Exercise 1 - Solutions

Time Series Models

2.1

y_t	-1	1
$P(Y_t = y_t)$	$\frac{1}{2}$	$\frac{1}{2}$

$$E[Y_t] = 0 \;\; ; \quad Var[Y_t] = 1$$

$$Z_t = \begin{cases} Y_t & \text{t even} \\ Y_{t-1} & \text{t odd} \end{cases}$$

a)

$$\left.\begin{array}{l} P(Z_t \le z) = P(Y_t \le z), \quad \text{t even} \\ P(Z_t \le z) = P(Y_{t-1} \le z) = P(Y_t \le z), \quad \text{t odd} \end{array}\right. \Rightarrow \text{first order stationary}$$

b)

$$E[Z_t Z_{t-1}] = \begin{cases} E[Y_t Y_{t-2}] = 0, & \text{t even} \\ E[Y_{t-1}^2] = 1, & \text{t odd} \end{cases} \Rightarrow \text{is not second order stationary}$$

2.2

$$Z_t = U \sin(2\pi t) + V \cos(2\pi t)$$

$$E[U] = E[V] = 0$$

$$Var[U] = Var[V] = 1$$

t is integer $\Rightarrow Z_t = V \quad ; \forall t \;\Rightarrow\; Z_t$ is strictly stationary

$E[Z_t, Z_{t-k}] = E[V^2] = 1 \Rightarrow Z_t$ is covariance stationary

2.3

$$Z_t = A \sin(\omega t + \theta)$$

$$\Rightarrow E[Z_t] = E[A]E[\sin(\omega t + \theta)]$$

$$E[Z_t Z_{t+k}] = E[A^2]E[\sin(\omega t + \theta)\sin(\omega(t+k) + \theta)] =$$

$$= \frac{E[A^2]}{2}\left(E[\cos(\omega t)] - E[\cos(2\omega t + \omega k + 2\theta)]\right)$$

a)

A constant, $\theta \sim U[0, 2\pi]$

$\Rightarrow E[Z_t] = 0,\ \gamma_k = \dfrac{A^2 \cos(\omega k)}{2} = \dfrac{A^2}{2}$ for $\omega = 2\pi \Rightarrow Z_t$ is covariance stationary

b)

θ constant, $E[A] = 0,\quad Var[A] = 1$

$\Rightarrow E[Z_t] = 0,\ \gamma_k = \dfrac{1}{2}\left[\cos(\omega k) - \cos\left(\omega(2t - k) + 2\theta\right)\right] =$

$\omega = 2\pi \Rightarrow \gamma_k = \dfrac{1}{2}\left[1 - \cos(2\theta)\right] = \sin^2(\theta) \Rightarrow Z_t$ is covariance stationary

c)

$Z_t = (-1)^t A\ ,\ E[A] = 0,\ Var[A] = 1$

$E[Z_t] = (-1)^t E[A] = 0$

$E[Z_t Z_{t+k}] = E[A^2](-1)^{2t+k} = 1.(-1)^k$

$\Rightarrow Z_t$ is covariance stationary

2.4

$$\rho_k = \begin{cases} 1 & k = 0 \\ \phi & |k| = 1 \quad \frac{1}{2} < \phi < 1 \\ 0 & |k| \geq 2 \end{cases}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i - t_j|} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 + \phi\alpha_2 \\ \phi\alpha_1 + \alpha_2 \end{bmatrix}$$

$$= \alpha_1^2 + \alpha_2^2 + 2\phi\alpha_1\alpha_2 = \mathrm{f}(\phi)$$

$$\left.\begin{array}{l} \mathrm{f}\left(\frac{1}{2}\right) = \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 > 0 \\ \mathrm{f}(1) = (\alpha_1 + \alpha_2)^2 \geq 0 \\ \mathrm{f}'(\phi) = 2\alpha_1\alpha_2 \quad \text{constant} \end{array}\right\} \Rightarrow \frac{1}{2} < \phi < 1\ OK$$

$$\left.\begin{array}{l} \mathrm{f}\left(-\frac{1}{2}\right) = \alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2 > 0 \\ \mathrm{f}(-1) = (\alpha_1 - \alpha_2)^2 \geq 0 \\ \mathrm{f}'(\phi) \quad \text{constant} \end{array}\right\} \Rightarrow -1 < \phi < -\frac{1}{2}\ OK$$

Yes it is a valid real-valued autocorrelation function because it is positive semidefinite.