A numerical scheme based on multipeakons for conservative solutions of the Camassa–Holm equation

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Summary. We present a convergent numerical scheme based on multipeakons to compute conservative solutions of the Camassa–Holm equation.

1 Introduction

The Camassa–Holm equation [3, 4] reads

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{1}$$

It has a bi-Hamiltonian structure, is completely integrable, and has infinitely many conserved quantities. In this paper we study the Cauchy problem $u|_{t=0} = \bar{u} \in H^1(\mathbb{R})$ for (1). A highly interesting property of the equation is that for a wide class of initial data the solution experiences wave breaking in finite time in the sense that the solution u remains bounded pointwise while the spatial derivative u_x becomes unbounded pointwise. However, the H^1 norm of u remains finite, see, e.g., [7, 8].

The extension of the solution beyond wave breaking is non-trivial and can be illustrated by studying multipeakons that are solutions of the form

$$u(t,x) = \sum_{i=1}^{n} p_i(t) \mathrm{e}^{-|x-q_i(t)|}, \tag{2}$$

where the $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot{q}_i = \sum_{j=1}^{n} p_j \mathrm{e}^{-|q_i-q_j|}, \quad \dot{p}_i = \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) \mathrm{e}^{-|q_i-q_j|}. \tag{3}$$

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Observe that the solution (2) is not smooth even with continuous functions $(p_i(t), q_i(t))$.

Peakons interact in a way similar to that of solitons of the Korteweg–de Vries equation, and wave breaking may appear when at least two of the q_i's coincide. If all the $p_i(0)$ have the same sign, the peakons move in the same direction and we have a global solution and no wave breaking. Higher peakons move faster than the smaller ones, and when a higher peakon overtakes a smaller, there is an exchange of mass, but no wave breaking takes place. Furthermore, the $q_i(t)$ remain distinct. However, if some of $p_i(0)$ have opposite sign, wave breaking may incur. For simplicity, consider the case with $n = 2$ and one peakon $p_1(0) > 0$ (moving to the right) and one antipeakon $p_2(0) < 0$ (moving to the left). In the symmetric case ($p_1(0) = -p_2(0)$ and $q_1(0) = -q_2(0) < 0$) the solution will vanish pointwise at the collision time t^* when $q_1(t^*) = q_2(t^*)$, that is, $u(t^*, x) = 0$ for all $x \in \mathbb{R}$. Clearly, at least two scenarios are possible; one is to let $u(t, x)$ vanish identically for $t > t^*$, and the other possibility is to let the peakon and antipeakon "pass through" each other in a way that is consistent with the Camassa–Holm equation. In the first case the energy $\int (u^2 + u_x^2)\, dx$ decreases to zero at t^*, while in the second case, the energy remains constant except at t^*. The first solution is denoted a dissipative solution, while the second one is called conservative. Other solutions are also possible. Global dissipative solutions of a more general class of equations were recently derived by Coclite, Holden, and Karlsen [6, 5], while global conservative solutions have been derived by Bressan, Fonte, and Constantin [1, 2], and Holden and Raynaud [9, 10].

Multipeakons are fundamental building blocks for general solutions. Indeed, if the initial data $\bar{u}$ is in H^1 and $\bar{m} := \bar{u} - \bar{u}''$ is a positive Radon measure, then it can proved, see [12], that one can construct a sequence of multipeakons that converges in $L^\infty_{\text{loc}}(\mathbb{R}; H^1_{\text{loc}}(\mathbb{R}))$ to the unique global solution of the Camassa–Holm equation. Here we extend this analysis to avoid the sign constraint on $\bar{m}$ for given initial data $\bar{u}$. We are thereby able to include the case where one may have wave breaking, and we select the conservative solution. This is in part based on [12] and a reformulation of the formulas for multipeakons [13]. In Bressan–Fonte [2] the approximation of general initial data by multipeakons is proved in an abstract way, while we in this paper provide a constructive approach amenable to numerical computations. The method is illustrated on concrete examples.

A completely different numerical approach based on a semi discrete finite difference scheme can be found in [11].

2 Description of the method

It turns out to be useful to reformulate the Camassa–Holm equation as the following system

$$u_t + uu_x + P_x = 0, \qquad P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{4}$$

In [9] a continuous semigroup of conservative solutions for the Camassa–Holm equation is constructed. The semigroup is defined on the set $\mathcal{D}$ of pair (u, μ) where $u \in H^1(\mathbb{R})$ and μ is a Radon measure satisfying $\mu_{\mathrm{ac}} = (u^2 + u_x)\,dx$. The conservative solutions are obtained by a change of variable to Lagrangian coordinates. The new set of coordinates (y, U, H) correspond to the characteristics, $y_t(t,\xi) = u(t, y(t,\xi))$, the Lagrangian velocity, $U(t,\xi) = u(t, y(t,\xi))$, and the Lagrangian energy distribution, $H(t,\xi) = \int_{-\infty}^{y(t,\xi)}(u^2 + u_x^2)\,dx$, respectively. They satisfy the following system of ordinary differential equations in some Banach space E:

$$y_t = U, \qquad U_t = -Q, \qquad H_t = U^3 - 2PU, \tag{5}$$

where

$$Q(t,\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2 y_\xi + H_\xi\big)(\eta)\,d\eta,$$

$$P(t,\xi) = \frac{1}{4}\int_{\mathbb{R}} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2 y_\xi + H_\xi\big)\,d\eta.$$

The set $\mathcal{D}$ is in bijection with a closed subset of E, see [9], and the topology on $\mathcal{D}$ is the topology of E transported to $\mathcal{D}$ by this bijection. In [9], we prove that if $u_n \to u$ in $H^1(\mathbb{R})$ then $(u_n, (u_n^2 + u_{n,x}^2)\,dx) \to (u, (u^2 + u_x^2)\,dx)$ and if $(u_n, \mu_n) \to (u, \mu)$ in $\mathcal{D}$ then $u_n \to u$ in $L^\infty(\mathbb{R})$ and $\mu_n \overset{*}{\rightharpoonup} \mu$.

Multipeakons are particular solutions of the Camassa–Holm equation given by (2)–(3) up to collision time, that is, a time t_c such that $q_i(t_c) = q_{i+1}(t_c)$ for some i. At collision time, the system (3) blows up and we have $\lim_{t\to t_c} p_i(t) = -\lim_{t\to t_c} p_{i+1}(t) = \infty$.

An equivalent definition of a multipeakon with n peaks is given by the set of pairs (q_i, u_i) for $i = 1, \dots, n$ where q_i represents the position of the ith peak and u_i denoted its height. Naturally, we must impose that if $q_i = q_j$ for some i and j, then $u_i = u_j$. The multipeakon function u corresponding to (2) for a given time is then given on the whole real axis as the solution of the boundary value problem: $u - u_{xx} = 0,\ u(x_i) = u_i,\ u(x_{i+1}) = u_{i+1}$. We denote by $\mathcal{M}_n$ the set of all such functions, i.e., functions that correspond to initial data of multipeakon solutions with n peaks. Furthermore, we set $\mathcal{M} = \cup_{n=1}^{\infty}\mathcal{M}_n$. The two representations are equivalent, and later we explain how one goes from one to the other. The advantage of the second representation is that it fits directly into the Lagrangian approach. In [13], we follow this direction and present a system of ordinary differential equations for the conservative multipeakon solutions of the Camassa–Holm equation. Conservative solutions require that we take into account the energy density $(u^2 + u_x^2)\,dx$. We introduce the variables H_i, for $i = 1, \dots, n$, which correspond to energy contained between $-\infty$ and q_i, i.e., $H_i = \int_{-\infty}^{q_i}(u^2 + u_x^2)\,dx$, and the differences $\delta H_i = H_{i+1} - H_i$. When $q_i \neq q_{i+1}$, δH_i is given by

$$\delta H_i = H_{i+1} - H_i = \int_{q_i}^{q_{i+1}} (u^2 + u_x^2)\,dx. \tag{6}$$

Due to the special structure of multipeakons, one can compute δH_i in terms of q_i and u_i. We have

$$\delta H_i = 2\bar{u}_i^2 \tanh(\delta q_i) + 2\delta u_i^2 \coth(\delta q_i) \tag{7}$$

where we for convenience have introduced the variables

$$\bar{q}_i = \frac{1}{2}(q_i + q_{i+1}), \quad \delta q_i = \frac{1}{2}(q_{i+1} - q_i), \quad \bar{u}_i = \frac{1}{2}(u_i + u_{i+1}), \quad \delta u_i = \frac{1}{2}(u_{i+1} - u_i).$$

In [13], we prove that the structure of multipeakons is preserved by the continuous semigroup of conservative solutions of the Camassa–Holm equation and the equations for (q_i, u_i, H_i) are given by the following system

$$\frac{dq_i}{dt} = u_i, \qquad \frac{du_i}{dt} = -Q_i, \qquad \frac{dH_i}{dt} = u_i^3 - 2P_i u_i, \tag{8}$$

with $P_i = \sum_{j=0}^{n} P_{ij}$, $Q_i = -\sum_{j=0}^{n} \kappa_{ij} P_{ij}$, and

$$P_{ij} = \begin{cases} \mathrm{e}^{(q_1 - q_i)} \frac{u_1^2}{4} & \text{for } j = 0, \\ \frac{\mathrm{e}^{-\kappa_{ij} q_i} \mathrm{e}^{\kappa_{ij} \bar{q}_j}}{8\cosh(\delta q_j)} \big[2\delta H_j \cosh^2(\delta q_j) & \\ \quad + 8\kappa_{ij} \bar{u}_j \delta u_j \sinh^2(\delta q_j) + 4\bar{u}_j^2 \tanh(\delta q_j)\big] & \text{for } j = 1, \dots, n-1, \\ \mathrm{e}^{(q_i - q_n)} \frac{u_n^2}{4} & \text{for } j = n. \end{cases} \tag{9}$$

The system (8) is well-posed and has global solutions.

Given initial data $\bar{u} \in H^1(\mathbb{R})$, we denote by (u, μ) the conservative solution of the Camassa–Holm equation with initial data $(\bar{u}, (\bar{u}^2 + \bar{u}_x^2)\,dx)$. We will prove that there exists a sequence $\bar{u}_n$ in $\mathcal{M}$ which converges to $\bar{u}$ in $H^1(\mathbb{R})$. We know that $(\bar{u}_n, (\bar{u}_n^2 + \bar{u}_{n,x}^2)\,dx)$ then converges to $(\bar{u}, (\bar{u}^2 + \bar{u}_x^2)\,dx)$ in $\mathcal{D}$. We compute the multipeakon solution (u_n, μ_n) with initial data $(\bar{u}_n, \bar{u}_n^2 + \bar{u}_{n,x}^2)\,dx)$ by solving the system of ordinary differential equations (8). Since the semigroup is continuous, $(u_n, \mu_n) \to (u, \mu)$ in $C([0, T], \mathcal{D})$ for any $T > 0$ and therefore $u_n \to u$ in $L^\infty(\mathbb{R})$.

3 Approximation of the initial data

In [2], Bressan and Fonte show that $\mathcal{M}$ is dense in $H^1(\mathbb{R})$. The proof is short and elegant but it is not constructive and therefore not suited to numerical applications. Here we define constructive approximation procedure.

For any integer n and any points x_i, $i = 1, \dots, n$, such that $x_i \leq x_{i+1}$, we denote by $\mathcal{P}$ the partition corresponding to those points. Let $L_{\mathcal{P}}$ be the span of the partition, i.e., $L_{\mathcal{P}} = x_n - x_1$ and $\Delta_{\mathcal{P}}$ the maximum step size, i.e.,

$\mathit{\Delta}_{\mathcal{P}} = \max_{i=1,\ldots,n-1}(x_{i+1} - x_i)$. Given a partition $\mathcal{P} = \{x_1, \ldots, x_n\}$, we define by $I_{\mathcal{P}}$ the operator from $H^1(\mathbb{R})$ to $\mathcal{M}$ defined as follows: For any $f \in H^1(\mathbb{R})$, we set $f_i = f(x_i)$ and $u = I_{\mathcal{P}}(f)$ where u is solution of the boundary problem

$$u - u_{xx} = 0,\ u(x_i) = f_i,\ u(x_{i+1}) = f_{i+1} \tag{10}$$

in (x_i, x_{i+1}). The definition (10) extends naturally to the intervals $(-\infty, x_1)$ and (x_n, ∞) by setting $x_0 = -x_{n+1} = -\infty$ and $u_0 = u_{n+1} = 0$. We have the following theorem.

Theorem 1. *For any $f \in H^1(\mathbb{R})$, $I_{\mathcal{P}}(f)$ converges to f in $H^1(\mathbb{R})$ when $\mathit{\Delta}_{\mathcal{P}}$ tends to zero and $L_{\mathcal{P}}$ tends to infinity.*

Proof. Given a partition $\mathcal{P}$, the map $I_{\mathcal{P}}\colon \mathcal{M} \to H^1(\mathbb{R})$ is linear and continuous. Let us prove that $I_{\mathcal{P}}$ is uniformly continuous, that is, there exists a constant C such that

$$\|I_{\mathcal{P}}(f)\|_{H^1(\mathbb{R})} \leq C \|f\|_{H^1(\mathbb{R})} \tag{11}$$

for any $f \in H^1(\mathbb{R})$ and any partition $\mathcal{P}$ such that $\mathit{\Delta}_{\mathcal{P}} \leq 1$. The $H^1(\mathbb{R})$ norm of $I_{\mathcal{P}}(f)$ can be obtained from (6) and (7):

$$\begin{aligned} \|I_{\mathcal{P}}(f)\|^2_{H^1(\mathbb{R})} = f_1^2 + f_n^2 + \sum_{i=1}^{n-1} \Big(2\Big(\frac{f_i + f_{i+1}}{2}\Big)^2 \tanh\Big(\frac{x_{i+1} - x_i}{2}\Big) \\ + 2\Big(\frac{f_{i+1} - f_i}{2}\Big)^2 \coth\Big(\frac{x_{i+1} - x_i}{2}\Big)\Big). \end{aligned} \tag{12}$$

Let $\alpha = \inf_{x \in [x_i, x_{i+1}]} f(x)^2$ and $x_{\min}$ be such that $f(x_{\min})^2 = \alpha$. Since f is continuous, $x_{\min}$ and α are well-defined. It follows from the definition of α that

$$\alpha(x_{i+1} - x_i) \leq \|f\|^2_{L^2(x_i, x_{i+1})}. \tag{13}$$

We have, for any $x \in [x_i, x_{i+1}]$,

$$\begin{aligned} f(x)^2 = \alpha + 2\int_{x_{\min}}^{x} f(t) f'(t)\, dt &\leq \alpha + 2\|f\|^2_{H^1(x_i, x_{i+1})} \\ &\leq \frac{1}{x_{i+1} - x_i} \|f\|^2_{L^2(x_i, x_{i+1})} + 2\|f\|^2_{H^1(x_i, x_{i+1})}, \end{aligned}$$

from (13), which implies $f(x)^2(x_{i+1} - x_i) \leq 3\|f\|^2_{H^1(x_i, x_{i+1})}$ since, by assumption, $x_{i+1} - x_i \leq 1$. Hence, $\|f\|^2_{L^\infty(x_i, x_{i+1})}(x_{i+1} - x_i) \leq 3\|f\|^2_{H^1(x_i, x_{i+1})}$. We can estimate the first term of the sum in (12):

$$2\Big(\frac{f_i + f_{i+1}}{2}\Big)^2 \tanh\Big(\frac{x_{i+1} - x_i}{2}\Big) \leq 2L_1(x_{i+1} - x_i)\|f\|^2_{L^\infty(x_i, x_{i+1})}$$

where $L_1 = \sup_{z \in [0,1]} \left(\frac{\tanh(z/2)}{z}\right)$ and therefore

$$2\Big(\frac{f_i+f_{i+1}}{2}\Big)^2\tanh\Big(\frac{x_{i+1}-x_i}{2}\Big)\leq 6L_1\left\|f\right\|_{H^1(x_i,x_{i+1})}^2. \tag{14}$$

As far as the other term in the sum is concerned, we have, by the Cauchy–Schwarz inequality, $|f_{i+1}-f_i| \leq \int_{x_i}^{x_{i+1}}|f_x|\,dx \leq \sqrt{x_{i+1}-x_i}\left\|f\right\|_{H^1(x_i,x_{i+1})}$ and therefore

$$2\Big(\frac{f_{i+1}-f_i}{2}\Big)^2\coth\Big(\frac{x_{i+1}-x_i}{2}\Big)\leq \frac{L_2}{2}\left\|f\right\|_{H^1(x_i,x_{i+1})}^2 \tag{15}$$

where $L_2=\sup_{z\in[0,1]}\coth(z/2)z$. Due to the Sobolev embedding $H^1(\mathbb{R})\subset L^\infty(\mathbb{R})$, there exists a constant L_0 such that $f_1^2+f_n^2\leq L_0\left\|f\right\|_{H^1(\mathbb{R})}^2$. Hence, gathering (12), (14) and (15), we obtain (11). Let us prove now that for any smooth function ϕ with compact support, $I_{\mathcal{P}}(\phi)\to\phi$ in $H^1(\mathbb{R})$ when $\Delta_{\mathcal{P}}\to 0$ and $L_{\mathcal{P}}\to\infty$. We denote $v_{\mathcal{P}}=\phi-I_{\mathcal{P}}(\phi)$. We have that $v_{\mathcal{P}}$ satisfies in each interval $[x_i,x_{i+1}]$, $v_{\mathcal{P}}-v_{\mathcal{P},xx}=\phi-\phi_{xx}$ and $v_{\mathcal{P}}(x_i)=v_{\mathcal{P}}(x_{i+1})=0$. By integration by part, we obtain

$$\begin{aligned}\left\|v_{\mathcal{P}}\right\|_{H^1(x_i,x_{i+1})}^2&=\int_{x_i}^{x_{i+1}}(v_{\mathcal{P}}-v_{\mathcal{P},xx})v_{\mathcal{P}}\,dx=\int_{x_i}^{x_{i+1}}(\phi-\phi_{xx})v_{\mathcal{P}}\,dx \\ &\leq\left\|\phi-\phi_{xx}\right\|_{L^2(x_i,x_{i+1})}\left\|v_{\mathcal{P}}\right\|_{L^2(x_i,x_{i+1})},\end{aligned} \tag{16}$$

after using Cauchy–Schwarz. Hence,

$$\left\|v_{\mathcal{P}}\right\|_{H^1(x_i,x_{i+1})}\leq\left\|\phi-\phi_{xx}\right\|_{L^2(x_i,x_{i+1})}\leq C\sqrt{\Delta_{\mathcal{P}}} \tag{17}$$

where $C=\sup_{x\in\mathbb{R}}(\phi-\phi_{xx})$. Since $v_{\mathcal{P}}(x_i)=0$, we have $v_{\mathcal{P}}(x)^2=2\int_{x_i}^{x}v_{\mathcal{P}}v_{\mathcal{P},x}\,dx \leq 2\left\|v_{\mathcal{P}}\right\|_{H^1(x_i,x_{i+1})}^2$ and therefore, after using (17), we obtain $\left\|v_{\mathcal{P}}\right\|_{L^\infty(x_i,x_{i+1})}\leq 2C\sqrt{\Delta_{\mathcal{P}}}$, which, together with (16), implies

$$\left\|v_{\mathcal{P}}\right\|_{H^1(x_i,x_{i+1})}^2\leq 2C\sqrt{\Delta_P}\int_{x_i}^{x_{i+1}}\left|\phi-\phi_{xx}\right|\,dx. \tag{18}$$

When $L_{\mathcal{P}}$ is large enough, ϕ and $I_{\mathcal{P}}(\phi)$ vanish in $(-\infty,x_1)$ and (x_n,∞) so that $\left\|v_{\mathcal{P}}\right\|_{H^1(\mathbb{R})}^2=\sum_{i=1}^{n-1}\left\|v_{\mathcal{P}}\right\|_{H^1(x_i,x_{i+1})}^2$. From (18), it follows that $\left\|v_{\mathcal{P}}\right\|_{H^1(\mathbb{R})}^2\leq 2C\sqrt{\Delta_{\mathcal{P}}}\left\|\phi-\phi_{xx}\right\|_{L^1(\mathbb{R})}$. Hence, we have proved that that $I_{\mathcal{P}}(\phi)\to\phi$ in $H^1(\mathbb{R})$ when $\Delta_{\mathcal{P}}\to 0$ and $L_{\mathcal{P}}\to\infty$. We conclude the proof of the theorem by a standard approximation argument. For any $\varepsilon>0$, there exists a smooth function ϕ with compact support such that $\left\|f-\phi\right\|_{H^1(\mathbb{R})}\leq\varepsilon$. For any $\Delta_{\mathcal{P}}\leq 1$ small enough and $L_{\mathcal{P}}$ large enough, we have $\left\|\phi-I_{\mathcal{P}}(\phi)\right\|_{H^1(\mathbb{R})}\leq\varepsilon$. Hence, by the uniform continuity of $I_{\mathcal{P}}$, we have

$$\begin{aligned}\left\|f-I_{\mathcal{P}}(f)\right\|_{H^1(\mathbb{R})}&\leq\left\|f-\phi\right\|_{H^1(\mathbb{R})}+\left\|\phi-I_{\mathcal{P}}(\phi)\right\|_{H^1(\mathbb{R})}+\left\|I_{\mathcal{P}}(\phi)-I_{\mathcal{P}}(f)\right\|_{H^1(\mathbb{R})}\\ &\leq(2+C)\varepsilon.\end{aligned}$$

We choose a sequence of partition $\mathcal{P}_n$ such that $\Delta_{\mathcal{P}_n}\to 0$ and $L_{\mathcal{P}_n}\to 0$. We denote by $x_{i,n}$ the points of this partition, for example we can take $x_{i,n}=i/n-n$ for $i=1,\ldots,n^2$. The numerical scheme described in the following theorem converges.

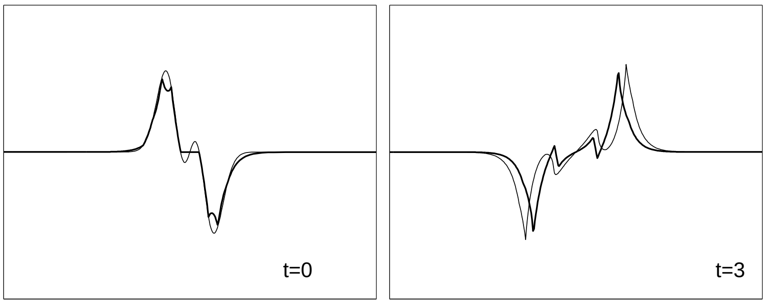

Fig. 1. Left: Approximation of the function $\bar{u}(x) = x(x^2-1)\mathrm{e}^{-x^2/4}$ by $n = 10$ equidistributed peakons. Right: Comparison of the computed solutions for $n = 10$ peakons (in bold) and $n = 100$ peakons at time $t = 3$.

Theorem 2. *For any initial data* $\bar{u} \in H^1(\mathbb{R})$, *let us define* $\bar{u}_n \in \mathcal{M}$ *by* $\bar{u}_n = I_{\mathcal{P}_n}(\bar{u})$. *We denote by* u_n *the solution of* (8) *with initial condition given by* $(\bar{q}_{i,n} = x_{i,n}, \bar{u}_{i,n} = \bar{u}_n(x_{i,n}), \bar{H}_{i,n})$ *where the* $\bar{H}_{i,n}$ *are computed from* $\bar{u}_{i,n}$ *and* $\bar{q}_{i,n}$ *using* (7)*. Then,* u_n *converges to the global conservative solution* u *of the Camassa–Holm equation in* $C([0,T], L^\infty(\mathbb{R}))$*, for any* $T > 0$.

Proof. By Theorem 1, we have that $\bar{u}_n \to \bar{u}$ in $H^1(\mathbb{R})$. From [9, Proposition 5.1] it follows that $(\bar{u}_n, (\bar{u}_n^2+\bar{u}_{x,n}^2)\,dx)$ converges to $(\bar{u}_n, (\bar{u}^2+\bar{u}_x^2)\,dx)$ in $\mathcal{D}$. The conservative solution of the Camassa–Holm equation constitues a continuous semigroup in $\mathcal{D}$. Hence, $(\bar{u}_n, (u_n^2 + u_{n,x}^2)\,dx)$ converges to $(u, (u^2+u_x^2)\,dx)$ in $C([0,T], \mathcal{D})$ and therefore in $C([0,T], L^\infty(\mathbb{R}))$, by [9, Proposition 5.2].

4 Numerical example

We consider the initial data $\bar{u}$ given by $\bar{u}(x) = x(x^2-1)\mathrm{e}^{-x^2/4}$. The computed solution is shown in Figure 2 for different times t. Figure 1 shows that the scheme is remarkably stable as, after a number of collisions, the solution computed with only $n = 10$ peakons remains close to the solution computed with much higher accuracy.

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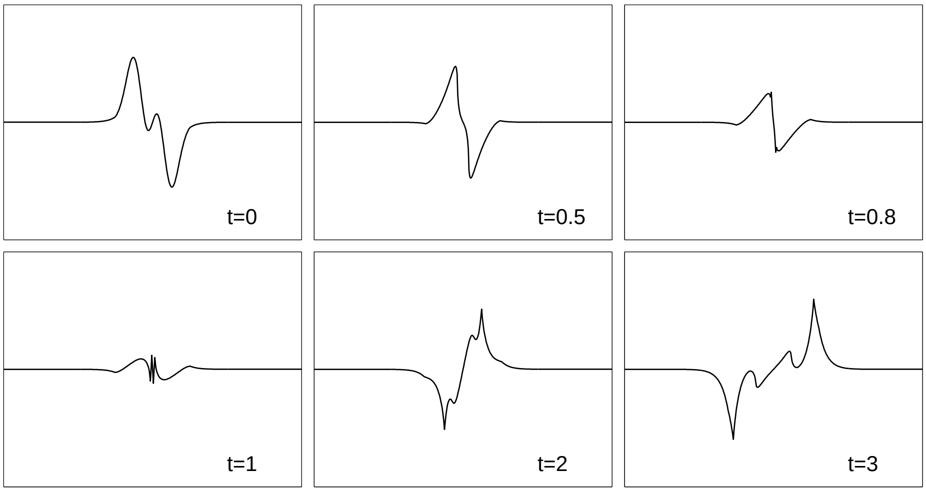

Fig. 2. The computed solution for $n = 100$ peakons shown at different times.

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