Convergence of a Spectral Projection of the Camassa-Holm Equation

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A spectral semi-discretization of the Camassa-Holm equation is defined. The Fourier-Galerkin and a de-aliased Fourier-collocation method are proved to be spectrally convergent. The proof is supplemented with numerical explorations which illustrate the convergence rates and the use of the dealiasing method. © 1994 John Wiley & Sons, Inc.

I. INTRODUCTION

In this article, consideration is given to the error analysis of a spectral projection of the periodic Camassa–Holm equation

$$u_t - u_{xxt} + \omega u_x + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx}) = 0 \tag{1.1}$$

on the interval $[0, 2\pi]$. Spectral discretizations of this equation have been in use ever since the work of Camassa and Holm [2] and Camassa, Holm and Hyman [3]. However, to the knowledge of the authors, no proof that such a discretization actually converges has appeared heretofore. Therefore, this issue is taken up here. Our method of proof is related to the work of Maday and Quarteroni on the convergence of a Fourier-Galerkin and collocation method for the Korteweg-de Vries equation [22]. While they were able to treat the unfiltered collocation approximation, we resort to proving the convergence of a de-aliased collocation projection which turns out to be equivalent to a Galerkin scheme. Before we get to the heart of the subject, a few words about the range of applicability of the equation are in order. The validity of the Camassa-Holm equation as a model for water waves in a channel of uniform width and depth has been a somewhat controversial subject. The discussion seems to have finally been settled in the recent articles of Johnson [16] and Kunze and Schneider [20]. One merit of the equation is the fact that it allows wave breaking typical of hyperbolic systems. Such wave breaking is

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observed in fluid flows, and under this aspect, the Camassa-Holm equation could be seen as a more suitable model than the well known Korteweg-de Vries equation, for instance. On the other hand, in the derivation of the Camassa-Holm equation, it is assumed that the solutions are more regular than breaking waves [20]. In this respect, smooth solutions are more closely related to the fluid flow problem than are irregular solutions. From this point of view, a spectral approximation seems a natural choice for a spatial discretization.

Another application of the Camassa-Holm equation arises when $\omega = 0$. In this case, the equation can be derived as a model equation for mechanical vibrations in a compressible elastic rod. As explained by Dai and Huo [8], the range of the parameter γ is roughly from -29.5 to 3.4. The equation has even found its place in the context of differential geometry, where it can be seen as a re-expression for geodesic flow on an infinite-dimensional Lie group [6, 13, 23].

Notwithstanding its importance as a model equation, one reason for the interest in the Camassa-Holm equation is its vast supply of novel mathematical issues, such as its integrable bi-Hamiltonian structure. This property alone has led to many interesting developments, a sample of which can be found in [2, 3, 4, 9, 10, 11], and the references contained therein. One aspect of the integrability of the equation in case $\gamma = 1$ is that the solitary-wave solutions are solitons [2, 4], similar to the solitary-wave solutions of the Korteweg-de Vries equation. However, the Camassa-Holm equation also admits solitary waves which are not smooth, but rather have a peak or even a cusp. These peaked solitary waves are well known, and owing to their soliton-like properties they have been termed peakons. In the case that $\omega = 0$ and $\gamma = 1$, they are of the form

$$u(x,t) = de^{-|x-dt|},$$

where $d \in \mathbb{R}$ is the wavespeed. For general ω and γ, a similar formula was found by one of the authors in [17]. Even more general shapes have been described in [21], where a classification of traveling-wave solutions is given. For the numerical approximation of these peaked or cusped waves, spectral methods may not be the best choice. Other methods based on finite-difference approximations have been used for instance in [1, 14, 15].

For the purpose of numerical study, it is important to have a satisfactory theory of existence of solutions, as well as uniqueness and continuous dependence with respect to the initial data. For the periodic case, an example of such well posedness results has been provided by Constantin and Escher in [5]. However, for our purposes, the available results are not quite strong enough. In particular, it appears that it is possible for solutions emanating from smooth initial data to form singularities in finite time. These singularities manifest themselves in the form of steepening up to the point where the first derivative becomes close to $-\infty$. In the context of one-dimensional water-wave theory, this may be understood as wave breaking which we have alluded to earlier. The idea of the proof of this phenomenon is to make use of a differential inequality which goes back at least to the work of Seliger [25] (see also Whitham [26]). It is clear that such singularity formation will prevent the spectral, or super-polynomial convergence usually exhibited by spectral discretizations. To circumvent this problem, we assume that the solution has at least four derivatives in the space of square-summable functions. This requirement turns out to be sufficient to obtain convergence of the spectral projection, with a convergence rate dependent upon the regularity of the solution to be approximated. Of course, this requirement also restricts the pool of possible solutions, and thus limits the applicability of our method. A spectral discretization may still be used for solutions that have lower regularity, but the convergence is then not known.

For the sake of simplicity, we will only give proofs in the case where $\omega = 0$ and $\gamma = 1$ but all the proofs extend to the general case with only small changes in the constants. To prepare the equation for the discretization, it is convenient to write (1.1) in the form

$$u_t + \tfrac{1}{2}(u^2)_x + K(u^2 + \tfrac{1}{2}(u_x)^2) = 0, \tag{1.2}$$

where K is a Fourier multiplier operator with the symbol

$$\widehat{Kf}(k) = \frac{ik}{1+k^2}\hat{f}(k),$$

which formally corresponds to

$$K = \frac{\partial_x}{1-\partial_x^2}.$$

This formulation reveals that it is natural to use a spectral Fourier discretization, as the symbol of the Green's function K is already known. In section 3, we define a Fourier-Galerkin approximation of the Camassa-Holm equation, and prove that this approximation converges if appropriate assumptions are made on the initial data and the solution. Indeed, it will transpire that for smooth solutions, the convergence is indeed spectral, i.e. super-polynomial. In section 4, a similar result will be proved for a de-aliased Fourier-collocation scheme. Finally, the last section contains some numerical computations, which illustrate the results obtained in sections 3 and 4, and which show that the de-aliased collocation scheme is preferable to an unfiltered collocation approximation, especially when approximating solutions which are not smooth.

II. NOTATION

In order to facilitate our study, we start by introducing some mathematical notation. Denote the inner product in $L^2(0, 2\pi)$ by

$$(f, g) = \int_0^{2\pi} f(x)\,\overline{g(x)}\,dx.$$

The Fourier coefficients $\hat{f}(k)$ of a function $f \in L^2(0, 2\pi)$ are defined by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx.$$

Recall the inversion formula

$$f(x) = \sum_{k\in\mathbb{Z}} e^{ikx}\hat{f}(k),$$

and the convolution formula

$$(\hat{f} * \hat{g})(k) = \widehat{fg}(k),$$

where the convolution of two functions $\hat{f}$ and $\hat{g}$ on $\mathbb{Z}$ is formally defined by

$$(\hat{f} * \hat{g})(k) = \sum_{m+n=k} \hat{f}(m)\,\hat{g}(n).$$

Denote by $\|\cdot\|_{H^m}$ the Sobolev norm, given by

$$\|f\|_{H^m}^2 = \sum_{k\in\mathbb{Z}} (1+|k|^2)^m |\hat{f}(k)|^2.$$

The space of periodic Sobolev functions on the interval $[0,2\pi]$ is defined as the closure of the space of smooth periodic functions with respect to the H^m-norm, and will be simply denoted by H^m. In particular, for $m=0$, we recover the space $L^2(0,2\pi)$ whose norm will be denoted by $\|\cdot\|_{L^2}$. The subspace of $L^2(0,2\pi)$ spanned by the set

$$\left\{ e^{ikx} \;\middle|\; k\in\mathbb{Z},\ -\frac{N}{2} \le k \le \frac{N}{2}-1 \right\}$$

for N even is denoted by S_N. In the following, it will always be assumed that N is even. The operator P_N denotes the orthogonal, self-adjoint, projection from L^2 onto S_N, defined by

$$P_N f(x) = \sum_{-N/2\le k\le N/2-1} e^{ikx}\hat{f}(k).$$

For $f \in H^m$, the estimates

$$\|f - P_N f\|_{L^2} \le C_P N^{-m} \|\partial_x^m f\|_{L^2}, \tag{2.1}$$

$$\|f - P_N f\|_{H^n} \le C_P N^{n-m} \|\partial_x^m f\|_{L^2} \tag{2.2}$$

hold for an appropriate constant C_P and a positive integer n. For the proof of these inequalities, the reader is referred to [7].

The space of continuous functions from the interval $[0,T]$ into the space H^n is denoted by $C([0,T],H^n)$. Similarly, we also consider the space $C([0,T],S_N)$, where the topology on the finite-dimensional space S_N can be given by any norm. Finally note the inverse inequality

$$\|\partial_x^m \phi\|_{L^2} \le N^m \|\phi\|_{L^2}, \tag{2.3}$$

which holds for integers $m>0$ and $\phi \in S_N$. A proof of this estimate can also be found in [7]. We will make use of the Sobolev lemma, which guarantees the existence of a constant c, such that

$$\sup_x |f(x)| \le c\,\|f\|_{H^1}. \tag{2.4}$$

Another standard result is that the assignment $(f,g)\mapsto fg$ is a continuous bilinear map from $H^1\times H^{-1}$ to H^{-1}, as shown by the estimate

$$\|fg\|_{H^{-1}} \le c\,\|f\|_{H^1}\,\|g\|_{H^{-1}}, \tag{2.5}$$

where the same constant c has been used for simplicity.

In order to obtain a unique solution, equation (1.1) has to be supplemented by appropriate boundary and initial conditions. For the purpose of numerical approximation, the problem will be studied on a finite interval with periodic boundary conditions. The periodic initial value problem associated to equation (1.1) is

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0 & x\in[0,2\pi]\ , t\ge 0,\\ u(0,t) = u(2\pi,t), & t\ge 0,\\ u(x,0) = u_0(x). \end{cases} \tag{2.6}$$

In the following, it will always be assumed that a solution of this problem exists on some time interval $[0, T]$, and with a certain amount of spatial regularity. In particular, we suppose that a solution exists in the space $C([0, T], H^4)$ for some $T > 0$. With these preliminaries in place, we are set to attack the problem of defining a suitable spectral projection of (2.6) and proving the convergence of such a projection. First, the Fourier-Galerkin method is presented and a proof of convergence given. Then in section 4, a de-aliased collocation scheme will be treated.

III. THE FOURIER-GALERKIN METHOD.

A space-discretization of (2.6) is defined by utilizing the equivalent formulation (1.2). Thus the problem is to find a function u_N from $[0, T]$ to S_N which satisfies

$$\begin{cases} \left(\partial_t u_N + \frac{1}{2}\partial_x (u_N)^2 + K\left(u_N^2 + \frac{1}{2}(\partial_x u_N)^2\right), \phi\right) = 0, & t \in [0, T], \\ & u_N(0) = P_N u_0, \end{cases} \tag{3.1}$$

for all $\phi \in S_N$. Since for each t, $u_N(\cdot, t) \in S_N$, u_N has the form

$$u_N(x,t) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_N(k,t) e^{ikx},$$

where $\hat{u}_N(k, t)$ are the Fourier coefficients of $u_N(\cdot, t)$.

Taking $\phi = e^{ikx}$ for $-N/2 \leq k \leq N/2 - 1$ in (3.1) yields the following system of equations for the Fourier coefficients of u_N.

$$\begin{cases} \frac{d}{dt}\hat{u}_N(k,t) = -\frac{1}{2} ik(\hat{u}_N * \hat{u}_N)(k,t) - \frac{ik}{1+k^2}\Big[(\hat{u}_N * \hat{u}_N)(k,t) \\ \qquad\qquad + \ \frac{1}{2}((ik\hat{u}_N) * (ik\hat{u}_N))(k,t)\Big], \\ \hat{u}_N(k,0) = \hat{u}_0(k), \end{cases} \tag{3.2}$$

for $-N/2 \leq k < N/2 - 1$..

The short-time existence of a maximal solution of (3.2) is proved using the contraction mapping principle, and the solution is unique on its maximal interval of definition, $[0, t_N^m)$, where t_N^m is possibly equal to T. Since the argument is standard, the proof is omitted here. The main result of this paper is the fact that the Galerkin approximation u_N converges to the exact solution u when u is smooth enough. This is stated in the next theorem.

Theorem 3.1. *Suppose that a solution u of the Camassa-Holm equation* (2.6) *exists in the space $C([0, T], H^m)$ for $m \geq 4$, and for some time $T > 0$. Then for N large enough, there exists a unique solution u_N of the finite-dimensional problem* (3.1)*. Moreover, there exists a constant λ such that*

$$\sup_{t \in [0,T]} \|u(\cdot, t) - u_N(\cdot, t)\|_{L^2} \leq \lambda N^{1-m}.$$

Before the proof is given, note that the assumptions of the theorem encompass the existence of a constant κ, such that

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^m} \le \kappa.$$

In particular, it follows then that there is another constant Λ, such that

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^2} \le \Lambda.$$

The main ingredient in the proof of the theorem is a local error estimate which will be established by the following lemma.

Lemma 3.2. *Suppose that a solution u_N of* (3.1) *exists on the time interval* $[0,t_N^*]$, *and that* $\sup_{t\in[0,t_N^*]}\|u_N(\cdot,t)\|_{H^2} \le 2\Lambda$. *Then the error estimate*

$$\sup_{t\in[0,t_N^*]} \|u(\cdot,t)-u_N(\cdot,t)\|_{L^2} \le \lambda N^{1-m} \tag{3.3}$$

holds for some constant λ which only depends on T, Λ and κ.

Proof. Let $h = P_N u - u_N$. We apply P_N to both sides of (1.2) and, since P_N commutes with derivation, we obtain

$$\partial_t P_N u + \tfrac{1}{2}P_N\partial_x(u^2) + KP_N(u^2+\tfrac{1}{2}u_x^2) = 0.$$

We multiply this equation by h, integrate over $[0,2\pi]$ and subtract the result from (3.1) where we have used h, which belongs to S_N, as a test function. We get

$$(h_t,h) = -\frac{1}{2}\left(P_N\partial_x u^2 - \partial_x u_N^2, h\right) - \left(KP_N\left[u^2+\frac{1}{2}(\partial_x u)^2\right],h\right) + \left(K\left[u_N^2+\frac{1}{2}(\partial_x u_N^2)^2\right],h\right)$$

Using the fact that P_N is self-adjoint on L^2, and $h\in S_N$, this may be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|h\|_{L^2}^2 = -\frac{1}{2}\left(\partial_x u^2 - \partial_x u_N^2, h\right) - \left(K\left[u^2-u_N^2\right],h\right) - \frac{1}{2}\left(K\left[(\partial_x u)^2-(\partial_x u_N)^2\right],h\right). \tag{3.4}$$

Let's estimate the three terms on the right-hand side of (3.4) in the time interval $[0,t_N^*]$ where the H^2-norm of u_N is bounded by 2Λ. We have

$$\begin{aligned}
\left(\partial_x u^2 - \partial_x u_N^2, h\right) &= \left(\partial_x\left\{(u+u_N)(u-u_N)\right\},h\right)\\
&= \left(\partial_x(u+u_N)\,(u-u_N),h\right) + \left((u+u_N)\,\partial_x(u-u_N),h\right)\\
&= \left(\partial_x(u+u_N)\,(u-u_N),h\right)\\
&\quad + \left((u+u_N)\partial_x(u-P_N u),h\right)\\
&\quad + \left((u+u_N)\partial_x(P_N u-u_N),h\right).
\end{aligned}$$

Consequently, there appears the estimate

$$\begin{aligned}\left|\left(\partial_x u^2-\partial_x u_N^2,h\right)\right| \le\ & \sup_x|\partial_x(u+u_N)|\,\|u-u_N\|_{L^2}\,\|h\|_{L^2}\\ &+\sup_x|u+u_N|\,\|\partial_x(u-P_Nu)\|_{L^2}\,\|h\|_{L^2}\\ &+\left|\int_0^{2\pi}(u+u_N)h_xh\,dx\right|\\ \le\ & c\,\|u+u_N\|_{H^2}\left(\|u-P_Nu\|_{L^2}+\|P_Nu-u_N\|_{L^2}\right)\|h\|_{L^2}\\ &+c\,\|u+u_N\|_{H^1}\,\|u-P_Nu\|_{H^1}\,\|h\|_{L^2}\\ &+\frac{1}{2}\int_0^{2\pi}h^2|\partial_x(u+u_N)|\,dx\\ \le\ & 3c\Lambda\left(C_PN^{-m}\|u\|_{H^m}+\|h\|_{L^2}\right)\|h\|_{L^2}\\ &+3c\Lambda C_PN^{1-m}\|u\|_{H^m}\,\|h\|_{L^2}\\ &+\frac{1}{2}\sup_x|\partial_x(u+u_N)|\int_0^{2\pi}h^2\,dx.\end{aligned}$$

Noting that the last integral is bounded by $\frac{1}{2}3c\Lambda\,\|h\|_{L^2}^2$, there appears the estimate

$$\left|\left(\partial_x u^2-\partial_x u_N^2,h\right)\right|\le 3c\Lambda\,\|h\|_{L^2}\left(\frac{3}{2}\|h\|_{L^2}+C_P\,\|u\|_{H^m}\left(N^{-m}+N^{1-m}\right)\right). \tag{3.5}$$

We turn to the second term in (3.4). The operator K is a continuous operator from H^{-1} to L^2. Therefore, after using the Cauchy-Schwartz inequality and (2.5), there appears

$$\begin{aligned}\left(K(u^2-u_N^2),h\right)&\le\left\|u^2-u_N^2\right\|_{H^{-1}}\|h\|_{L^2}\\ &\le\|(u-u_N)(u+u_N)\|_{H^{-1}}\,\|h\|_{L^2}\\ &\le c\,\|u+u_N\|_{H^1}\,\|u-u_N\|_{H^{-1}}\,\|h\|_{L^2}\,.\end{aligned}\tag{3.6}$$

Then, since

$$\begin{aligned}\|u-u_N\|_{H^{-1}}&\le\|u-u_N\|_{L^2}\\ &\le\|u-P_Nu\|_{L^2}+\|h\|_{L^2}\\ &\le C_PN^{-m}\,\|u\|_{H^m}+\|h\|_{L^2}\,,\end{aligned}$$

and $\|u+u_N\|_{H^1}$ is bounded (recall that the estimates are established on $[0,t_N^*]$ where the H^2-norm of u_N is bounded by 2Λ), we get from (3.6)

$$\left(K(u^2-u_N^2),h\right)\le 3c\Lambda\,\|h\|_{L^2}\left(C_PN^{-m}\,\|u\|_{H^m}+\|h\|_{L^2}\right). \tag{3.7}$$

Similarly for the remaining term in (3.4) we have

$$\begin{aligned}\left(K((\partial_xu)^2-(\partial_xu_N)^2),h\right)&\le\|h\|_{L^2}\left\|(\partial_xu)^2-(\partial_xu_N)^2\right\|_{H^{-1}}\\ &\le c\,\|\partial_x(u+u_N)\|_{H^1}\,\|\partial_x(u-u_N)\|_{H^{-1}}\,\|h\|_{L^2}\\ &\le 3c\Lambda\,\|h\|_{L^2}\left(C_P\,\|u\|_{H^m}N^{-m}+\|h\|_{L^2}\right).\end{aligned}\tag{3.8}$$

Gathering the estimates (3.5), (3.7) and (3.8), it transpires that

$$\frac{d}{dt}\|h\|_{L^2}\le\frac{27}{4}c\Lambda\,\|h\|_{L^2}+\frac{15}{2}c\Lambda C_P\kappa N^{1-m}.$$

Consequently, Gronwall's inequality gives

$$\sup_{t\in[0,t_N^*]} \|h(\cdot,t)\|_{L^2} \leq \lambda N^{1-m} \tag{3.9}$$

for an appropriate constant λ which depends on T, Λ and κ. After decomposing $u - u_N$ as the sum $u - P_N u + h$ and using (2.1) and the triangle inequality, (3.9) yields

$$\sup_{t\in[0,t_N^*]} \|u(\cdot,t) - u_N(\cdot,t)\|_{L^2} \leq \lambda N^{1-m}$$

for another constant λ which again only depends on T, Λ and κ. ∎

Lemma 3.3. *Suppose that a solution u_N of* (3.1) *exists on the time interval* $[0, t_N^*]$, *and that* $\sup_{t\in[0,t_N^*]} \|u_N(\cdot,t)\|_{H^2} \leq 2\Lambda$. *Then the error estimate*

$$\sup_{t\in[0,t_N^*]} \|u(\cdot,t) - u_N(\cdot,t)\|_{H^2} \leq \lambda N^{3-m} \tag{3.10}$$

holds for some constant λ which only depends on T, Λ and κ.

The proof of this lemma follows from (3.9) after application of the triangle inequality and the inverse inequality (2.3).

Proof of Theorem 3.1. We want to extend the estimate (3.3) to the time interval $[0,T]$. Note that the time t_N^* appearing in Lemma 3.2 has so far been unspecified. We now define t_N^* by

$$t_N^* = \sup\{t \in [0,T] \mid \text{ for all } t' \leq t, \|u_N(\cdot,t')\|_{H^2} \leq 2\Lambda\}. \tag{3.11}$$

Thus the time t_N^* corresponds to the largest time in $[0,T]$ for which the H^2-norm of u_N is uniformly bounded by 2Λ. Since $\|u_N(\cdot,0)\|_{H^2} = \|P_N u(\cdot,0)\|_{H^2}$, we have

$$\|u_N(\cdot,0)\|_{H^2} \leq \|u(\cdot,0)\|_{H^2} \leq \Lambda.$$

Hence, $t_N^* > 0$ for all N. Note that t_N^* is necessarily smaller than the maximum time of existence t_N^m. On the other hand, we are going to prove that there exists N_* such that

$$t_N^* = T \quad \text{for all } N \geq N_*, \tag{3.12}$$

and therefore the supremum in (3.3) holds on $[0,T]$. By definition (3.11), we either have $t_N^* = T$ or $t_N^* < T$ and in this case, since $\|u_N(t)\|_{H^2}$ is a continuous function in time, $\|u_N(t_N^*)\|_{H^2} = 2\Lambda$. Suppose that $t_N^* < T$. Then using the triangle inequality yields

$$\begin{aligned} 2\Lambda &= \|u_N(\cdot,t_N^*)\|_{H^2} \\ &\leq \|(u_N(\cdot,t_N^*) - u(\cdot,t)\|_{H^2} + \sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^2} \\ &= \|(u_N(\cdot,t_N^*) - u(\cdot,t_N^*)\|_{H^2} + \Lambda, \end{aligned}$$

by the definition of Λ. Hence,

$$\Lambda \leq \|(u_N(\cdot,t_N^*) - u(\cdot,t_N^*)\|_{H^2}.$$

By Lemma 3.3, it follows that

$$\Lambda \leq \lambda N^{3-m}$$

or

$$N \leq \left(\frac{\lambda}{\Lambda}\right)^{\frac{1}{m-3}}.$$

In conclusion, for $N^* > \left(\frac{\lambda}{\Lambda}\right)^{\frac{1}{m-3}}$, we cannot have $t_N^* < T$ and the claim (3.12) holds. It follows that for $N \geq N_*$ the solution u_N of (3.2) is defined on $[0, T]$ because, as we noticed earlier, $t_N^* < t_N^m$ and, from (3.3), we get

$$\sup_{t\in[0,T]} \|u(\cdot,t) - u_N(\cdot,t)\|_{L^2} \leq \lambda N^{1-m}.$$

■

The following corollary is immediate from the estimate (3.9) and the inequalities (2.2) and (2.3).

Corollary. *Suppose that a solution u of the Camassa-Holm equation* (2.6) *exists in the space $C([0,T], H^m)$ for $m \geq 4$, and for some time $T > 0$. Then for N large enough, there exists a unique solution u_N of the finite-dimensional problem* (3.1)*. Moreover, there exists a constant λ such that*

$$\sup_{t\in[0,T]} \|u(\cdot,t) - u_N(\cdot,t)\|_{H^2} \leq \lambda N^{3-m}.$$

IV. THE FOURIER-COLLOCATION METHOD

The Galerkin method is not very attractive from a computational point of view, because the computation of the convolution sums in (3.2) is very expensive. If the convolution is computed by means of the Fast Fourier Transform (FFT), the computational time is minimized, but an additional error known as *aliasing* is introduced. This means that high wavenumbers are projected back into low wavenumber modes, causing spurious oscillations. We refer to [7, 12] for more details about the aliasing phenomenon. Methods that use the FFT are often called *collocation* methods since they are sometimes algebraically equivalent to a collocation scheme in the case of a Fourier basis. The problem of aliasing can be somewhat alleviated if enough modes are used to resolve all frequencies. However, this remedy is mostly applicable to the study of one-dimensional problems, and it supposes that the amplitudes decay in a reasonable fashion. In the case of the Camassa-Holm equation, one may want to use the spectral discretization to study the peakon solutions mentioned in the introduction. In the case that $\omega = 0$ and $\gamma = 1$, the representation

$$u(x,t) = de^{-|x-dt|},$$

reveals that u has an elementary Fourier transform, given for instance at time $t = 0$ by

$$\hat{u}(k,0) = d\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}. \tag{4.1}$$

As this expression shows, the Fourier amplitudes decay only quadratically, and therefore, it will be nearly impossible to avoid aliasing, even when using a large number of modes. In light of this problem, we have chosen to treat the case of a de-aliased scheme. In fact, it will be shown that the dealiasing we choose yields a scheme which is equivalent to the Galerkin scheme treated in the previous section.

The collocation operator in S_N denoted I_N is defined as follows. Let the collocation points be $x_j = \frac{2\pi j}{N}$ for $j = 0, 1, ..., N-1$. Then, given a continuous and periodic function f, $I_N f$ is the unique element in S_N, such that

$$I_N f(x_j) = f(x_j),$$

for $j = 0, 1, ..., N-1$. $I_N f$ is also called the N-th trigonometric interpolant of f. When restricted to S_N, the collocation operator reduces to the identity operator, as highlighted by the identity

$$I_N \phi = \phi \quad \text{for all } \phi \in S_N. \tag{4.2}$$

It has been proved in [19, 24] that when $f \in H^m$ with $m \geq 1$, there exists a constant C_I, such that

$$\|f - I_N f\|_{L^2} \leq C_I N^{-m} \|\partial_x^m f\|_{L^2}, \tag{4.3}$$

and more generally

$$\|f - I_N f\|_{H^n} \leq C_I N^{n-m} \|\partial_x^m f\|_{L^2}. \tag{4.4}$$

The collocation approximation to (2.6) is given by a function u_N from $[0, T]$ to S_N, such that

$$\begin{cases} \partial_t u_N + \frac{1}{2}\partial_x I_N[u_N^2] + K I_N\left[u_N^2 + \frac{1}{2}(\partial_x u_N)^2\right] = 0, & t \in [0, T], \\ & u_N(0) = I_N u_0. \end{cases} \tag{4.5}$$

Note that (4.5) reduces to (3.1) if the interpolation operator I_N is replaced by the projection operator P_N. In order to apply the FFT, we take the *discrete Fourier transform*, denoted here by $\mathcal{F}_N$. We again refer to [7, 12] for more details on this algorithm and its properties. The discrete Fourier transform $\mathcal{F}_N$ of a continuous function u is the vector in $\mathbb{C}^N$ defined as

$$\mathcal{F}_N(u)(k) = \widehat{I_N(u)}(k), \text{ for } -N/2 \leq k < N/2 - 1. \tag{4.6}$$

Applying the discrete Fourier transform to each term in (4.5), and using the definition (4.6) of $\mathcal{F}_N$, there appears the equation

$$\left(\partial_t \mathcal{F}_N u_N + \frac{ik}{2}\mathcal{F}_N(u_N^2) + \frac{ik}{1+k^2}\mathcal{F}_N\left[u_N^2 + \tfrac{1}{2}(\partial_x u_N)^2\right]\right)(k) = 0. \tag{4.7}$$

Let $\tilde{u}_N \in \mathbb{C}^N$ denote $\mathcal{F}_N(u_N)$. The solutions of (4.5), or equivalently of (4.7), are obtained by solving the following system of ordinary differential equations:

$$\begin{cases} \dfrac{d}{dt}\tilde{u}_N(k) + \dfrac{ik}{2}\mathcal{F}_N((\mathcal{F}_N^{-1}\tilde{u}_N)^2)(k) + \dfrac{ik}{1+k^2}\mathcal{F}_N\left[(\mathcal{F}_N^{-1}\tilde{u}_N)^2 + \tfrac{1}{2}(\mathcal{F}_N^{-1}(ik\tilde{u}_N))^2\right](k) = 0, \\ \tilde{u}_N(k, 0) = \mathcal{F}_N u(k, 0) \end{cases}$$

for $-N/2 \leq k < N/2 - 1$. This method is appealing because of the efficiency of the FFT which allows us to rapidly compute the discrete Fourier transform and its inverse. It has to be compared to (3.2) where the computation of the convolution is extremely expensive. Nevertheless the FFT introduces errors due to aliasing. In order to avoid aliasing, we apply the well known 2/3-rule. Thus we consider instead the following initial value problem: Find $u_N \in C([0,T], S_N)$ such that

$$\begin{cases} \partial_t u_N + \frac{1}{2}\partial_x I_N[(P_{\frac{2N}{3}} u_N)^2] + K I_N \left[(P_{\frac{2N}{3}} u_N)^2 + \frac{1}{2}(\partial_x(P_{\frac{2N}{3}} u_N))^2\right] = 0, & t \in [0,T], \\ & u_N(0) = I_N u_0. \end{cases} \tag{4.8}$$

The corresponding system of ordinary differential equation satisfied by $\tilde{u}_N = \mathcal{F}_N u_N$ is

$$\begin{cases} \dfrac{d}{dt}\tilde{u}_N(k) + \dfrac{ik}{2}\mathcal{F}_N((P_{\frac{2N}{3}}\mathcal{F}_N^{-1}\tilde{u}_N)^2)(k) \\ \qquad + \dfrac{ik}{1+k^2}\mathcal{F}_N\left[(P_{\frac{2N}{3}}\mathcal{F}_N^{-1}\tilde{u}_N)^2 + \frac{1}{2}(P_{\frac{2N}{3}}\mathcal{F}_N^{-1}(ik\tilde{u}_N))^2\right](k) = 0, \\ \tilde{u}_N(k,0) = \mathcal{F}_N u(k,0) \end{cases} \tag{4.9}$$

which again can be solved efficiently by the use of the FFT. When implementing (4.9) numerically, it is important to note that $P_{\frac{2N}{3}}\mathcal{F}_N^{-1}\tilde{u}_N$ (and similarly $P_{\frac{2N}{3}}\mathcal{F}_N^{-1}(ik\tilde{u}_N)$) can be rewritten according to

$$P_{\frac{2N}{3}}\mathcal{F}_N^{-1}\tilde{u}_N = \mathcal{F}_N^{-1}\tilde{u}_N^c,$$

where $\tilde{u}_N^c$ is obtained by cutting off the frequencies higher than M, i.e.

$$\tilde{u}_N^c(k) = \begin{cases} \tilde{u}_N(k) & \text{if } \dfrac{2N}{3} - 1 \le k \le \dfrac{2N}{3}, \\ 0 & \text{otherwise,} \end{cases}$$

thus making these quantities easy to compute. The use of the projection $P_{\frac{2N}{3}}$ is justified by the following identity which is derived in [7],

$$P_M I_N\left(P_M f \cdot P_M g\right) = P_M\left(P_M f \cdot P_M g\right), \tag{4.10}$$

and which holds for any continuous functions f, g and any $M \le \frac{2N}{3}$. The identity (4.10) essentially means that the interpolation operator I_N, which generally introduces aliasing, becomes harmless when we apply an M-filter, that is when we cut off the frequencies higher than M.

In order to prove convergence of the scheme, we introduce $v_N = P_{\frac{2N}{3}} u_N$ and $h = v_N - P_{\frac{2N}{3}} u$. Note that $v_N, \partial_x v_N, h \in S_{\frac{2N}{3}}$. After applying $P_{\frac{2N}{3}}$ in (4.8) and taking the scalar product with h, we get

$$(\partial_t v_N, h) + \frac{1}{2}\left(P_{\frac{2N}{3}}\partial_x I_N(v_N^2), h\right) + \left(P_{\frac{2N}{3}} K I_N\left[(v_N)^2 + \tfrac{1}{2}(\partial_x v_N)^2\right], h\right) = 0. \tag{4.11}$$

The projection operator $P_{\frac{2N}{3}}$ commutes with ∂_x and K. Using (4.10) with $M = \frac{2N}{3}$, we get $P_{\frac{2N}{3}} I_N(v_N^2) = P_{\frac{2N}{3}}(v_N^2)$ and $P_{\frac{2N}{3}} I_N(\partial_x v_N^2) = P_{\frac{2N}{3}}(\partial_x v_N^2)$ since $v_N, \partial_x v_N \in S_{\frac{2N}{3}}$. Hence, from (4.11), we get

$$(\partial_t v_N, h) + \frac{1}{2}\left(P_{\frac{2N}{3}}\partial_x(v_N^2), h\right) + \left(P_{\frac{2N}{3}} K\left[(v_N)^2 + \tfrac{1}{2}(\partial_x v_N)^2\right], h\right) = 0.$$

Then using the fact that $P_{\frac{2N}{3}}$ is self-adjoint and $P_{\frac{2N}{3}} h = h$, there obtains

$$(\partial_t v_N, h) + \frac{1}{2}\left(\partial_x(v_N^2), h\right) + \left(K\left[(v_N)^2 + \tfrac{1}{2}(\partial_x v_N)^2\right], h\right) = 0. \tag{4.12}$$

After applying $P_{\frac{2N}{3}}$ to (1.2) and taking the scalar product with h, we get

$$\left(P_{\frac{2N}{3}}\partial_t u, h\right) + \frac{1}{2}\left(\partial_x(u^2), h\right) + \left(K\left[u^2 + \tfrac{1}{2}(\partial_x u)^2\right], h\right) = 0, \tag{4.13}$$

where we have used again the fact that $P_{\frac{2N}{3}}$ is self-adjoint and $P_{\frac{2N}{3}} h = h$. Subtracting (4.13) from (4.12), we obtain

$$(\partial_t h, h) + \tfrac{1}{2}\left(\partial_x(v_N^2) - \partial_x u^2, h\right) + \left(K\left[(v_N)^2 - u^2\right], h\right) + \tfrac{1}{2}\left(K\left[(\partial_x v_N)^2 - (\partial_x u)^2\right], h\right) = 0. \tag{4.14}$$

We now proceed exactly in the same way as for the Galerkin method in the previous section. After introducing

$$t_N^* = \sup\{t \in [0,T] \mid \text{ for all } t' \le t, \|v_N(t', \cdot)\|_{H^2} \le 2\Lambda\}$$

is appears that the estimate

$$\sup_{t\in[0,t_N^*]} \|v_N - v\|_{L^2} \le \lambda \left(\frac{2N}{3}\right)^{1-m} \tag{4.15}$$

holds for some λ depending only on T, Λ and κ. The factor $2N/3$ in (4.15) comes from the fact that we used the projection on $S_{\frac{2N}{3}}$ instead of the projection on S_N that was used to derive (3.3). As in the previous section, we can prove that for N large enough the inequality (4.15) holds when taking the supremum over all t in $[0,T]$. Thus we are led to the following theorem.

Theorem 4.1. *Suppose that a solution u of the Camassa-Holm equation* (1.1) *exists in the space $C([0,T], H^m)$ for $m \ge 4$ and for some time $T > 0$. Then for N large enough, there exists a unique solution u_N to the finite-dimensional problem* (4.8)*. Moreover, there exists a constant λ such that,*

$$\sup_{t\in[0,T]} \|u(\cdot,t) - v_N(\cdot,t)\|_{L^2} \le \lambda \left(\frac{2N}{3}\right)^{1-m},$$

where $v_N = P_{\frac{2N}{3}} u_N$.

V. NUMERICAL EXPERIMENTS

It appears that spectral discretizations have been widely used to study mathematical properties of the Camassa-Holm equation. In particular, the interaction of two or more peakon solution has been a topic of intense interest. Here, we restrict ourselves to the computation of single traveling waves in order to validate the results of the previous sections. Traveling waves have the form

$$u(x,t) = \phi(x - dt),$$

were ϕ is either known exactly, or can be approximated, and where d is the wave speed. On the real line $\mathbb{R}$, the so-called peakons are well known. In the case that $\omega = 0$ and $\gamma = 1$, they are given by

$$\phi(x) = de^{-|x|}. \tag{5.1}$$

Since we consider periodic boundary conditions, it is convenient to know that there are also periodic peaked traveling waves. These are given by

$$\phi(x) = d\frac{\cosh(\frac{1}{2} - x)}{\cosh(\frac{1}{2})} \tag{5.2}$$

on the interval $[0, 1]$, and are periodic with period 1.

The Camassa-Holm equation (1.1) also admits smooth periodic traveling-wave solutions. These are not known in closed form, but they can be approximated. In this case, ϕ is given implicitly by

$$|x - x_0| = \int_{\phi_0}^{\phi} \frac{\sqrt{d-y}}{\sqrt{(M-y)(y-m)(y-z)}} dy, \tag{5.3}$$

where $\phi(x_0) = \phi_0$, and $z = d - M - m$. If $z < m < M < d$, then ϕ is a smooth function with $m = \min_{x\in\mathbb{R}} \varphi(x)$ and $M = \max_{x\in\mathbb{R}} \varphi(x)$. Once this integral is evaluated for a sufficient number of values, the function is inverted, and an approximation is found via a spline interpolation. This procedure for finding smooth traveling waves is explained in more detail in [18, 21].

For the purpose of numerical integration, the Fourier discretization is supplemented with the well known explicit four-stage Runge-Kutta scheme. Since the equation is only mildly stiff, an explicit method appears to be more advantageous than an implicit method. The scheme is explained as follows. If the wave profile $v_N(\cdot, t_i)$ is known at a particular time t_i, the four-stage Runge-Kutta method consists of letting

$$V_1 = \hat{v}_N(\cdot, t_i), \qquad \Gamma_1 = F(V_1),$$

$$V_2 = V_1 + \frac{\Delta t}{2}\Gamma_1, \qquad \Gamma_2 = F(V_2),$$

$$V_3 = V_2 + \frac{\Delta t}{2}\Gamma_2, \qquad \Gamma_3 = F(V_3),$$

$$V_4 = V_3 + \Delta t \Gamma_3, \qquad \Gamma_4 = F(V_4).$$

Finally, these functions are combined to compute $v_N(\cdot, t_{i+1}) = v_N(\cdot, t_i + \Delta t)$, according to

$$\hat{v}_N(\cdot, t_i + \Delta t) = \hat{v}_N(\cdot, t_i) + \frac{\Delta t}{6}\left(\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4\right).$$

This scheme is formally fourth-order convergent, meaning that if the time step Δt is halved, the error should decrease by a factor of 16.

The traveling-wave solutions can be used to test the numerical algorithm, because their time evolution is simply given by translation. If $v_N(\cdot, T)$ is the result of a numerical computation with initial data $\phi(x)$, it can be compared with the translated function $\phi(x - dT)$. In this way, the error produced by the discretization can be calculated. In particular, the smooth traveling wave shown in Figure 1 can be used to exhibit the convergence rate of the numerical approximation. In Table 1, it can be seen that the 4-th order convergence of the temporal integration scheme is approximately achieved. The spectral convergence of the spatial discretization is visible in Table 2. Note however, that

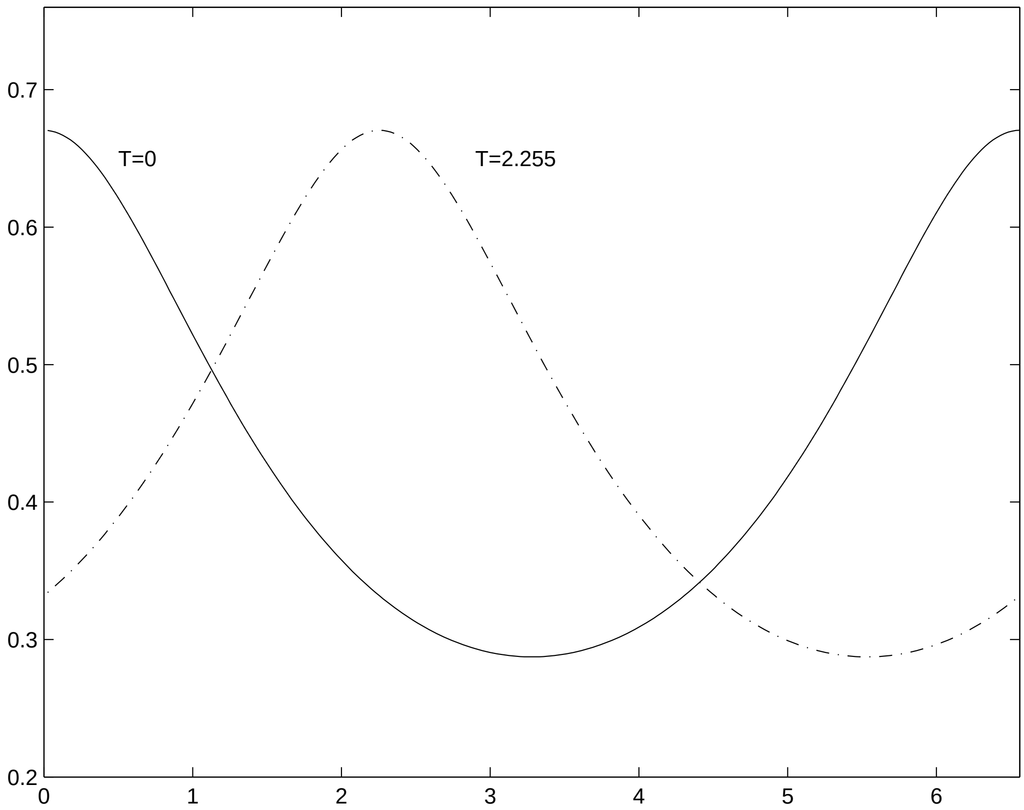

FIG. 1. A smooth periodic traveling wave. The solid curve shows the initial data, while the dashed-dotted curve shows the computed solution at $T = 2.255$.

TABLE I. Temporal discretization error for a smooth periodic traveling wave over the time domain $[0, 2.255]$. The number of grid points is $N = 256$.

Δt	L^2-Error	Ratio
0.036	1.50e-07	
0.018	9.30e-09	16.08
0.009	5.58e-10	16.66
0.0045	1.82e-11	30.74
0.00225	2.52e-11	7.20

TABLE II. Spatial discretization error for a smooth periodic traveling wave over the time domain $[0, 3.28]$. The time step is $\Delta t = 0.0002$.

N	L^2-Error	Ratio
4	8.67e-02	
8	8.12e-03	10.67
16	1.11e-04	73.14
32	7.98e-08	1391.0
64	1.49e-11	5354.1
128	1.49e-11	0.9945

TABLE III. Discretization error for a peaked traveling wave on the real line over the time domain $[0, 3.2]$. The time step is $\Delta t = 0.0002$, and the size of the spatial domain is 50.

N	L^2-Error	Ratio
512	7.26e-02	
1024	3.29e-02	2.20
2048	1.45e-02	2.27
4096	6.35e-03	2.28
8192	2.81e-03	2.26
16384	1.28e-03	2.18
32768	6.20e-04	2.06

TABLE IV. Discretization error for a peaked periodic traveling wave over the time domain $[0, 3.2]$. The time step is $\Delta t = 0.003$.

N	L^2-Error	Ratio
32	8.89e-01	
64	3.95e-01	2.25
128	1.86e-01	2.12
512	9.36e-02	1.98
1024	4.66e-02	2.01
2048	2.31e-02	2.02
4096	1.15e-02	2.01

the error is limited below because the smooth traveling waves are not known in closed form.

For the Camassa-Holm equation, the peakon solutions are of special importance. For these, we cannot expect spectral convergence. However, as shown in Table 3, the spectral discretization converges nevertheless, albeit at a lower rate. The Fourier representation (4.1) reveals that the peakon is in the Sobolev space $H^s(\mathbb{R})$ for any $s < \frac{3}{2}$. Thus it appears that the convergence at rate of N^{1-m} is probably not an optimal result. We hasten to mention however that this result as it stands is not applicable to peakons, because the proof makes use of H^4-regularity.

In order to analyze the advantage of the dealiasing scheme, we compared our calculations with computations performed using an unfiltered scheme. For both the smooth approximate traveling waves, and the peaked waves, the differences in convergence were minute on small time scales. In fact the observed rate of convergence was almost exactly the same. However, as shown in Figure 3, the dealiasing did reduce some spurious oscillations in the approximations. The advantage of the dealiasing became apparent when integrating over intermediate time scales. The periodic peaked traveling waves then suffered serious destabilization when computed with an aliased scheme. As shown in Figure 4, this problem was completely avoided by the use of an appropriate filter. This shows in fact that it is preferable to perform de-aliased computations, especially for solutions with low regularity.

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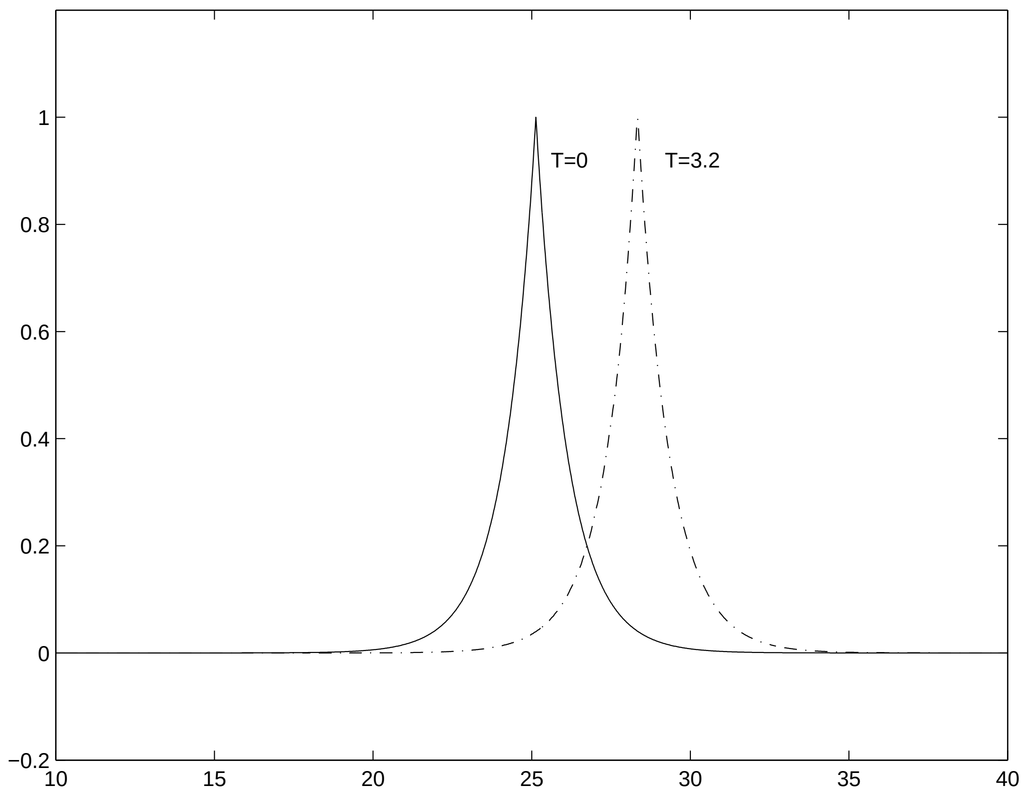

FIG. 2. A peakon. The solid curve shows the initial data, while the dashed-dotted curve shows the computed solution at $T = 3.2$. The size of the domain is $L = 50$.

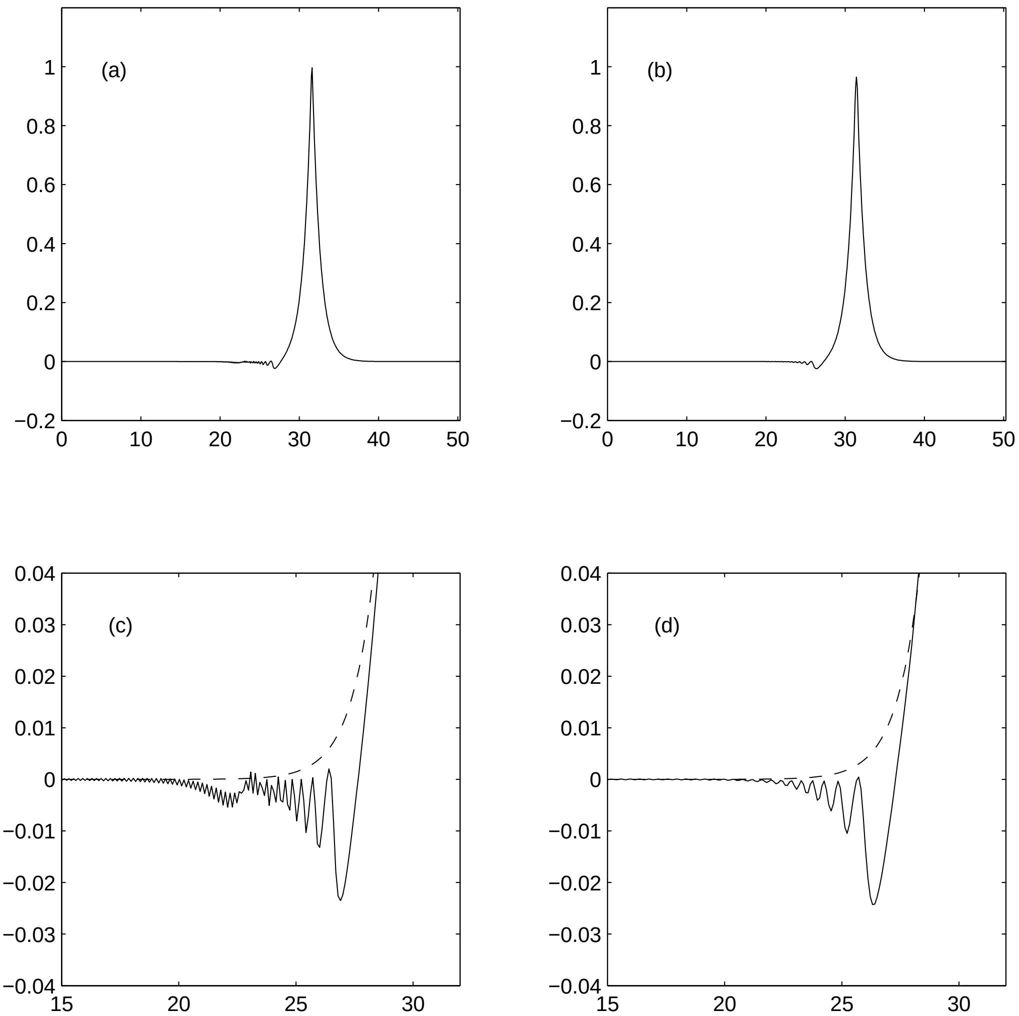

FIG. 3. A peakon, computed with and without dealiasing. The number of grid points is $N = 512$. The left column shows the aliased calculations. (a) $T = 6.4$, (c) $T = 6.4$, close-up. The right column shows the de-aliased calculations. (b) $T = 6.4$, (d) $T = 6.4$, close-up. The dashed curve shows the exact peakon, translated by an appropriate amount according to the time $T = 6.4$ and the speed $d = 1$. It appears that the de-aliased computation shown on the right is more advantageous.

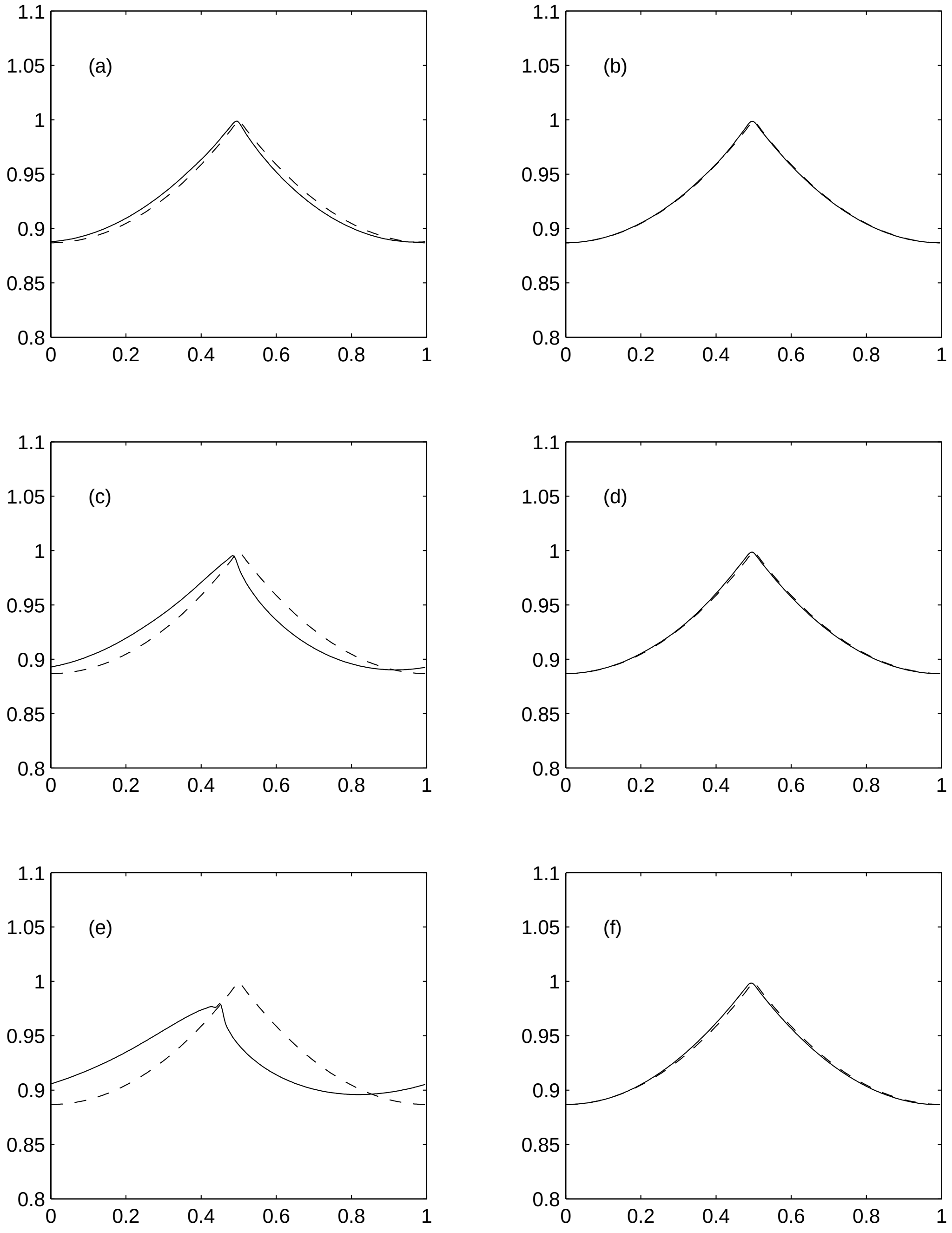

FIG. 4. Comparison of aliased and de-aliased calculations of a periodic peakon with $N = 256$ and $\Delta t = 0.003$. The left column shows the aliased calculations. (a) $T = 4$, (c) $T = 6$, (e) $T = 8$. The right column shows the de-aliased calculations. (b) $T = 4$, (d) $T = 6$, (f) $T = 8$. Solid curves are computed waves after 4, 6 and 8 periods. Dashed curves are the initial data, translated by an appropriate amount. In the de-aliased computation, a 2/3 filter was used, so that the effective number of modes is only 170.

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