INTERPOLATION BY DIRICHLET SERIES IN H^∞

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Abstract. We present some basic results about interpolating sequences for $\mathscr{H}^\infty$, the algebra of Dirichlet series in H^∞ of the right half-plane. As a consequence of a theorem of B. Berndtsson, S.-Y. A. Chang, and K.-C. Lin on interpolating sequences in polydiscs, we obtain a description of bounded interpolating sequences for $\mathscr{H}^\infty$. We show how interpolating sequences for $\mathscr{H}^\infty$ can be described via interpolating sequences for bounded analytic functions in polydiscs. Finally, we deduce a necessary density condition from a theorem on the existence of a linear operator of interpolation.

1. Introduction

This note presents a few basic results about interpolating sequences for the algebra $\mathscr{H}^\infty$, which consists of those bounded analytic functions f on $\mathbb{C}_+ = \{s = \sigma + \mathrm{i}\, t : \ \sigma > 0\}$ that can be represented by an ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (1)$$

in some half-plane. We cannot expect to obtain a complete geometric description of the interpolating sequences in this case, but even partial results can give interesting function theoretic information about $\mathscr{H}^\infty$.

We note that $\mathscr{H}^\infty$ is a closed subalgebra of $H^\infty(\mathbb{C}_+)$, the algebra of bounded analytic functions, assuming both spaces are equipped with the supremum norm:

$$\|f\|_\infty = \sup_{\sigma>0} |f(\sigma + \mathrm{i}\, t)|.$$

In general, if A is an algebra of bounded analytic functions on some domain Ω, a sequence $S = (s_j)$ of distinct points $s_j = \sigma_j + \mathrm{i}\, t_j$ in Ω (possibly finite) is said to be an interpolating sequence for A if every interpolation problem

$$f(s_j) = a_j \qquad (2)$$

has a solution f in A whenever (a_j) is a bounded sequence of complex numbers. Our first theorem is the following local result.

Theorem 1. *A bounded sequence S in $\mathbb{C}_+$ is an interpolating sequence for $\mathscr{H}^\infty$ if and only if it is an interpolating sequence for $H^\infty(\mathbb{C}_+)$.*

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In other words, for bounded sequences, L. Carleson's characterization of interpolating sequences for $H^\infty(\mathbb{C}_+)$ [5] applies to $\mathscr{H}^\infty$ as well. The result is similar to the main result of [13]. It is quite illuminating because it somehow suggests that, locally, functions in $\mathscr{H}^\infty$ look like functions in $H^\infty(\mathbb{C}_+)$.

Theorem 1 is a direct consequence of a theorem of B. Berndtsson, S.-Y. A. Chang, and K.-C. Lin [3] concerning interpolating sequences in the unit polydisc. This means that Theorem 1 is really a result about a much smaller algebra than $\mathscr{H}^\infty$. Indeed, for m a positive integer, let $\mathscr{H}_m^\infty$ be the closed subalgebra of $\mathscr{H}^\infty$ consisting of Dirichlet series of the form (1) involving only integers n generated by the first m primes $2, 3, ..., p_m$. We will see below that in Theorem 1 we may replace $\mathscr{H}^\infty$ by $\mathscr{H}_2^\infty$.

The next theorem shows how to connect the finite-dimensional case (corresponding to $\mathscr{H}_m^\infty$) to the infinite-dimensional one (corresponding to $\mathscr{H}^\infty$). To state that result, we need to introduce what is known as the constant of interpolation. To this end, let A be any of the algebras considered so far. Using the open mapping theorem, we find that if S is an interpolating sequence for A, then we can always solve (2) with an f in A such that

$$\|f\|_\infty \leq C\|(a_j)\|_\infty$$

for some $C < \infty$ depending only on S. The constant of interpolation $M = M(S)$ is the smallest such C; we set $M(S) = \infty$ if S is not an interpolating sequence for A. For $\delta > 0$, let also S_δ be the subsequence of S consisting of points from the halfplane $\Re s > \delta$. We now have:

Theorem 2. *The sequence S is an interpolating sequence for $\mathscr{H}^\infty$ if and only if there exist numbers $M > 1$ and $c > 2$ such that, for every $\delta > 0$ and $m > c^{1/\delta}$, the constant of interpolation for S_δ, considered as an interpolating sequence for $\mathscr{H}_m^\infty$, is at most M.*

It is a rather simple fact that each of the sequences S_δ in Theorem 2 is finite. The following theorem gives a much more precise quantitative result.

Theorem 3. *If $S = (s_j)$ is an interpolating sequence for $\mathscr{H}^\infty$ with constant of interpolation M, then*

$$\sum_j M^{-c/\sigma_j} < \infty$$

whenever $c > 1/2$.

We will exhibit an example showing that the theorem fails if we replace c/σ_j by $\alpha/(\sigma_j|\log\sigma_j|)$ for any positive constant α. We will also see that there is a huge gap between the condition of Theorem 3 and the weaker one that we get from the mere fact that the sequence S has to be separated with respect to Gleason distance (to be defined below). Here the contrast to the situation for $H^\infty(\mathbb{C}_+)$ or indeed any of the spaces $\mathscr{H}_m^\infty$ is quite striking.

The proofs of the three theorems stated above are relatively short, but somewhat elaborate preparations are required. It is essential that we associate $\mathscr{H}^\infty$ with H^∞ of the infinite-dimensional polydisc and $\mathscr{H}_m^\infty$ with H^∞ of the m-disc, following a fundamental idea of H. Bohr. We explain this viewpoint in the next section. Section 3 presents a well-known perturbation argument that (1) shows that interpolating sequences are stable under small perturbations and (2) leads to a proof of Theorem 2. In Section 4, we present the theorem of Berndtsson–Chang–Lin, and we show how it gives Theorem 1. In Section 5, we obtain Theorem 3 as a consequence of a general result about the existence of a linear operator of interpolation. Finally, Section 6 contains the discussion around Theorem 3 alluded to above.

2. $\mathscr{H}_m^\infty$ AND $\mathscr{H}^\infty$ AS SPACES ON POLYDISCS

Following Bohr [4], we set

$$z_1 = 2^{-s}, z_2 = 3^{-s}, ..., z_m = p_m^{-s}, ...,$$

where p_m denotes the m-th prime number. Thus, in view of the fundamental theorem of arithmetic, Dirichlet series in $\mathscr{H}_m^\infty$ and $\mathscr{H}^\infty$ can be viewed as power series in respectively m variables and infinitely many variables. Indeed, if the prime factorization of the positive integer n is written as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, we obtain from (1) the formal power series

$$F(\mathbf{z}) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_r^{\alpha_r},$$

where $\mathbf{z} = (z_1, z_2, ..., z_m)$ or $\mathbf{z} = (z_1, z_2, ...)$ depending on whether f is a function in $\mathscr{H}_m^\infty$ or in $\mathscr{H}^\infty$. The key point is that, thanks to Kronecker's theorem on diophantine approximation [10, Ch. XXIII], this is not just a formal correspondence:

Lemma 1. *Let $2, 3, ..., p_m$ be the first m primes ($m \geq 2$). Then the map*

$$t \mapsto (2^{-\mathrm{i}t}, 3^{-\mathrm{i}t}, ..., p_m^{-\mathrm{i}t})$$

from $\mathbb{R}^+$ to $\mathbb{T}^m$ has dense range in $\mathbb{T}^m$.

It is immediate from Lemma 1 and the maximum principle that

$$\|f\|_\infty = \|F\|_\infty, \tag{3}$$

where the norm on the right-hand side is the $H^\infty(\mathbb{D}^m)$ norm. The result is the same in the infinite-dimensional case, but some care has to be taken when defining the norm in the polydisc. First, we restrict point evaluation to the set $\mathbb{D}^\infty \cap c_0$, where $c_0 = c_0(\mathbb{N})$ is the space of sequences tending to zero. Indeed, for a point $\mathbf{z} = (z_1, z_2, ..., z_m, ...)$ in $\mathbb{D}^\infty \cap c_0$ we put

$$\mathbf{z}^{(m)} = (z_1, z_2, ..., z_m, 0, 0, ...),$$

meaning that all coordinates z_j for $j > m$ are replaced by zeros. Plugging $\mathbf{z}^{(m)}$ into F, we obtain a function $F(\mathbf{z}^{(m)})$ in the polydisc $\mathbb{D}^m$, the "m-te Abschnitt", as Bohr called it. We declare F to be in $H^\infty(\mathbb{D}^\infty)$ if the H^∞ norm of these functions are uniformly bounded, and denote the smallest uniform bound on these norms by $\|f\|_\infty$. By the Schwarz lemma for the polydisc, we have for $m < l$

$$|F(\mathbf{z}^{(m)}) - F(\mathbf{z}^{(l)})| \leq 2\|F\|_\infty \max\{|z_j| : \ m < j \leq l\},$$

so that we may define

$$F(\mathbf{z}) = \lim_{m\to\infty} F(\mathbf{z}^{(m)}).$$

We may view $\mathbb{C}_+$ as a one-dimensional complex variety in $\mathbb{D}^\infty \cap c_0$ via the map

$$s \mapsto (2^{-s}, 3^{-s}, ..., p_n^{-s}, ...).$$

It was proved in [11] that (3) remains true in the infinite-dimensional case. In other words, we may associate $\mathscr{H}^\infty$ with $H^\infty(\mathbb{D}^\infty)$. We will in what follows call F the Bohr extension of f.

The considerations above imply that our interpolation problems are special versions of interpolation problems on finite or infinite polydiscs. This does not mean that our interpolation problems are much easier than their counterparts on the larger domains, but we have at least the option of applying methods from either setting. For instance, as already mentioned, Theorem 1 is really a result for polydiscs, while the one-dimensional Phragmén–Lindelöf principle is an essential ingredient in the proof of Theorem 3.

3. Perturbation arguments and proof of Theorem 2

The existence of a constant of interpolation gives a way of solving interpolation problems that are somehow small perturbations of problems for which solutions are known to exist. We will now give two versions of this kind of reasoning, one of which constitutes the proof of Theorem 2.

We begin with distortions with respect to Gleason distance. We declare the Gleason distance between two points $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{D}^m$ to be the supremum of $|f(\mathbf{w})|$ when f is a function in the unit ball of $H^\infty(\mathbb{D}^m)$ such that $f(\mathbf{z}) = 0$. It follows from the Schwarz lemma that the Gleason distance between $\mathbf{z} = (z_1, ..., z_m)$ and $\mathbf{w} = (w_1, ..., w_m)$ in $\mathbb{D}^m$ is

$$d_m(\mathbf{z}, \mathbf{w}) = \max_{1\le j\le m} \left| \frac{z_j - w_j}{1 - \overline{w_j} z_j} \right|.$$

Likewise, the Gleason distance between two points $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{D}^\infty \cap c_0$ is the supremum of $|f(\mathbf{w})|$ when f is a function in the unit ball of $H^\infty(\mathbb{D}^\infty)$ such that $f(\mathbf{z}) = 0$. Using again the Schwarz lemma, we find that the Gleason distance between $\mathbf{z} = (z_1, z_2, ...)$ and $\mathbf{w} = (w_1, w_2, ...)$ in $\mathbb{D}^\infty \cap c_0$ is

$$d_\infty(\mathbf{z}, \mathbf{w}) = \max_{j} \left| \frac{z_j - w_j}{1 - \overline{w_j} z_j} \right|.$$

If $Z = (\mathbf{z}_j)$ is an interpolating sequence for $H^\infty(\mathbb{D}^m)$ ($m \ge 1$ or $m = \infty$) with constant of interpolation M, then it follows from the definition of Gleason distance that

$$d_m(\mathbf{z}_j, \mathbf{z}_l) \ge M^{-1}$$

whenever $j \ne l$. When we have such a uniform lower estimate for the Gleason distance between distinct points, we say that Z is separated with respect to Gleason distance.

The following lemma is folklore. A proof is included for the sake of completeness and to demonstrate another usage of the iteration scheme appearing in the proof of Theorem 2.

Lemma 2. *If $Z = (\mathbf{z}_j)$ is an interpolating sequence for $H^\infty(\mathbb{D}^m)$ ($m \geq 1$ or $m = \infty$) with constant of interpolation M, then any sequence $Z' = (\mathbf{z}'_j)$ satisfying*

$$\sup_j d_m(\mathbf{z}_j, \mathbf{z}'_j) < 1/(M+1)$$

is also an interpolating sequence for $H^\infty(\mathbb{D}^m)$.

Proof. Set $\varepsilon = \sup_j d_m(\mathbf{z}_j, \mathbf{z}'_j)$. We want to solve $f(\mathbf{z}'_j) = a_j$. We first solve $f_0(\mathbf{z}_j) = a_j$ with $\|f_0\|_\infty \leq M\|(a_j)\|_\infty$. We set $a_j^{(1)} = a_j - f_0(\mathbf{z}'_j)$ and note that we have $\|(a_j^{(1)})\|_\infty \leq (M+1)\varepsilon\|(a_j)\|_\infty$. We next solve $f_1(\mathbf{z}_j) = a_j^{(1)}$ with $\|f_1\|_\infty \leq M(M+1)\varepsilon\|(a_j)\|_\infty$. We iterate this procedure: In the k-th step, we set $a_j^{(k)} = a_j^{(k-1)} - f_{(k-1)}(\mathbf{z}'_j)$ and solve the interpolation problem $f_k(\mathbf{z}_j) = a_j^{(k)}$ with the bound $\|f_k\|_\infty \leq M(M+1)^k\varepsilon^k\|(a_j)\|_\infty$. Therefore,

$$f = \sum_{k=0}^{\infty} f_k$$

solves the problem $f(\mathbf{z}'_j) = a'_j$ whenever $\varepsilon < 1/(M+1)$. ■

We mentioned in Section 2 that $F(\mathbf{z})$ could be defined on $\mathbb{D}^\infty \cap c_0$ as a limit of the "m-te Abschnitt" $F(\mathbf{z}^{(m)})$ when $m \to \infty$. This construction relies on the Schwarz lemma. The starting point for the proof of Theorem 2 is the restriction of this reasoning to $\mathbb{C}_+$, which should now be viewed as a subset of $\mathbb{D}^\infty \cap c_0$.

For the rest of this section we let f_m ($m \geq 2$) denote the element in $\mathscr{H}_m^\infty$ whose Bohr extension is $F(\mathbf{z}^{(m)})$. The Schwarz lemma implies that

$$|f_m(s) - f(s)| \leq 2p_{m+1}^{-\sigma}\|f\|_\infty.$$

A loose interpretation of this inequality is that when we restrict attention to a smaller half-plane than $\mathbb{C}_+$, say $\sigma \geq \delta > 0$, then only a finite number of the "variables" $2^{-s}, 3^{-s}, \ldots$ plays a "significant" role. Theorem 2 can be viewed as a precise quantitative version of this statement.

Proof of Theorem 2. The sufficiency is an immediate consequence of Montel's theorem. To prove the necessity, we begin by applying the Schwarz lemma in the following way. Assuming S is an interpolating sequence with constant of interpolation L, we solve $f(s_j) = a_j$ for s_j in S_δ. Then

$$|f(s) - f_m(s)| \leq 2p_{m+1}^{-\sigma}L\|(a_j)\|_\infty$$

and $\|f_m\|_\infty \leq L\|(a_j)\|_\infty$. We define a new sequence $a_j^{(1)} = a_j - f_m(s_j)$ and solve $f^{(1)}(s_j) = a_j^{(1)}$, which then satisfies

$$\|f^{(1)}\|_\infty \leq 2p_{m+1}^{-\delta}L^2\|(a_j)\|_\infty.$$

We take again the m-te Abschnitt and obtain the corresponding Dirichlet series $f_m^{(1)}$ for which we have $\|f_m^{(1)}\|_\infty \leq 2p_{m+1}^{-\delta}L^2\|(a_j)\|_\infty$. We iterate this procedure: In the k-th step, we set $a_j^{(k)} = a_j^{(k-1)} - f_m^{(k-1)}(s_j)$ and solve the interpolation problem $f^{(k)}(s_j) = a_j^{(k-1)}$ with the bound

$$\|f^{(k)}\|_\infty \le 2^k p_{m+1}^{-k\delta} L^{k+1} \|(a_j)\|_\infty.$$

If $p_{m+1}^\delta \ge (2+\varepsilon)L$, then the interpolation problem $g(s_j) = a_j$ can be solved in $\mathscr{H}_m^\infty$ by the function

$$g = f_m + \sum_{k=1}^\infty f_m^{(k)}.$$

For the constant of interpolation M we get the estimate

$$M \le L(2+\varepsilon)/\varepsilon.$$

■

The main obstacle for applying Theorem 2 is of course that it is hard to control the constant of interpolation in nontrivial situations.

4. The Berndtsson–Chang–Lin condition and proof of Theorem 1

Our starting point is the following theorem of Berndtsson, Chang, and Lin [3].

Lemma 3 (Berntsson–Chang–Lin theorem). *If a sequence $Z = (\mathbf{z}_j)$ in $\mathbb{D}^m$ ($1 \le m < \infty$) satisfies*

$$\inf_j \prod_{l \ne j} d_m(\mathbf{z}_j, \mathbf{z}_l) > 0,$$

then $Z = (\mathbf{z}_j)$ is an interpolating sequence for $H^\infty(\mathbb{D}^m)$.

We note that the proof relies on a one-variable result of P. Jones [12]. It would be interesting to know whether this theorem remains valid for $m = \infty$ when Z is a sequence of distinct points from $\mathbb{D}^\infty \cap c_0$.

We will now switch to the half-plane $\mathbb{C}_+$. We set

$$\varrho_2(s_1, s_2) = d_2((2^{-s_1}, 3^{-s_1}), (2^{-s_2}, 3^{-s_2}))$$

and

$$\varrho(s_1, s_2) = \left| \frac{s_1 - s_2}{s_1 + \overline{s_2}} \right|,$$

which is the Gleason distance for $H^\infty(\mathbb{C}_+)$. Clearly,

$$\varrho_2(s_1, s_2) \le \varrho(s_1, s_2).$$

Carleson's theorem [5] says that

$$\inf_j \prod_{l \ne j} \varrho(s_j, s_l) > 0 \tag{4}$$

is a necessary and sufficient condition for $S = (s_j)$ to be an interpolating sequence for $H^\infty(\mathbb{C}_+)$. Consequently, in view of Lemma 3, Theorem 1 will follow once we have established the following lemma.

Lemma 4. *For every $M > 0$, there exists a positive constant γ such that if s_1 and s_2 are points in C_+ satisfying $|s_1|, |s_2| \le M$, then*

$$\varrho_2(s_1, s_2) \ge [\varrho(s_1, s_2)]^\gamma.$$

Proof. It suffices to show that there exist positive constants c and C such that

$$\varrho_2(s_1, s_2) \ge c\varrho(s_1, s_2) \quad \text{and} \quad 1 - \varrho_2(s_1, s_2) \le C(1 - \varrho(s_1, s_2)). \tag{5}$$

To prove the first inequality in (5), we set $K = \{s = \sigma + \mathrm{i}\,t : \ \sigma \ge 0, \ |s| \le M\}$ and define the following function $\psi(s_1, s_2)$ on the compact set $K \times K$. We set $\psi(s_1, s_2) = \varrho(s_1, s_2)/\varrho_2(s_1, s_2)$ if $s_1 \ne s_2$ and both s_1 and s_2 are in $\mathbb{C}_+$. If at least one of the points s_1 and s_2 lie on the imaginary axis, we set $\psi(s_1, s_2) = 1$. Finally, if s is in $\mathbb{C}_+$, we put

$$\psi(s, s) = \frac{2^\sigma - 2^{-\sigma}}{2\sigma \log 2}.$$

We observe that ψ is a continuous function, and so we may choose for c the reciprocal of its maximum on $K \times K$.

To prove the second inequality in (5), we begin by noting that

$$1 - [\varrho_2(s_1, s_2)]^2 = \frac{(1 - p^{-2\sigma_1})(1 - p^{-2\sigma_2})}{(1 - p^{-(\sigma_1+\sigma_2)})^2 + 2p^{-(\sigma_1+\sigma_2)}(1 - \cos((\log p)(t_1 - t_2)))},$$

where p is either 2 or 3. Hence

$$1 - [\varrho_2(s_1, s_2)]^2 \le \frac{C\,\sigma_1\sigma_2}{(\sigma_1 + \sigma_2)^2 + (t_1 - t_2)^2}$$

for a constant C depending on M. The result follows since

$$1 - [\varrho(s_1, s_2)]^2 = \frac{4\sigma_1\sigma_2}{(\sigma_1 + \sigma_2)^2 + (t_1 - t_2)^2}.$$

∎

Defining $\varrho_m(s_1, s_2)$ for $m > 2$ or $m = \infty$ in the obvious way, we note that for $m < \infty$ we also have the reverse estimate

$$1 - [\varrho_m(s_1, s_2)]^2 \ge \frac{1}{4}(1 - p_m^{-2\sigma_1})(1 - p_m^{-2\sigma_2}).$$

This means that if $S = (\sigma_j + \mathrm{i}\,t_j)$, viewed as a sequence in $\mathbb{D}^m$ via the map $s \mapsto (2^{-s}, 3^{-s}, ..., p_m^{-s})$, meets the Berndtsson–Chang–Lin condition, then

$$\sum_j \sigma_j < \infty.$$

This summability condition is not even necessary for S to be an interpolating sequence for $\mathscr{H}_2^\infty$ (see Section 6).

5. P. Beurling functions and proof of Theorem 3

A theorem of P. Beurling [6] says that if $S = (s_j)$ is an interpolating sequence for $H^\infty(\mathbb{C}_+)$ with constant of interpolation M, then there are functions f_j in $H^\infty(\mathbb{C}_+)$ such that $f_j(s_k) = \delta_{j,k}$ and

$$\sum_j |f_j(s)| \le M$$

holds for all s in $\mathbb{C}_+$. In other words, we may solve the interpolation problem $f(s_j) = a_j$ by means of the linear operator

$$f(s) = \sum_j a_j f_j(s),$$

and the norm of this operator is optimal.

We cannot expect such linear operators of interpolation to exist for general Banach algebras. However, for the spaces $\mathscr{H}^\infty_m$ and $\mathscr{H}^\infty$, linear operators of interpolation are known to exist, thanks to a general theorem from the theory of uniform algebras. (See J. B. Garnett's book [9, p. 288], where the result is attributed to N. Th. Varopoulos [14] and A. Bernard [2].) Explicitly, we have the following version of this result:

Lemma 5. *Let A be any of the algebras $\mathscr{H}^\infty$ or $\mathscr{H}^\infty_m$. Then if S is an interpolating sequence for A with constant of interpolation M, then there are functions f_j in A satisfying $f_j(s_k) = \delta_{j,k}$, $\|f_j\|_\infty \le M$, and*

$$\sum_j |f_j(s)| \le M^2.$$

The proof relies on normal families along with a general argument valid for uniform algebras on compact sets. See [9, pp. 288–289] for details. We see that we get a poorer bound on the norm than in the P. Beurling theorem. We will still refer to the functions f_j of Lemma 5 as P. Beurling functions.

To place our application of Lemma 5 in context, we begin with some remarks on how it applies to $H^\infty(\mathbb{C}_+)$ and $\mathscr{H}^\infty_m$. Suppose first that S is an interpolating sequence for $H^\infty(\mathbb{C}_+)$ with constant of interpolation M. Let I be an arbitrary interval on the imaginary axis, and let $Q(I)$ be the square in $\mathbb{C}_+$ such that I is one of its sides. For an arbitrary positive number ξ, let also ξI be the interval of length $\xi|I|$ concentric with I. Then starting from the Poisson formula

$$1 = f_j(s_j) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_j(\mathrm{i}\,t)\frac{\sigma_j}{(t-t_j)^2+\sigma_j^2}\,\mathrm{d}\,t,$$

we get the following estimate for $s_j = \sigma_j + \mathrm{i}\,t_j$ in $Q(I)$:

$$\sigma_j \le \int_{3MI} |f_j(\mathrm{i}\,t)|\,\mathrm{d}\,t.$$

Summing over j and applying Lemma 5, we get

$$\sum_{s_j\in Q(I)} \sigma_j \le 3M^3|I|,$$

which is Carleson's condition. (Using P. Beurling's theorem, we could replace M^3 by M^2 in the latter estimate.)

We may act similarly when we consider $\mathscr{H}_m^\infty$, but we need to switch to the polydisc. To this end, let $Z = (\mathbf{z}_j)$ be an interpolating sequence for $H^\infty(\mathbb{D}^m)$, and let R_j be the "rectangular base" of $\mathbf{z}_j$ on the distinguished boundary $\mathbb{T}^m$ of $\mathbb{D}^m$. Then using Poisson integrals for each f_j and summing as above, we get

$$\sum_{R_j \subset \Omega} |R_j| \le C|\Omega|$$

for every open subset of the distinguished boundary. Here the constant C depends only on M. This result is stated in [3] and is obtained there as a consequence of work of Varopoulos [15].

Returning to the half-plane and taking Ω to be the entire distinguished boundary, we get the necessary condition

$$\sum_j \sigma_j^m < \infty \tag{6}$$

for S to be an interpolating sequence for $\mathscr{H}_m^\infty$. This condition is only slightly more restrictive than the one we get from the fact that the sequence has to be separated with respect to Gleason distance. In particular, the latter condition implies that

$$\sum_j \sigma_j^{m+\varepsilon} < \infty$$

for every $\varepsilon > 0$.

We now turn to $\mathscr{H}^\infty$. Instead of using Poisson integrals as above, we will make use of the link between $\mathscr{H}^\infty$ and the Banach space $\mathscr{H}^1$, which we choose to define in the following way. For Dirichlet polynomials

$$f(s) = \sum_{n=1}^{N} a_n n^{-s},$$

we set

$$\|f\|_1 = \lim_{T\to\infty} \frac{1}{T}\int_0^\infty |f(\mathrm{i}\,t)|\,\mathrm{d}\,t,$$

where the limit exists thanks to a classical theorem of F. Carlson [7]. The space $\mathscr{H}^1$ is obtained by taking the closure of the set of Dirichlet polynomials with respect to this norm. For the details of this construction, we refer to [1].

We will need the following fact from [8]. A function f represented by a Dirichlet series in $\mathscr{H}^1$ is analytic in the half-plane $\sigma > 1/2$. Indeed, we have the following estimate for point evaluation:

$$|f(\sigma + \mathrm{i}\,t) \le C\frac{\|f\|_1}{\sigma - 1/2}$$

for a constant C independent of f. We will also need the fact that we can compute $\|f\|_1$ for f in $\mathscr{H}^\infty$ in the following way:

$$\|f\|_1 = \lim_{\sigma\to 0}\lim_{T\to\infty}\frac{1}{T}\int_0^T |f(\sigma+\mathrm{i}\,t)|\,\mathrm{d}\,t. \tag{7}$$

Thus, in particular,

$$\|f\|_1 \le \|f\|_\infty$$

for functions f in $\mathscr{H}^\infty$. This inequality can be seen as a consequence of a much stronger result, proved in [1], which says that $\mathscr{H}^\infty$ is in fact the multiplier algebra for $\mathscr{H}^1$.

We are now ready to prove Theorem 3.

Proof of Theorem 3. Set $M(\sigma) = \sup_t |f(\sigma+\mathrm{i}\,t)|$. Then the Phragmén-Lindelöf principle gives for $\sigma_0 \ge \sigma$

$$|f(\sigma+\mathrm{i}\,t)| \le \|f\|_\infty^{1-\sigma/\sigma_0}[M(\sigma_0)]^{\sigma/\sigma_0}.$$

If $\sigma_0 > 1/2$, we can use the bound for point evaluation in $\mathscr{H}^1$ to obtain from this

$$|f(\sigma+\mathrm{i}\,t)| \le [C/(\sigma_0-1/2)]^{\sigma/\sigma_0}\|f\|_\infty^{1-\sigma/\sigma_0}\|f\|_1^{\sigma/\sigma_0}.$$

Applying this inequality to $f = f_j$ at the point s_j, we get

$$\|f_j\|_1 \ge [(\sigma_0-1/2)/C]\,M^{-\sigma_0/\sigma_j+1}.$$

But in view of Lemma 5 and formula (7), we also have

$$\sum_j \|f_j\|_1 \le M^2,$$

and so we conclude that

$$\sum_j M^{-c/\sigma_j} < \infty$$

whenever $c > 1/2$. ■

6. Remarks on Theorem 3

The following result on separation of points on a vertical line shows that the condition of Theorem 3 is extremely restrictive compared with what we get from the mere fact that an interpolating sequence is separated with respect to Gleason distance.

Proposition 1. *Suppose $0 < \varepsilon < 1$ and a positive number $\gamma < 2$ are given. For sufficiently small σ we can find at least $2^{(1/\varepsilon)^{\gamma/\sigma}}$ distinct points $\sigma + \mathrm{i}\,t_j$ such that*

$$\varrho_\infty(\sigma+\mathrm{i}\,t_j, \sigma+\mathrm{i}\,t_k) \ge \varepsilon$$

for $j \ne k$.

Proof. It suffices with a rough estimate. We begin by finding the smallest radius r such that we can find two distinct points on the circle $|z| = r$ the pseudohyperbolic distance between which is at least ε. We obtain the equation

$$\frac{2r}{1+r^2} = \varepsilon,$$

which gives $r = 1 - \sqrt{1-\varepsilon^2}$. We now estimate the number of primes p for which

$$p^{-\sigma} > 1 - \sqrt{1-\varepsilon^2},$$

or, in other words, for which

$$\log p < \frac{1}{\sigma} \log \frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon^2}.$$

For sufficiently small σ there are at least $(1/\varepsilon)^{\gamma/\sigma}$ such primes in view of the prime number theorem. The claim now follows by an application of Lemma 1. ■

The proposition could have been stated in a slightly sharper form, but we note that it would not hold if we allowed γ to be any number larger than 2. We may see this by essentially repeating the argument above, this time starting from the observation that the maximal number of points pseudohyperbolically separated by ε on the circle $|z| = 2^{-\sigma}$ is bounded by a constant times $1/(\varepsilon\sigma)$. We then get $(c/\varepsilon\sigma)^{(1/\varepsilon)^{2/\sigma}}$ as an upper bound for the number of t_j that can be chosen.

To see to what extent (6) and Theorem 3 are sharp, we consider two examples. Set

$$\Sigma_l = \{s = 2^{-l} + \mathrm{i}\, 2\pi k \log 2,\ k \text{ an integer}\}$$

for l a positive integer and look for an interpolating sequence S being a subsequence of $\bigcup_l \Sigma_l$. Then the problem reduces to the problem of solving the interpolation problem $f_l(s_j) = a_j$ for s_j in $\Sigma_l \cap S$ with uniform control of norms, because the global interpolation problem can be solved as

$$f(s) = \sum_l g_l(2^{-s}) f_l(s),$$

with g_l being P. Beurling functions for the sequence $2^{-2^{-l}}$ in the unit disc. By Lemma 1, we may, for each l, choose 2^l points s_j from Σ_l such that the points 3^{-s_j} are uniformly separated with respect to pseudohyperbolic distance on the unit disc. This means that the f_l can be chosen as functions of the "variable" 3^{-s}, where the functions in question are the P. Beurling functions associated with the finite sequence (3^{-s_j}). Thus we have an example showing that we may have

$$\sum \sigma_j = \infty$$

even in $\mathscr{H}_2^\infty$. This argument can be further developed to a proof that an interpolating sequence S for $\mathscr{H}_m^\infty$ may have

$$\sum \sigma_j^{m-1} = \infty.$$

Let us now turn to $\mathscr{H}^\infty$. We begin by considering the two points $z_{n,0} = p_n^{-\sigma}$ and $z_{n,1} = -p_n^{-\sigma}$ on each of the circles $|z_n| = p_n^{-\sigma}$. Then by Pick's theorem (see [9, p. 7]), the constant of interpolation for the two-point sequence $z_{n,0}, z_{n,1}$ is p_n^σ. In the $(m-1)$-disk, we take the cross product of these sequences

$$Z_m(\sigma) = (\mathbf{z_r}),$$

where $\mathbf{z_r} = (z_{2,r_2}, z_{3,r_3}, ..., z_{m,r_m})$ and $\mathbf{r} = (r_2, r_3, ..., r_m)$ ranges over $\{0,1\}^{m-1}$. Thus $Z_m(\sigma)$ is a sequence consisting of 2^{m-1} points. The interpolation problem $f(\mathbf{z_r}) = a_\mathbf{r}$ for $Z_m(\sigma)$ can be solved by the function

$$f(z) = \sum_{\mathbf{r}} a_\mathbf{r} f_{r_2}(z_2) f_{r_3}(z_3) \cdots f_{r_m}(z_m)$$

with the functions f_{r_n} being the P. Beurling functions for the respective two-point problems. The constant of interpolation M_m therefore satisfies

$$M_m \le \prod_{n=2}^{m} p_n^\sigma = e^{\sigma \sum_{n=2}^m \log p_m}.$$

This expression is uniformly bounded if

$$m \le \frac{c}{\sigma \log(1/\sigma)}$$

for some positive constant c. By Lemma 1, we can find points on Σ_l that are arbitrarily close to the corresponding points on $Z_m(2^{-l})$. By Lemma 2, this means that we can find a sequence of 2^m points from Σ_l with a constant of interpolation arbitrarily close to M_m. In other words, we may place at least

$$2^{-\frac{c}{\sigma_l \log \sigma_l}}$$

points on Σ_l, where $\sigma_l = 2^{-l}$. We have therefore constructed a sequence which satisfies

$$\sum_j 2^{\frac{c}{\sigma_j \log \sigma_j}} = \infty.$$

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