UU_\xi - Qy_\xi)y_\xi - (\frac{1}{2}U^2 - P)y_{\xi\xi}, \tag{3.5}$$

$$P_{\xi\xi} = Qy_{\xi\xi} - \frac{1}{2}H_\xi y_\xi - (\frac{1}{2}U^2 - P)y_\xi^2. \tag{3.6}$$

Since $Q_{\xi\xi}$ and $P_{\xi\xi}$ are given as sums and products of B-Lipschitz maps from $\bar{E}$ into $C(\mathcal{A})$, we have that $\mathcal{Q}$ and $\mathcal{P}$ are B-Lipschitz from $\bar{E}$ into $C^2(\mathcal{A})$. The system of equation (2.9) can be written in the condensed form

$$X_t = F(X)$$

where $F\colon \bar{E} \to \bar{E}$ is given by $F(X) = [U, -Q, U^3 - 2PU]$. We can see that each component of $F$ consist of products and sums of B-Lipschitz maps from $\bar{E}$ into $C^2(\mathcal{A})$. Hence, $F$ is B-Lipschitz from $\bar{E}$ to $\bar{E}$ and, by the standard contraction argument, we obtain the short-time existence of solutions in $\bar{E}$. As far as global existence is concerned, we know that, for initial data in $W^{1,\infty}(\mathbb{R})$, $\|X\|_{W^{1,\infty}(\mathbb{R})}$ does not blow up, see [18, Lemma 2.4]. For the second derivative, we have, for any $\xi \in \mathcal{A}$, that

$$\begin{aligned} y_{\xi\xi t} &= U_{\xi\xi}, \\ U_{\xi\xi t} &= \frac{1}{2}H_{\xi\xi} + \left[\frac{1}{2}U^2 - P\right] y_{\xi\xi} + \left[UU_\xi y_\xi - Qy_\xi^2\right], \\ H_{\xi\xi t} &= \left[-2QU\right] y_{\xi\xi} + \left[3U^2 - 2P\right] U_{\xi\xi} \\ &\quad + \left[Uy_\xi H_\xi + U^3 y_\xi - 2PUy_\xi^2 + 6UU_\xi^2 - 4QU_\xi y_\xi\right]. \end{aligned} \tag{3.7}$$

The system (3.7) is affine (it equals the sum of a linear transformation and a constant) with respect to $y_{\xi\xi}$, $U_{\xi\xi}$ and $H_{\xi\xi}$. Hence, on any time interval $[0, T)$, we have

$$\|X_{\xi\xi}(t,\,\cdot\,)\|_{L^\infty(\mathcal{A})} \leq \|X_{\xi\xi}(0,\,\cdot\,)\|_{L^\infty(\mathcal{A})} + C + C\int_0^t \|X_{\xi\xi}(\tau,\,\cdot\,)\|_{L^\infty(\mathcal{A})}\,d\tau$$

where $C$ is a constant that only depends on $\sup_{t\in[0,T)}\|X(t,\,\cdot\,)\|_{W^{1,\infty}(\mathbb{R})}$, which is bounded. Gronwall's lemma allows us to conclude that $\|X(t,\,\cdot\,)\|_{W^{2,\infty}(\mathcal{A})}$ does not blow up, and therefore the solution is globally defined in $\bar{E}$. □

Next we want to prove that the solution given by Theorem 2.3 with initial data (3.3) satisfies $u - u_{xx} = 0$ between the peaks. Assuming that $y_\xi(t,\xi) \neq 0$, we formally have

$$u_x \circ y = \frac{U_\xi}{y_\xi}$$

and

$$u_{xx} \circ y = \left(\frac{U_\xi}{y_\xi}\right)_\xi \frac{1}{y_\xi} = \frac{U_{\xi\xi}y_\xi - y_{\xi\xi}U_\xi}{y_\xi^3}.$$

Hence,

$$(u - u_{xx}) \circ y = \frac{Uy_\xi^3 - U_{\xi\xi}y_\xi + y_{\xi\xi}U_\xi}{y_\xi^3}, \tag{3.8}$$

and we are naturally led to analyze the quantity

$$A = Uy_\xi^3 - U_{\xi\xi}y_\xi + y_{\xi\xi}U_\xi. \tag{3.9}$$

For a given *fixed* $\xi \in \mathcal{A}$, we differentiate (3.9) with respect to time and, after using (2.9), (2.11), and (3.7), we obtain

$$\begin{aligned}\frac{dA}{dt} &= 3UU_\xi y_\xi^2 - Qy_\xi^3 - U_\xi U_{\xi\xi} - y_\xi\big(\frac{1}{2}H_{\xi\xi} + UU_\xi y_\xi + \frac{1}{2}U^2 y_{\xi\xi} - Qy_\xi^2 - Py_{\xi\xi}\big)\\ &\quad + \left(\frac{1}{2}H_\xi + (\frac{1}{2}U^2 - P)y_\xi\right) y_{\xi\xi} + U_\xi U_{\xi\xi}\\ &= 2U_\xi U y_\xi^2 - \frac{1}{2}y_\xi H_{\xi\xi} + \frac{1}{2}H_\xi y_{\xi\xi}. \end{aligned}\tag{3.10}$$

We differentiate (2.12c) with respect to $\xi$ and get

$$y_{\xi\xi}H_\xi + y_\xi H_{\xi\xi} = 2y_\xi y_{\xi\xi}U^2 + 2y_\xi^2 UU_\xi + 2U_\xi U_{\xi\xi}. \tag{3.11}$$

After inserting the value of $y_\xi H_{\xi\xi}$ given by (3.11) into (3.10) and multiplying the equation by $y_\xi$, we get

$$y_\xi\frac{dA}{dt} = y_\xi^3 U_\xi U + (H_\xi y_\xi y_{\xi\xi} - y_\xi^2 y_{\xi\xi}U^2) - U_\xi y_\xi U_{\xi\xi}.$$

Hence, by (2.12c),

$$y_\xi\frac{dA}{dt} = U_\xi A$$

or, since $y_{\xi t} = U_\xi$,

$$y_\xi \frac{dA}{dt} = y_{\xi t} A. \tag{3.12}$$

Let us prove that $\frac{A}{y_\xi}$ is $C^1$ in time (we recall that we keep $\xi$ fixed in $\mathcal{A}$). We have

$$\frac{A}{y_\xi} = U y_\xi^2 - U_{\xi\xi} + \frac{y_{\xi\xi} U_\xi}{y_\xi} \tag{3.13}$$

$$= U y_\xi^2 - U_{\xi\xi} + \frac{y_{\xi\xi} U_\xi}{y_\xi + H_\xi} + \frac{y_{\xi\xi} H_\xi U_\xi}{(y_\xi + H_\xi) y_\xi}. \tag{3.14}$$

After multiplying (3.11) by $\frac{U_\xi}{y_\xi}$, we obtain

$$\frac{y_{\xi\xi} H_\xi U_\xi}{y_\xi} = -H_{\xi\xi} U_\xi + 2 y_{\xi\xi} U^2 U_\xi + 2 y_\xi U U_\xi^2 + 2 \frac{U_\xi^2}{y_\xi} U_{\xi\xi} \tag{3.15}$$

$$= -H_{\xi\xi} U_\xi + 2 y_{\xi\xi} U^2 U_\xi + 2 y_\xi U U_\xi^2 + 2 (H_\xi - y_\xi U^2) U_{\xi\xi}$$

because $\frac{U_\xi^2}{y_\xi} = H_\xi + y_\xi U^2$, from (2.12c). Hence, we can rewrite (3.14) as

$$\frac{A}{y_\xi} = \frac{J(X, X_\xi, X_{\xi\xi})}{y_\xi + H_\xi}$$

for some polynomial $J$. Since $X \in C^1(\mathbb{R}, \bar{E})$, we have $X$, $X_\xi$ and $X_{\xi\xi}$ are $C^1$ in time. Since $X(t)$ remains in $\mathcal{G}$ for all $t$, from (2.12b), we have $y_\xi + H_\xi > 0$ and therefore $1/(y_\xi + H_\xi)$ is $C^1$ in time. Hence, $A/y_\xi$ is $C^1$ in time. For any time $t$ such that $y_\xi(t) \neq 0$, that is, for almost every $t$ (see [18, Lemma 2.7]) we have

$$\frac{d}{dt}\left(\frac{A}{y_\xi}\right) = \frac{A_t y_\xi - y_{\xi t} A}{y_\xi^2} = 0$$

from (3.12). Hence, $\frac{A}{y_\xi}$ is constant in time, i.e.,

$$A(t, \xi) = K(\xi) y_\xi(t, \xi), \tag{3.16}$$

for some constant $K(\xi)$ independent of time. This leads to

$$y_\xi^2 (u - u_{xx}) \circ y = K(\xi)$$

which corresponds to the conservation of spatial angular momentum as defined in [1], see [14]. For the multipeakons at time $t = 0$, we have $y(0, \xi) = \xi$ and $(u - u_{xx})(0, \xi) =$ for all $\xi \in \mathcal{A}$. Hence,

$$\frac{A}{y_\xi}(t, \xi) = 0 \tag{3.17}$$

for all time $t$ and all $\xi \in \mathcal{A}$.

**Proposition 3.2.** *The energy $\mu$ admits a singular part $\mu_s$ only when two peaks collide and the support of $\mu_s$ corresponds to the points of collision of the peaks. Moreover, no more than two peaks can collide at the same time.*

**Proof.** Let $x$ be a singular point of $\mu$. We claim that $y^{-1}(\{x\})$ then is a closed interval of length $\mu_{\rm s}(\{x\})$. Let us prove this. For any $\xi$, from the definition (2.16a) of $y$, there exists an increasing sequence $x_i$ such that $\lim_{i\to\infty} x_i = y(\xi)$ and

$$\mu((-\infty, x_i)) + x_i \le \xi. \tag{3.18}$$

Since $(-\infty, x_i)$ is an increasing sequence of sets and $(-\infty, y(\xi)) = \cup_{i\in\mathbb{N}}(-\infty, x_i)$, we have $\lim_{i\to\infty} \mu((-\infty, x_i)) = \mu((-\infty, y(\xi)))$, and it follows from (3.18) that

$$\mu((-\infty, y(\xi))) + y(\xi) \le \xi. \tag{3.19}$$

We set $\bar\xi = \mu((-\infty, x)) + x$ and, using (3.19), it is not hard to prove that $\bar\xi$ is the smallest element of $y^{-1}(\{x\})$. Let $\xi \in y^{-1}(\{x\})$, by definition of $y$, there exists a decreasing sequence $x_i$ which converges to $x$ such that

$$\mu((-\infty, x_i)) + x_i > \xi.$$

Letting $i$ tend to infinity, we obtain

$$\begin{aligned}\xi &\le \mu((-\infty, x]) + x\\ &\le \mu((-\infty, x)) + \mu_{\rm s}(\{x\}) + x\\ &\le \bar\xi + \mu_{\rm s}(\{x\}).\end{aligned}$$

Hence, $\xi \in [\bar\xi, \bar\xi + \mu_{\rm s}(\{x\})]$ and $y^{-1}(\{x\}) \subset [\bar\xi, \bar\xi + \mu_{\rm s}(\{x\})]$. Conversely, let us consider $\xi \in [\bar\xi, \bar\xi + \mu_{\rm s}(\{x\})]$. Since $y$ is increasing, $y(\xi) \ge y(\bar\xi) = x$. Assume that $y(\xi) > x$. Then, it follows from the definition of $y$ that there exists $x' > x$ such that

$$\mu((-\infty), x') + x' \le \xi.$$

Since $x' > x$, we have

$$\begin{aligned}\mu((-\infty, x')) &\ge \mu((-\infty, x])\\ &= \mu((-\infty, x)) + \mu_{\rm s}(\{x\})\\ &= \bar\xi - x + \mu_{\rm s}(\{x\}).\end{aligned}$$

Hence, $\bar\xi - x + \mu_{\rm s}(\{x\}) + x' \le \xi$ which implies $\bar\xi + \mu_{\rm s}(\{x\}) < \xi$. This contradicts the fact that $\xi \in [\bar\xi, \bar\xi + \mu_{\rm s}(\{x\})]$. Our claim is proved. This claim is a general result and does not depend on the multipeakon structure of the initial data. For solutions with multipeakon initial data, we have the following result.

**Lemma 3.3.** *If $y_\xi(t,\xi)$ vanishes at some point $\bar\xi$ in the interval $(\xi_i, \xi_{i+1})$, then $y_\xi(t,\xi)$ vanishes everywhere in $(\xi_i, \xi_{i+1})$.*

**Proof of Lemma 3.3.** Let $B$ be the set

$$B = \{\xi \in (\xi_i, \xi_{i+1}) \mid y_\xi(t,\xi) = 0\}.$$

The set $B$ is not empty as $\bar\xi \in B$. Since $y_\xi(t,\,\cdot\,) \in C(\mathcal{A})$, $B$ is closed (relatively in $(\xi_i, \xi_{i+1})$). Let us prove that $B$ is also open. Take a point $\xi_0 \in B$. We have $y_\xi(t,\xi_0) = 0$ and, by (2.12b), it implies $H_\xi(t,\xi_0) > 0$. Since $H_\xi(t,\,\cdot\,) \in C(\mathcal{A})$, there exists an open interval $I$ around $\xi_0$ such that $H_\xi(t,\xi) > 0$ for all $\xi \in I$. After multiplying (3.17) by $U_\xi$ and using (2.12c), we obtain

$$UU_\xi y_\xi^2 - U_\xi U_{\xi\xi} + y_{\xi\xi} H_\xi - y_{\xi\xi} y_\xi U^2 = 0. \tag{3.20}$$

We differentiate (2.12c) with respect to $\xi$ and obtain

$$U_{\xi\xi} U_\xi = \frac{1}{2}(y_{\xi\xi} H_\xi + H_{\xi\xi} y_\xi) - y_\xi y_{\xi\xi} U^2 - y_\xi^2 U_\xi U. \tag{3.21}$$

Inserting this into (3.20), we end up with an equation of the form

$$y_{\xi\xi} H_\xi = f(\xi) y_\xi \tag{3.22}$$

where $f(\xi)$ is a continuous function of $\xi$. Since $H_\xi \neq 0$ on $I$, we obtain

$$\begin{aligned} y_{\xi\xi}(\xi) &= \frac{f(\xi)}{H_\xi(\xi)} y_\xi(\xi), \\ y_\xi(\xi_0) &= 0. \end{aligned}$$

The unique solution of this ordinary differential equation where $y_\xi(\xi)$ plays the role of the unknown, is $y_\xi(\xi) = 0$. Hence, $y_\xi(\xi) = 0$ for all $\xi \in I$. This implies that $I \subset B$ and therefore $B$ is open. Thus $B$ is an open and closed set, relatively in $(\xi_i, \xi_{i+1})$. Since $(\xi_i, \xi_{i+1})$ is a connected set, it implies that $B = (\xi_i, \xi_{i+1})$, which concludes the proof of the lemma. □

Let us consider a time $T$ when $\mu$ admits a singular point that we denote $\{x\}$. Then, the interval of strictly positive length $y^{-1}(\{x\})$ intersects $\mathcal{A}$ and there exists a point $\bar{\xi} \in (\xi_i, \xi_{i+1})$ for some $i \in \{1, \dots, n\}$ such that $y_\xi(T, \bar{\xi}) = 0$ (with the convention $\xi_0 = -\infty$ and $\xi_{n+1} = \infty$). From Lemma 3.3, we get that $[\xi_i, \xi_{i+1}] \subset y^{-1}(\{x\})$. In particular, $y(\xi_i) = y(\xi_{i+1}) = x$, which means that the point $x$ where the energy concentrates, is located at the collision point between two peaks. We claim that

$$[\xi_i, \xi_{i+1}] = y^{-1}(\{x\}), \tag{3.23}$$

which in particular means that no other peak than the ones originating from $\xi_i$ and $\xi_{i+1}$ can be found at $x$. Assume that (3.23) is not true, then, due to Lemma 3.3, $y^{-1}(\{x\})$ must take the form

$$y^{-1}(\{x\}) = [\xi_j, \xi_k]$$

where $j \leq i$, $k \geq i+1$ and $k - j \geq 2$. We introduce $\bar{X} = (\bar{y}, \bar{U}, \bar{H})$ defined as $\bar{y}(\xi) = y(T, \xi)$, $\bar{U}(\xi) = U(T, \xi)$, and

$$\bar{H}(\xi) = \begin{cases} H(T, \xi_j) \dfrac{\xi_k - \xi}{\xi_k - \xi_j} + H(T, \xi_k) \dfrac{\xi - \xi_j}{\xi_k - \xi_j} & \text{when } \xi \in (\xi_j, \xi_k), \\ H(T, \xi) & \text{otherwise}, \end{cases}$$

so that $\bar{H}$ is linear in $(\xi_j, \xi_k)$ and continuous. Since $y_\xi(T, \xi) = U_\xi(T, \xi) = 0$ and $H_\xi(T, \xi) > 0$ in $(\xi_j, \xi_j)$, it is not hard to check that all the conditions (2.12) are fulfilled and $\bar{X} \in \mathcal{F}$. Let us look at $X(T)$ and $\bar{X}$ in Eulerian coordinates. We write $(u,\mu) = M([X(T)])$ and $(\bar{u},\bar{\mu}) = M([\bar{X}])$. Since $\bar{y}(\xi) = y(T,\xi)$ and $\bar{U}(\xi) = U(T,\xi)$, it is clear that $\bar{u} = u$. We have, using (2.17b),

$$\mu(\{x\}) = \int_{[\xi_j,\xi_k]} H_\xi \, d\xi = H(\xi_k) - H(\xi_j) = \int_{[\xi_j,\xi_k]} \bar{H}_\xi \, d\xi = \bar{\mu}(\{x\}).$$

Hence, for any Borel set $A$,

$$\mu(A) = \mu(A \setminus \{x\}) + \mu(\{x\}) = \bar{\mu}(A \setminus \{x\}) + \bar{\mu}(\{x\}) = \bar{\mu}(A)$$

and $\bar{\mu} = \mu$. Since $M$ is injective, we have $[X(T)] = [\bar{X}]$, which means that $X(T)$ and $\bar{X}$ are equivalent and there exists $f \in \mathcal{F}$ such that

$$X(T) \circ f = \bar{X}. \tag{3.24}$$

The point is that $\bar{X}$ is linear in $(\xi_j,\xi_k)$, and therefore it possesses a priori more regularity than $X(T)$ on this interval. Introduce $\tilde{\mathcal{A}} = \mathbb{R} \setminus \{\xi_1,\ldots,\xi_j,\xi_k,\ldots,\xi_n\}$. We can solve (2.9) backward in time and, slightly abusing the notation, we denote $\bar{X}(t)$ the solution which satisfies $\bar{X}(T) = \bar{X}$ at time $T$. Proposition 3.1 gives us that $\bar{X}(t) \in [C^2(\tilde{\mathcal{A}})]^3$ for all time $t$. Since $X(T)$ and $\bar{X}(T)$ are equivalent and satisfy (3.24), by (2.15), we obtain that $X(t) \circ f = \bar{X}(t)$ for all time $t$. At time $t = 0$, it yields

$$f(\xi) = \bar{y}(0,\xi)$$

because $y(0,\xi) = \xi$. Since $\bar{y}(0,\xi) \in C^2((\xi_j,\xi_k))$, $f \in C^2((\xi_j,\xi_k))$. By definition, see (2.14), the derivative of $f^{-1}$ is bounded. It implies that $f_\xi$ is bounded strictly away from zero, see [18, Lemma 3.2] for a detailed proof of this result. Hence, $f_\xi > 0$ in $(\xi_j,\xi_k)$ and, by the implicit function theorem, $f^{-1}$ belongs to $C^2((\xi_j,\xi_k))$. Hence,

$$u(0,\xi) = U(0,\xi) = \bar{U}(0,f^{-1}(\xi))$$

also belongs to $C^2((\xi_j,\xi_k))$. This contradicts the fact that $(\xi_j,\xi_k)$ contains either $\xi_i$ or $\xi_{i+1}$, which are points where the derivative of $u(0,\xi)$ is discontinuous.

Given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, there exists $\xi$, which may not be unique, such that $x = y(t,\xi)$. If $\xi \in \mathcal{A}^c$, then $x$ corresponds to the position of a peak. For $\xi \in \mathcal{A}$, if $y_\xi(t,\xi) = 0$, then, by Lemma 3.3, $y_\xi(t,\xi') = 0$ for all $\xi' \in (\xi_i,\xi_{i+1})$ where $i$ is such that $\xi \in (\xi_i,\xi_{i+1})$, and $x$ again corresponds to a peak. If $y_\xi(t,\xi) \neq 0$ then, using again Lemma 3.3, we have $y_\xi(t,\xi') \neq 0$ for all $\xi' \in (\xi_i,\xi_{i+1})$. By the implicit function theorem, we obtain that $y(t,\cdot\,)$ is invertible in $(\xi_i,\xi_{i+1})$ and its inverse is $C^2$. It follows that $u(t,x) = U(t,y^{-1}(t,x'))$ is $C^2$ with respect to the spatial variable and the quantity $(u - u_{xx})(t,x)$ is defined in the classical sense. Moreover, by (3.17) and (3.8), we have

$$(u - u_{xx})(t,x) = \frac{A(t,\xi)}{y_\xi^3(t,\xi)} = 0. \tag{3.25}$$

We summarize our results in the following theorem.

**Theorem 3.4.** *Given an initial multipeakon solution* $\bar{u}(x) = \sum_{i=1}^n p_i e^{-|x-\xi_i|}$*, let* $(y, U, H)$ *be the solution of the system* (2.9) *with initial data* $(\bar{y}, \bar{U}, \bar{H})$ *given by* (3.4)*. Between adjacent peaks, say* $x_i = y(t, \xi_i) \neq x_{i+1} = y(t, \xi_{i+1})$*, the solution* $u(t, x)$ *is twice differentiable with respect to the space variable, and we have*

$$(u - u_{xx})(t, x) = 0 \text{ for } x \in (x_i, x_{i+1}).$$

We are now in position to start the derivation of a system of ordinary differential equations for multipeakons.

## 4. A system of ordinary differential equations for multipeakons

For each $i \in \{1, \dots, n\}$, we have, from (2.9),

$$\begin{cases} \dfrac{dy_i}{dt} = u_i, \\ \dfrac{du_i}{dt} = -Q_i, \\ \dfrac{dH_i}{dt} = u_i^3 - 2P_i u_i \end{cases} \tag{4.1}$$

where $y_i$, $u_i$, $H_i$, $P_i$ and $Q_i$ denote $y(t, \xi_i)$, $U(t, \xi_i)$, $H(t, \xi_i)$, $P(t, \xi_i)$ and $Q(t, \xi_i)$, respectively. For almost every $t$, the function $y(t, \cdot\,)$ is invertible. We can make the change of variables $x = y(t, \xi)$ so that $P_i$ and $Q_i$ can be rewritten as

$$P_i = \frac{1}{2} \int_{\mathbb{R}} e^{-|y_i - x|} (u^2 + \frac{1}{2} u_x^2)\, dx, \tag{4.2}$$

$$Q_i = -\frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(y_i - x) e^{-|y_i - x|} (u^2 + \frac{1}{2} u_x^2)\, dx. \tag{4.3}$$

Theorem 3.4 gives us a priori the shape of $u$ and allows us to express $P_i$ and $Q_i$ as a function of the variables $u_i$, $H_i$ and $y_i$ only, thereby transforming (4.1) into a well-posed $3n$-dimensional system of ordinary differential equations.

For almost every time $t$, $y_\xi(t, \xi) > 0$ for almost every $\xi$ and $\xi \mapsto y(t, \xi)$ is invertible, see Sect. 2 . From now on, we will consider such time $t$ and omit it in the notation when there is no ambiguity. For such time, by Theorem 3.4, no peak coincide. From the same theorem, we know that between two adjacent peaks located at $y_i$ and $y_{i+1}$, $u$ satisfies $u - u_{xx} = 0$ and therefore $u$ can be written as

$$u(x) = A_i e^x + B_i e^{-x} \text{ for } x \in [y_i, y_{i+1}], \quad i = 1, \dots, n-1. \tag{4.4}$$

The constants $A_i$ and $B_i$ depend on $u_i$, $u_{i+1}$, $y_i$ and $y_{i+1}$ and read

$$A_i = \frac{e^{-\bar{y}_i}}{2} \left[ \frac{\bar{u}_i}{\cosh(\delta y_i)} + \frac{\delta u_i}{\sinh(\delta y_i)} \right], \tag{4.5}$$

$$B_i = \frac{e^{\bar{y}_i}}{2} \left[ \frac{\bar{u}_i}{\cosh(\delta y_i)} - \frac{\delta u_i}{\sinh(\delta y_i)} \right], \tag{4.6}$$

where we for convenience have introduced the variables

$$\bar{y}_i = \frac{1}{2}(y_i + y_{i+1}), \quad \delta y_i = \frac{1}{2}(y_{i+1} - y_i), \tag{4.7}$$
$$\bar{u}_i = \frac{1}{2}(u_i + u_{i+1}), \quad \delta u_i = \frac{1}{2}(u_{i+1} - u_i).$$

The constants $A_i$ and $B_i$ uniquely determine $u$ on the interval $[y_i, y_{i+1}]$. Thus, we can compute

$$\begin{aligned}\delta H_i = H_{i+1} - H_i &= \int_{y_i}^{y_{i+1}} (u^2 + u_x^2)\, dx \\ &= 2\bar{u}_i^2 \tanh(\delta y_i) + 2\delta u_i^2 \coth(\delta y_i). \end{aligned}\tag{4.8}$$

At this point, we can get some more understanding of what is happening at a time of collision. Let $t^*$ be a time when the two peaks located at $y_i$ and $y_{i+1}$ collide, i.e., such that $\lim_{t\uparrow t^*} \delta y_i(t) = 0$. Since the solution $u$ remains in $H^1$ for all time, the function $u$ remains continuous so that we have $\lim_{t\uparrow t^*} \delta u_i = 0$. Still, $A_i$ and $B_i$ may have a finite limit when $t$ tends to $t^*$. However, we know that the first derivative blows up (see [5]), and this implies $\lim_{t\uparrow t^*} B_i = -\lim_{t\uparrow t^*} A_i = \infty$. Thus $\delta u_i$ tends to zero but slower than $\delta y_i$. We can now be more precise: Letting $t$ tend to $t^*$ in (4.8), we obtain, to first order in $\delta y_i$, that

$$\delta u_i = \sqrt{\frac{\delta H_i}{2}}\sqrt{\delta y_i} + o(\delta y_i).$$

Recall that $H$ and $y$ are increasing functions, and therefore $\delta H_i$ and $\delta y_i$ are positive ($\delta H_i$ is even strictly positive in this case). Hence, we see that $\delta u_i$ tends to zero at the same rate as $\sqrt{\delta y_i}$. Let us now turn to the computation of $P_i$ as given by (4.2). This computation is quite long but not difficult. We will not give all the intermediate steps but enough so that a courageous reader will have no problems filling in the gaps. We start by writing $u$ as

$$u(t,x) = \sum_{j=0}^{n} (A_j e^x + B_j e^{-x}) \chi_{(y_j, y_{j+1})}(x).$$

We have set $y_0 = -\infty$, $y_{n+1} = \infty$, $u_0 = u_{n+1} = 0$, and $A_0 = u_1 e^{-y_1}$, $B_0 = 0$, $A_n = 0$, $B_n = u_n e^{y_n}$. We have

$$u^2 + \frac{1}{2}u_x^2 = \sum_{j=0}^{n} \left( \frac{3}{2} A_j^2 e^{2x} + A_j B_j + \frac{3}{2} B_j^2 e^{-2x} \right) \chi_{(y_j, y_{j+1})}. \tag{4.9}$$

Introduce

$$\kappa_{ij} = \begin{cases} -1 & \text{if } j \geq i, \\ 1 & \text{otherwise.} \end{cases}$$

Inserting (4.9) into (4.2), we obtain

$$P_i = \frac{1}{2} \sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-\kappa_{ij}(y_i - x)} \left( \frac{3}{2} A_j^2 e^{2x} + A_j B_j + \frac{3}{2} B_j^2 e^{-2x} \right) dx. \tag{4.10}$$

We have

$$\int_{y_j}^{y_{j+1}} e^{-\kappa_{ij}(y_i-x)}A_j^2e^{2x}\,dx = e^{-\kappa_{ij}y_i}A_j^2\frac{e^{(2+\kappa_{ij})y_{j+1}}-e^{(2+\kappa_{ij})y_j}}{2+\kappa_{ij}}$$
$$= e^{-\kappa_{ij}y_i}A_j^2\exp\left((2+\kappa_{ij})\bar{y}_j\right)\frac{2\sinh\left((2+\kappa_{ij})\delta q_j\right)}{2+\kappa_{ij}}. \tag{4.11}$$

From (4.5) and (4.8), we get

$$A_j^2 = \frac{e^{-2\bar{y}_j}}{\sinh^2(2\delta y_i)}\left[\bar{u}_j^2\sinh^2(\delta y_j)+2\bar{u}_j\delta u_j\sinh(\delta y_j)\cosh(\delta y_j)+\delta u_j^2\cosh^2(\delta y_j)\right]$$

and

$$A_j^2 = \frac{e^{-2\bar{y}_j}}{4\sinh(2\delta y_i)}\left[\delta H_j+4\bar{u}_j\delta u_j\right]. \tag{4.12}$$

Similarly, we obtain

$$B_j^2 = \frac{e^{2\bar{y}_j}}{4\sinh(2\delta y_i)}\left[\delta H_j-4\bar{u}_j\delta u_j\right], \tag{4.13}$$

and

$$A_jB_j = \frac{1}{4\sinh(2\delta y_i)}\left[4\bar{u}_j^2\tanh(\delta y_j)-\delta H_j\right]. \tag{4.14}$$

Hence, inserting (4.12) into (4.11), we get

$$\int_{y_j}^{y_{j+1}} e^{-\kappa_{ij}(y_i-x)}A_j^2e^{2x}\,dx$$
$$= \frac{e^{-\kappa_{ij}y_i}e^{\kappa_{ij}\bar{y}_j}}{2(2+\kappa_{ij})\sinh(2\delta y_j)}\sinh\left((2+\kappa_{ij})\delta y_j\right)[\delta H_j+4\bar{u}_j\delta u_j]. \tag{4.15}$$

In the same way we find

$$\int_{y_j}^{y_{j+1}} e^{-\kappa_{ij}(y_i-x)}A_jB_j\,dx = \frac{e^{-\kappa_{ij}y_i}e^{\kappa_{ij}\bar{y}_j}}{2\sinh(2\delta y_j)}\sinh(\delta y_j)[4\bar{u}_j^2\tanh(\delta y_j)-\delta H_j)], \tag{4.16}$$

and

$$\int_{y_j}^{y_{j+1}} e^{-\kappa_{ij}(y_i-x)}B_j^2e^{-2x}\,dx$$
$$= \frac{e^{-\kappa_{ij}y_i}e^{\kappa_{ij}\bar{y}_j}}{2(\kappa_{ij}-2)\sinh(2\delta y_j)}\sinh\left((\kappa_{ij}-2)\delta y_j\right)[\delta H_j-4\bar{u}_j\delta u_j]. \tag{4.17}$$

After collecting (4.15), (4.16) and (4.17), we can rewrite $P_i$ in (4.10) as

$$\begin{aligned} P_i = \sum_{j=0}^{n} & \frac{e^{-\kappa_{ij} y_i} e^{\kappa_{ij}\bar{y}_j}}{4\sinh(2\delta y_j)} \\ & \times \Bigg[ \delta H_j \Big[ \frac{3}{2}\Big( \frac{\sinh\left((2+\kappa_{ij})\delta y_j\right)}{2+\kappa_{ij}} + \frac{\sinh\left((\kappa_{ij}-2)\delta y_j\right)}{\kappa_{ij}-2} \Big) - \sinh(\delta y_j) \Big] \\ & \quad + 6\bar{u}_j \delta u_j \Big[ \frac{\sinh\left((2+\kappa_{ij})\delta y_j\right)}{2+\kappa_{ij}} - \frac{\sinh\left((\kappa_{ij}-2)\delta y_j\right)}{\kappa_{ij}-2} \Big] \\ & \quad + 4\bar{u}_j^2 \sinh(\delta y_j)\tanh(\delta y_j) \Bigg]. \end{aligned} \tag{4.18}$$

By using only trigonometric manipulations and the fact that $\kappa_{ij}^2 = 1$, we get the following two identities

$$\frac{3}{2}\Big( \frac{\sinh\left((2+\kappa_{ij})\delta y_j\right)}{2+\kappa_{ij}} + \frac{\sinh\left((\kappa_{ij}-2)\delta y_j\right)}{\kappa_{ij}-2} \Big) - \sinh(\delta y_j) = 2\sinh(\delta y_j)\cosh^2(\delta y_j)$$

and

$$\frac{\sinh\left((2+\kappa_{ij})\delta y_j\right)}{2+\kappa_{ij}} - \frac{\sinh\left((\kappa_{ij}-2)\delta y_j\right)}{\kappa_{ij}-2} = \frac{4\kappa_{ij}}{3}\sinh^3(\delta y_j)$$

that we use to simplify (4.18). We end up with

$$P_i = \sum_{j=0}^{n} \frac{e^{-\kappa_{ij} y_i} e^{\kappa_{ij}\bar{y}_j}}{8\cosh(\delta y_j)} \left[ 2\delta H_j \cosh^2(\delta y_j) + 8\kappa_{ij}\bar{u}_j\delta u_j \sinh^2(\delta y_j) + 4\bar{u}_j^2\tanh(\delta y_j) \right],$$

or

$$P_i = \sum_{j=0}^{n} P_{ij} \tag{4.19}$$

with

$$P_{ij} = \begin{cases} e^{(y_1 - y_i)}\frac{u_1^2}{4} & \text{for } j = 0, \\ \frac{e^{-\kappa_{ij} y_i} e^{\kappa_{ij}\bar{y}_j}}{8\cosh(\delta y_j)} \left[ 2\delta H_j \cosh^2(\delta y_j) \right. & \text{for } j = 1, \dots, n-1, \\ \quad \left. +8\kappa_{ij}\bar{u}_j\delta u_j \sinh^2(\delta y_j) + 4\bar{u}_j^2\tanh(\delta y_j) \right] & \\ e^{(y_i - y_n)}\frac{u_n^2}{4} & \text{for } j = n. \end{cases} \tag{4.20}$$

The term $Q_i$ can be computed in the same way. We have

$$\begin{aligned} Q_i &= \sum_{j=0}^{n} -\frac{1}{2}\int_{y_j}^{y_{j+1}} \operatorname{sgn}(q_i - x) e^{-\kappa_{ij}(y_i - x)} (u^2 + \frac{1}{2}u_x^2)\, dx \\ &= \sum_{j=0}^{n} -\kappa_{ij} \int_{y_j}^{y_{j+1}} e^{-\kappa_{ij}(y_i - x)} \left( \frac{3}{2}A_j^2 e^{2x} + A_j B_j + \frac{3}{2}B_j^2 e^{-2x} \right) dx, \end{aligned}$$

so that we end up with

$$Q_i = -\sum_{j=0}^{n} \kappa_{ij} P_{ij}, \tag{4.21}$$

where $P_{ij}$ is given by (4.20).

We summarize the result in the following theorem.

**Theorem 4.1.** *Given a multipeakon initial data $\bar{u}$, as given by (3.3), let $\bar{y}_i = \xi_i$, $\bar{u}_i = \bar{u}(\xi_i)$ and $\bar{H}_i = \int_{-\infty}^{\xi_i}(\bar{u}^2 + \bar{u}_x^2)\,dx$ for $i = 1,\ldots,n$. Then, there exists a global in time solution $(y_i, u_i, H_i)$ of (4.1), (4.19)–(4.21) with initial data $(\bar{y}_i, \bar{u}_i, \bar{H}_i)$. For each time $t$, we define $u(t,x)$ as the solution of the Dirichlet problem*

$$u - u_{xx} = 0 \text{ with boundary conditions } u(t, y_i(t)) = u_i(t),\ \ u(t, y_{i+1}(t)) = u_{i+1}(t)$$

*on each interval $[y_i(t), y_{i+1}(t)]$. Then, $u$ is a conservative solution of the Camassa–Holm equation, and we denote it the multipeakon solution.*

The simplest cases can be computed explicitly, and for completeness we include the cases $n = 1, 2$. In addition we present the case $n = 4$ numerically (with and without collisions).

**Example 4.2.** (i) Let $n = 1$. Here we find that $P_1 = \frac{1}{2}u_1^2$ and $Q_1 = 0$. Thus $u_1 = c$ and $y_1 = ct + a$ for constants $a, c$, and we finally find the familiar one peakon $u(t,x) = ce^{-|x-ct-a|}$. Note that $H_1 = c^2$ is constant. However, we did not use the energy to compute the solution. This is a general result; when there is no collision, the first two equations in (4.1) decouple from the last one, and the energy equation is not needed.

(ii) Let $n = 2$. We will solve analytically the case of an antisymmetric pair of peakons when the two peakons collide. In this case, at the collision point the energy concentrates in a single point, see [5]. We take the origin of time equal to the time of collision. The initial conditions are

$$y_1(0) = y_2(0) = u_1(0) = u_2(0) = \delta H_0(0) = \delta H_2(0) = 0, \quad \delta H_1(0) = E^2$$

where $E > 0$ corresponds to the energy of the system. The solution remains antisymmetric. Let us assume this for the moment and write

$$\begin{aligned} y &= y_2 = -y_1, \\ u &= u_2 = -u_1, \\ h &= \delta H_1, \end{aligned} \tag{4.22}$$

and, since the total energy is preserved ($H(t,\infty)$ is constant), we have $\delta H_0 = \delta H_2 = \frac{1}{2}(E^2 - h)$. We compute $P_i$ and $Q_i$ using (4.19) and (4.21). After some calculations, we obtain that, whenever the solution is antisymmetric, $P_1 = P_2 = P$

and $Q_1 = -Q_2 = -Q$ where

$$\begin{aligned} P &= (2u^2 + h)\frac{1+e^{-2y}}{8}, \\ Q &= u^2\frac{1-e^{-2y}}{4} - h\frac{1+e^{-2y}}{8}. \end{aligned} \tag{4.23}$$

We are led to the following system of ordinary differential equations

$$\begin{aligned} y_t &= u, \\ u_t &= -Q, \\ h_t &= 2(u^3 - 2Pu), \end{aligned} \tag{4.24}$$

with initial conditions $y(0) = u(0) = 0$ and $h(0) = E^2$. This system can be solved and, after retrieving the original variables by (4.22), since the identities $P_1 = P_2$ and $Q_1 = -Q_2$ hold, $(y_1, y_2, u_1, u_2, H_1, H_2)$ is the unique solution of (4.1) and therefore it is antisymmetric. From (4.23), we get $Q = \frac{1}{2}u^2 - P$. Hence, $h_t = 4uQ = -4uu_t$ and, after integration,

$$h = -2u^2 + E^2.$$

We insert this in (4.24) which yields the following second-order differential equation

$$y_{tt} + \frac{y_t^2}{2} = \frac{E^2}{8}(1 + e^{-2y})$$

with initial data $y(0) = y_t(0) = 0$. We can get rid of the factor $E^2$ by rescaling the time variable, $t \mapsto Et$, and the equation we have to solve is

$$y_{tt} + \frac{y_t^2}{2} = \frac{1}{8}(1 + e^{-2y}), \tag{4.25}$$

$$y(0) = y_t(0) = 0. \tag{4.26}$$

By a phase-plane analysis, one can prove that $y_t(t) < 0$ for $t < 0$, $y_t(t) > 0$ for $t > 0$ and $y(t) > 0$ for all $t \neq 0$. We make the change of variables $z = e^{-2y}$ and (4.25) becomes

$$-4zz_{tt} + 5z_t^2 = z^2(1 + z). \tag{4.27}$$

We multiply the equation by $z^\alpha z_t$ where $\alpha$ is a constant to be determined and get

$$-4z^{\alpha+1}z_t z_{tt} + 5z_t^3 z^\alpha = z^{2+\alpha}(1 + z)z_t. \tag{4.28}$$

The term on the left is the derivative of $z^\alpha z_t^2$ if $\alpha = -\frac{5}{2}$. Taking this value for $\alpha$, (4.28) can be integrated and we obtain, after some calculations,

$$z_t^2 = z^2(1 - z). \tag{4.29}$$

Hence,

$$z_t = -\varepsilon z\sqrt{1 - z} \tag{4.30}$$

where $\varepsilon = \operatorname{sgn}(t)$. We use the change of variables $v = \sqrt{1-z}$ and obtain $v_t = -\varepsilon/2(1-v^2)$, which can be integrated and gives $v(t) = \varepsilon \tanh(\frac{t}{2})$. Finally, going back to the original variables, we obtain

$$y(t) = \ln\cosh(\frac{Et}{2}),$$
$$u(t) = \frac{E}{2}\tanh(\frac{Et}{2}).$$

Note that the ordinary differential equation (4.30) does not satisfy the Lipschitz condition and therefore does not have a unique solution. However, the solution we are looking for is in fact the solution of the second-order ordinary differential equation

$$z_{tt} = z - \frac{3}{2}z^2 \tag{4.31}$$

which is obtained from (4.27) by inserting (4.29), which is perfectly well-posed. It is not hard to check that the solution $z(t)$ we obtained indeed satisfies (4.31). In Fig. 1c, we plot $\delta H_0$, $\delta H_1$, and $\delta H_2$ which represent the energy contained between $-\infty$ and $y_1$, $y_1$ and $y_2$, $y_2$ and $+\infty$, respectively. We see how the energy concentrates at collision time.

The case with two peakons has been computed by Wahlén [20] (see also [2,3,4]). For completeness, we reproduce his results here. We have[a]

$$\begin{aligned} y_1 &= \ln\left(\frac{c_1 - c_2}{c_1e^{-c_1t} - c_2e^{-c_2t}}\right), & y_2 &= \ln\left(\frac{c_1e^{c_1t} - c_2e^{c_2t}}{c_1 - c_2}\right), \\ u_1 &= \frac{c_1^2 - c_2e^{(c_1-c_2)t}}{c_1 - c_2e^{(c_1-c_2)t}}, & u_2 &= \frac{c_2^2 - c_1e^{(c_1-c_2)t}}{c_2 - c_1e^{(c_1-c_2)t}}, \\ H_1 &= u_1^2, & H_2 &= 2c_1^2 + 2c_2^2 - u_2^2, \end{aligned} \tag{4.32}$$

where $c_1, c_2$ denotes the speed of the peaks $y_1$ and $y_2$, respectively, when $t$ tends to infinity. The initial data is set so that, if there is a collision, it occurs at time $t = 0$.

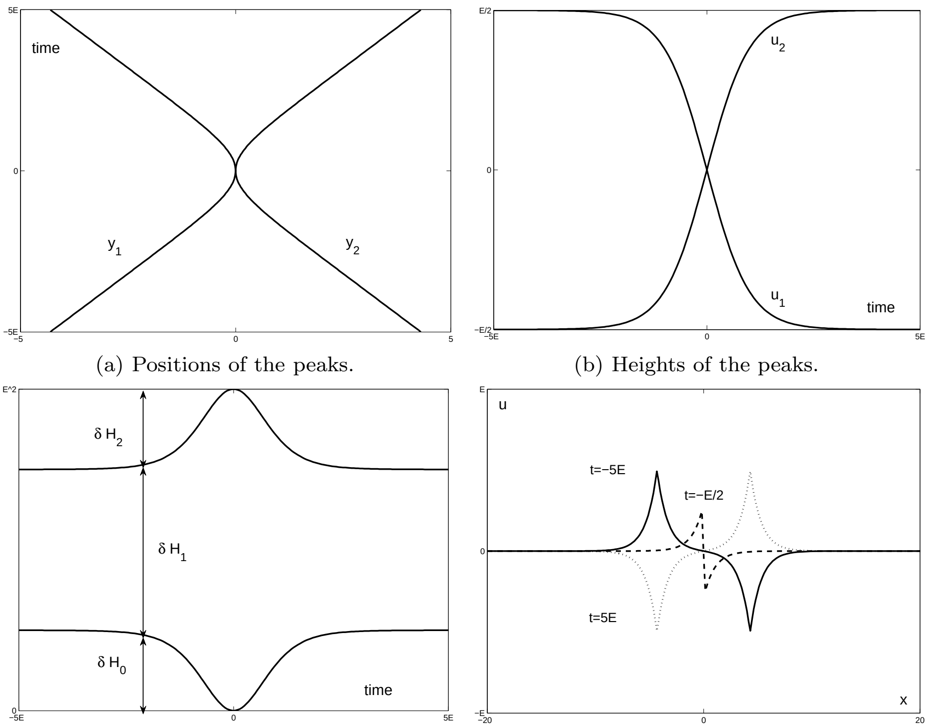
(iii) Let $n = 4$. Consider first the case where there is no wave breaking with all $p_i(0)$ positive for $i = 1, 2, 3, 4$. We take

$$y_1(0) = -10,\ y_2(0) = -5,\ y_3(0) = 0,\ y_4(0) = 5,$$
$$u_1(0) = 4,\ u_2(0) = u_3(0) = u_4(0) = 2.$$

The results are plotted in Fig. 2. Note that the characteristics do not intersect.

Consider finally the case when $p_i(0)$ is positive for $i = 1, 2, 3$, but $p_4(0)$ is negative. The system (4.1) of ordinary differential equations can be solved numerically.

[a] The expressions in (4.32) differ slightly from [20] where two different expressions are given for positive and negative time. This is due to the fact that a relabeling of the solution is implicitly made at collision time so that the two peaks interchange their role at that time. This has no consequence in the Eulerian picture and the resulting function $u$ in Eulerian coordinates is in both cases a conservative solution of the Camassa–Holm equation.




(a) Positions of the peaks.

(b) Heights of the peaks.

(c) Redistribution of the energy between the peaks.

(d) Plot of the solution at different times

Fig. 1: antisymmetric multipeakon collision

We use the explicit Runge–Kutta solver `ode45` for ordinary differential equations from MATLAB. In Fig. 3, we present the results obtained for the initial data

$$y_1(0) = -10,\ y_2(0) = -5,\ y_3(0) = 0,\ y_4(0) = 5,$$
$$u_1(0) = u_2(0) = u_3(0) = 2,\ u_4(0) = -2.$$

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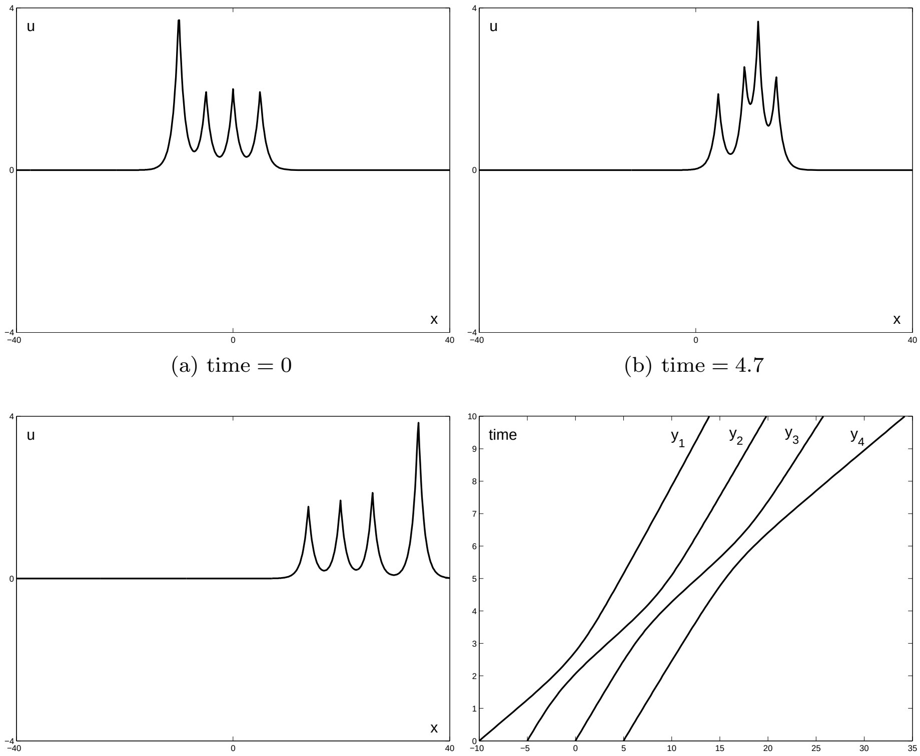


(a) time = 0 (b) time = 4.7

(c) time = 10 (d) positions of the peaks

Fig. 2: Example of multipeakon without collision.


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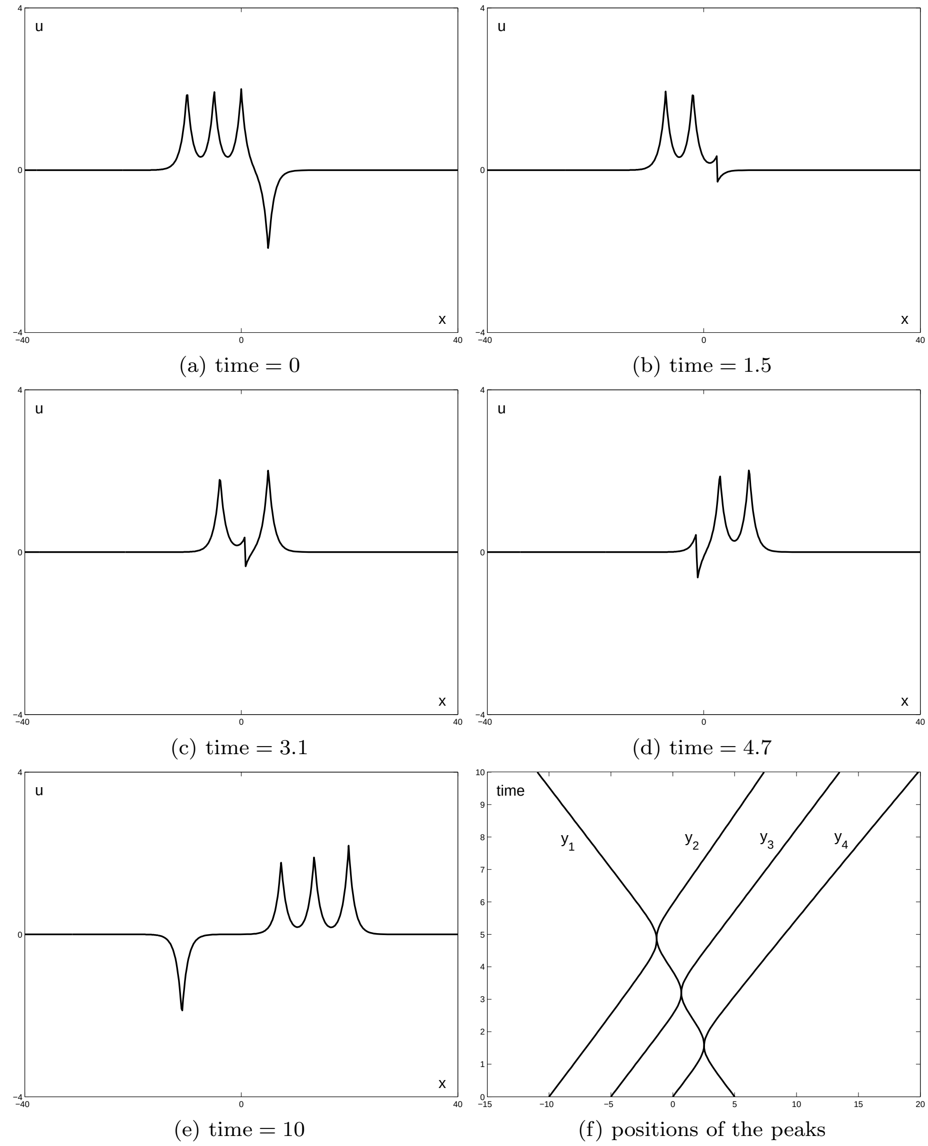


(a) time = 0 (b) time = 1.5

(c) time = 3.1 (d) time = 4.7

(e) time = 10 (f) positions of the peaks

Fig. 3: Example of multipeakon with collision

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