INTEGRAL MEANS AND BOUNDARY LIMITS OF DIRICHLET SERIES

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Abstract. This paper deals with the boundary behavior of functions in the Hardy spaces $\mathscr{H}^p$ for ordinary Dirichlet series. The main result, answering a question of H. Hedenmalm, shows that the classical F. Carlson theorem on integral means does not extend to the imaginary axis for functions in $\mathscr{H}^\infty$, i.e., for ordinary Dirichlet series in H^∞ of the right half-plane. We discuss an important embedding problem for $\mathscr{H}^p$, the solution of which is only known when p is an even integer. Viewing $\mathscr{H}^p$ as Hardy spaces of the infinite-dimensional polydisc, we also present analogues of Fatou's theorem.

1. Introduction

A classical theorem of F. Carlson [5] says that if an ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

converges in the right half-plane $\Re s > 0$ and is bounded in every half-plane $\Re s \geq \delta > 0$, then for each $\sigma > 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}. \tag{2}$$

From a modern viewpoint, Carlson's theorem is a special case of the general ergodic theorem, as will be explained below.

A natural question, first raised by H. Hedenmalm [7], is whether the identity (2) remains valid when $\sigma = 0$, provided $f(s)$ is a bounded function in $\Re s > 0$. The problem makes sense because we may replace $f(\sigma+it)$ by the nontangential limit $f(it)$, which in this case exists for almost every t. We note that the general ergodic theorem is of no help for this problem.

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We denote by $\mathscr{H}^\infty$ the class of functions $f(s)$ that are bounded in $\Re s > 0$ with f represented by an ordinary Dirichlet series (1) in some half-plane. We will use the notation

$$\|f\|_\infty = \sup_{\sigma>0} |f(\sigma+it)| \quad \text{and} \quad \|f\|_2^2 = \sum_{n=1}^\infty |a_n|^2.$$

Our main result is that there is no "boundary version" of Carlson's theorem:

Theorem 1. *The following two statements hold:*

(i) *There exists a function f in $\mathscr{H}^\infty$ such that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f(it)|^2 dt$$

does not exist.

(ii) *Given $\varepsilon > 0$, there exists a singular inner function g in $\mathscr{H}^\infty$ such that $\|g\|_2 \le \varepsilon$.*

In other words, the limit on the left-hand side of (2) may fail to exist, and even if it does exist, the identity (2) need not hold. In the next section, we will see that both parts of the theorem rely on a basic construction of W. Rudin [13] concerning radial limits of analytic functions in polydiscs.

To see how to obtain Carlson's theorem as a special case of the general ergodic theorem, we resort to a fundamental observation of Bohr [4]. We put

$$z_1 = 2^{-s},\ z_2 = 3^{-s},\ ...,\ z_j = p_j^{-s},\ ...,$$

where p_j denotes the j-th prime; then, in view of the fundamental theorem of arithmetic, the Dirichlet series (1) can be considered as a power series in infinitely many variables. For a given Dirichlet series f we denote by F the corresponding extension to the infinite polydisc $\mathbb{D}^\infty$; then if F happens to be a function of only n variables, it is immediate from Kronecker's theorem and the maximum principle that

$$\|f\|_\infty = \|F\|_\infty, \tag{3}$$

where the norm on the right-hand side is the $H^\infty(\mathbb{D}^n)$ norm. The result is the same in the infinite-dimensional case, but some care has to be taken when defining the norm in the polydisc. (See [8] for details.) We can now think of any vertical line $t \mapsto \sigma + it$ as an ergodic flow on the infinite-dimensional torus $\mathbb{T}^\infty$:

$$(\tau_1, \tau_2, \dots) \mapsto (p_1^{-it}\tau_1, p_2^{-it}\tau_2, ...) \quad \text{for} \ \ (\tau_1, \tau_2, \dots) \in \mathbb{T}^\infty.$$

If $F(p_1^{-\sigma}z_1, p_2^{-\sigma}z_2, ...)$ is continuous on $\mathbb{T}^\infty$, then the general ergodic theorem yields (2).

A similar problem concerning integral means of nontangential limits can be stated for the closely related space $\mathscr{H}^2$, which consists of those Dirichlet series of the form (1) for which $\|f\|_2 < \infty$. In this case, $f(s)/s$ belongs to the Hardy space H^2 of the half-plane $\sigma > 1/2$, thanks to the following embedding (see [12, p. 140], [8, Theorem 4.1]):

$$\int_\theta^{\theta+1} \left| f\left(\frac{1}{2}+it\right)\right|^2 dt \le C\|f\|_2^2, \tag{4}$$

with C an absolute constant independent of θ. It follows immediately that we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left|f\left(\frac{1}{2}+it\right)\right|^2dt=\sum_{n=1}^{\infty}|a_n|^2n^{-1} \tag{5}$$

for every function f in $\mathscr{H}^2$, since the space of Dirichlet polynomials is dense in $\mathscr{H}^2$ and the identity holds trivially when f is a Dirichlet polynomial.

It is interesting to compare Theorem 1 with what has been proved about pointwise convergence of Dirichlet series in $\mathscr{H}^2$ and in $\mathscr{H}^\infty$. Hedenmalm and Saksman [9] showed that the Dirichlet series of a function in $\mathscr{H}^2$ converges almost everywhere on the vertical line $\sigma = 1/2$. (See [11] for a short proof that gives the result as a corollary of L. Carleson's theorem on almost everywhere convergence of Fourier integrals.) On the other hand, F. Bayart, S. V. Konyagin, and H. Queffélec [2] exhibited an example of a function f in $\mathscr{H}^\infty$, continuous in the closed half-plane $\sigma \geq 0$, whose Dirichlet series diverges everywhere on the imaginary axis $\sigma = 0$. Our result is consistent with these findings: "less" remains of the Dirichlet series on the boundary in the $\mathscr{H}^\infty$ setting than in the $\mathscr{H}^2$ setting.

In Section 3 of this paper, after the proof of Theorem 1, we will discuss the curious situation that occurs when we replace $\mathscr{H}^2$ by the spaces $\mathscr{H}^p$ ($1 \leq p < \infty$), which were introduced and studied by Bayart [1]. The question of whether there is a p-analogue of (4) for every $p \geq 1$ appears as the most important problem regarding $\mathscr{H}^p$. This problem seems to require quite nontrivial analytic number theory. At present, beyond $p = 2$, the result is known only when p is an even integer, which is just a trivial extension of (4). Our discussion of this problem will draw attention to those properties of $\mathscr{H}^p$ that force us to abandon the standard analytical approach involving interpolation techniques.

Finally, in Section 4, we will present certain analogues of the Fatou theorem for Hardy spaces of the infinite-dimensional polydisc. Our version of the Fatou theorem for $\mathscr{H}^p$ gives sense to the statement that the p-analogue of (4) holds if and only if the p-analogue of (5) holds.

2. Proof of Theorem 1

Before embarking on the proof of Theorem 1, we make some simple observations in order to clarify what our problem is really about. We note that another way of phrasing Hedenmalm's question is to ask whether we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|f(it)|^2dt=\lim_{\sigma\to0^+}\lim_{T\to\infty}\frac{1}{T}\int_0^T|f(\sigma+it)|^2dt$$

for every f in $\mathscr{H}^\infty$. We observe that for a finite interval, say for $t_1 < t_2$, we have indeed

$$\int_{t_1}^{t_2}|f(it)|^2dt=\lim_{\sigma\to0^+}\int_{t_1}^{t_2}|f(\sigma+it)|^2dt,$$

as follows by Lebesgue's dominated convergence theorem. Similarly, by applying Cauchy's integral theorem and again Lebesgue's dominated convergence theorem, we get

$$a_n = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(it)n^{it}dt,$$

for every positive integer n. Let us also note that the upper estimate

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T |f(it)|^2 dt \ge \lim_{\sigma\to 0^+}\lim_{T\to\infty}\frac{1}{T}\int_0^T |f(\sigma+it)|^2 dt = \|f\|_2^2$$

may be obtained from the Poisson integral representation of $[f(\sigma+it)]^2$, i.e.,

$$[f(\sigma+it)]^2 = \frac{1}{\pi}\int_{-\infty}^{\infty}[f(i\tau)]^2\frac{\sigma}{(t-\tau)^2+\sigma^2}dt.$$

We conclude from these observations that the counterexamples of Theorem 1 should be functions whose nontangential limits have increasing oscillations when the argument t tends to ∞.

We begin by recalling some terminology and briefly reviewing Rudin's method for constructing real parts of analytic functions in the polydisc $\mathbb{D}^n$ with given boundary values almost everywhere on the distinguished boundary $\mathbb{T}^n$. Rudin treats $\mathbb{D}^n$ with arbirary $n \ge 1$, but we shall need only the case $n = 2$. We refer to [13, pp. 34–36] for full details of the construction.

We employ the complex notation for points on the distinguished boundary $\mathbb{T}^2$ of the bidisc $\mathbb{D}^2$. The normalized Lebesgue measure on $\mathbb{T}^2$ is denoted by m_2. The distance between $\tau = (\tau_1, \tau_2)$ and $\tau' = (\tau_1', \tau_2')$ is

$$d(\tau, \tau') := \max(|\tau_1 - \tau_1'|, |\tau_2 - \tau_2'|),$$

and $B(\tau, r)$ stands for the ball with center τ and radius r. We set

$$P_r(\tau) := \frac{(1-r^2)^2}{|1-r\tau_1|^2|1-r\tau_2|^2}, \quad 0 < r < 1,$$

where $\tau = (\tau_1, \tau_2)$ is a point in $\mathbb{T}^2$. In particular, the Poisson integral of a measure μ on $\mathbb{T}^2$ can then be expressed in the form

$$P\mu(r\tau) = \int_{\mathbb{T}^2} P_r(\tau\overline{w})\mu(dw),$$

where $\tau\overline{w} := (\tau_1\overline{w_1}, \tau_2\overline{w_2})$. For every finite Borel measure μ and every $\tau \in \mathbb{T}^2$, the Poisson maximal operator is defined by setting $P_*|\mu|(\tau) := \sup_{r\in(0,1)} P_r|\mu|(\tau)$. The following estimate is immediate.

Lemma 1. *We have* $P_r(\tau) \le 16(d(\tau,(1,1)))^{-2}$ *for* $r \in (0,1)$. *In particular, if* $s = d(\tau, \operatorname{supp}(\mu)) > 0$, *then* $P_*\mu(\tau) \le 16s^{-2}\|\mu\|$.

Let $g : \mathbb{T}^2 \to \mathbb{R}$ be a strictly positive, integrable, and lower semicontinuous function. Following Rudin, we may express it as

$$g = \sum_{j=1}^{\infty} p_j,$$

where the p_j are non-negative trigonometric polynomials on $\mathbb{T}^2$. For each $j \geq 1$, Rudin shows that one may choose a positive singular measure μ_j with $\mu_j(\mathbb{T}^2) = \int_{\mathbb{T}^2} p_j \, dm_2$ and so that $P(p_j - \mu_j)$ is the real part of an analytic function on $\mathbb{D}^2$. More specifically, μ_j is chosen to be of the form $p_j \lambda_{k_j}$, where $k_j \geq \deg(p_j)$ and for any positive integer k the measure λ_k has the Fourier series expansion

$$\lambda_k = \sum_{j=-\infty}^{\infty} \exp(ikj(\theta_1 + \theta_2)) \tag{6}$$

on $\mathbb{T}^2$, where (θ_1, θ_2) corresponds to the point $(e^{i\theta_1}, e^{i\theta_2})$ on $\mathbb{T}^2$. This measure is positive, has mass one, and with respect to the standard Euclidean identification $\mathbb{T}^2 = [0, 2\pi)^2$ of the 2-torus, it is just the normalized 1-measure supported on $2k - 1$ line segments of $\mathbb{T}^2 = [0, 2\pi)^2$ parallel to the direction $(1, -1)$. On the torus, its support consists of k equally spaced closed "rings".

For s in the right half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z \geq 0\}$, we set $\phi(s) := (2^{-s}, 3^{-s})$. The induced boundary map takes the form $\phi(it) = (\exp(-i \log(2)t), \exp(-i \log(3)t))$. We denote the image of the boundary by L. Thought of as a subset of $[0, 2\pi)^2$, L consists of a dense set of segments that have common direction vector $v_0 := (\log(2), \log(3))$.

Lemma 2. *Let a summable sequence of nonnegative numbers a_k $(k = 1, 2, \ldots)$ be given. If the measure μ satisfies*

$$0 \leq \mu \leq \sum_{k=1}^{\infty} a_k \lambda_k,$$

then $\lim_{r \to 1^-} P\mu(\tau) = 0$ for almost every $\tau \in L$.

Proof. It is enough to prove the claim for $\mu = \sum_{k=1}^{\infty} a_k \lambda_k$. By [13, Theorem 2.3.1], we know that $\lim_{r \to 1^-} P\mu(\tau) = 0$ for m_2-a.e. $\tau \in \mathbb{T}^2$. Pick any segment $J \subset L$ of length $1/2$, say. By Fubini's theorem we see that for almost every $s \in [0, 1/2]$ the claim holds for almost every $\tau \in J + s(1, -1)$. However, since the measure μ is invariant with respect to the translation $\tau \to \tau + s(1, -1)$, we see that the statement is true for every $s \in [0, 1/2]$. In particular, we have $\lim_{r \to 1^-} P\mu(\tau) = 0$ for almost every $\tau \in J$. By expressing L as a countable union of such segments, we obtain the conclusion of the lemma. □

Part (i) of Theorem 1 will be deduced from the following lemma.

Lemma 3. *Given $\varepsilon > 0$, there is an open set $U \subset \mathbb{T}^2$ with $m_2(U) < \varepsilon/2$ and a probability measure μ on $\mathbb{T}^2$ such that the function*

$$h = P(\chi_U + (1/2)\chi_{U^c}) - P\mu,$$

is the real part of a function in the unit ball of $H^\infty(\mathbb{D}^2)$. Moreover, $\lim_{r\to 1^-} h(r\tau) = 1$ for almost every $\tau \in L$ with respect to the Hausdorff 1-measure on L.

Proof. We begin by covering L with a thin open strip U that becomes thinner and thinner so that $m_2(U) < \varepsilon/2$. For example, we may take

$$U := \bigcup_{t\in\mathbb{R}} B\big(\phi(it), \frac{\varepsilon}{100(1+|t|)^2}\big).$$

The next step is to run Rudin's construction with respect to the positive and lower semicontinuous function $\chi_U + (1/2)\chi_{U^c}$. Thus we choose strictly positive trigonometric polynomials $p_1, p_2, \ldots$ on $\mathbb{T}^2$ in such a way that $\sum_{j=1}^\infty p_j = \chi_U + (\varepsilon/2)\chi_{U^c}$ at every point of $\mathbb{T}^2$. Moreover, by a compactness argument, we observe that we may perform the selection in such a way that

$$0 < p_j(\tau) \le j^{-2} \quad \text{if} \quad d(\tau, \partial U) \ge j^{-1}. \tag{7}$$

We may also require that $\int_{\mathbb{T}^2} p_j\, dm_2 \le j^{-2}$. We set $\mu_j = p_j \lambda_{k(j)}$ and observe that

$$\|\mu_j\| = \int_{\mathbb{T}^2} p_j\, dm_2 \le j^{-2}. \tag{8}$$

Write

$$\lambda_0 := \sum_{j=1}^\infty j^{-2}\lambda_{k(j)}.$$

Then, according to Lemma 2, we have

$$\lim_{r\to 1^-} P\lambda_0(r\tau) = 0 \quad \text{for } \tau \in L\setminus E, \tag{9}$$

where E has linear measure zero. A fortiori, we have in particular that

$$\lim_{r\to 1^-} P\mu_j(r\tau) = 0 \quad \text{for } \tau \in L\setminus E. \tag{10}$$

We now set $\mu = \sum_{j=1}^\infty \mu_j$. The fact that $h := P(\chi_U + (\varepsilon/2)\chi_{U^c}) - P\mu$ is the real part of an analytic function in the unit ball of $H^\infty(\mathbb{D}^2)$ is immediate from Rudin's theorem [13, Theorem 3.5.2]. Since U is open and the mass of the two-dimensional Poisson kernel concentrates on any neighborhood of the origin as $r \to 1^-$, we see that $\lim_{r\to 1^-} P(\chi_U + (\varepsilon/2)\chi_{U^c})(rw) = 1$ for every $w \in U$. Hence it remains to verify that $\lim_{r\to 1^-} P\mu(r\tau) \to 0$ for almost every $\tau \in L$ with respect to Hausdorff 1–measure on L. In fact, we will show that

$$\lim_{r\to 1^-} P\mu(r\tau) = 0 \quad \text{if} \quad \tau \in L\setminus E, \tag{11}$$

which is clearly sufficient.

Fix an arbitrary $\tau \in L \setminus E$. Write $s = d(\tau, \partial U) > 0$, $B = B(\tau, s/2)$, and set

$$\mu_k^a := \chi_B \mu_k \quad \text{and} \quad \mu_k^b := \mu_k - \mu_k^a.$$

Pick $k_0 \geq (s/2)^{-1}$. We clearly have

$$\sum_{k=k_0}^{\infty} \mu_k^a \leq \lambda_0 \tag{12}$$

so that (9) implies that

$$\lim_{r \to 1^-} P(\sum_{k=k_0}^{\infty} \mu_k^a)(r\tau) = 0. \tag{13}$$

On the other hand, we have $d(\tau, \mathrm{supp}\,(\mu_k^b)) \geq s/2$ and $\|\mu_k^b\| \leq \|\mu_k\| \leq k^{-2}$. Hence Lemma 2 yields

$$P_*(\sum_{k=k_0}^{\infty} \mu_k^b)(r\tau) \leq 64 s^{-2} \sum_{k=k_0}^{\infty} k^{-2} \leq C(\tau) k_0^{-1}. \tag{14}$$

By (10), we have

$$\lim_{r \to 1^-} P(\sum_{k=1}^{k_0-1} \mu_k)(r\tau) = 0. \tag{15}$$

As k_0 can be chosen arbitrarily large, we obtain the desired conclusion by combining this fact with (13) and (14). □

Proof of Theorem 1. We begin by proving part (ii) of the theorem. Let h be the function given in Lemma 3, and assume that it is the real part of the analytic function H on $\mathbb{D}^2$. When k is large enough, the function $R := \exp(k(H-1))$ satisfies $\|R\|_{H^\infty(\mathbb{D}^2)} = 1$ and $\|R\|_{H^2(\mathbb{D}^2)} \leq \varepsilon$. Moreover, its modulus has radial boundary values 1 at almost every point of the set L with respect to linear measure. It is almost immediate from this that the function

$$g(s) := R(\phi(s)) = R(2^{-s}, 3^{-s})$$

is, by construction, a singular inner function in $\mathbb{C}^+$ with $\|g\|_{\mathscr{H}^2} < \varepsilon$. The only matter that requires a little attention, is how we conclude that $|g|$ has unimodular boundary values almost everywhere. The point is that horizontal boundary approach in $\mathbb{C}^+$ does not transfer exactly via ϕ to radial approach, but instead to what we will call quasi-radial approach. This means that $(r_1 w_1, r_2 w_2) \to (w_1, w_2)$, where $r_1 \to 1^-$ and $r_2 \to 1^-$ in such a way that the ratio $(1-r_1)/(1-r_2)$ stays uniformly bounded from above and below. However, apart from a change of non-essential constants, our proof of Lemma 3 remains valid for quasi-radial approach. This is easily verified for Lemma 1, and it remains true for the basic theorem [13, Theorem 2.3.1] on radial limits of singular measures (see [13, Exercise 2.3.2.(d)]). These remarks conclude the proof of part (ii) of Theorem 1.

We now turn to the proof of part (i) of Theorem 1. The basic construction is similar to the one in the proof of part (ii), so we only indicate the required changes. To simplify the notation, we identify the imaginary axis with L. Lebesgue measure on the imaginary axis is denoted by ν. This time we cover only part of the image of the imaginary axis L by an open set U. To this end, given $\varepsilon > 0$, we first construct by induction a sequence of open subsets $U_1, U_2, \ldots \subset T^2$ with the following properties for each $n \geq 1$:

(1) There is $t_n \geq n$ so that $\nu(U_n \cap [0, it_n]) > (1 - \varepsilon/2)t_n$.
(2) The closures $\overline{U_1}, \overline{U_2}, \ldots, \overline{U_n}$ are disjoint.
(3) The set U_n is a finite union of open dyadic squares and

$$\sum_{j=1}^{n} m_2(U_j) < \varepsilon/2.$$

In the first step, we set $t_1 = 1$ and, apart from a finite number of points, we cover $[0, it_1]$ by a finite union of dyadic open cubes U_1 with $m_2(U_1) = m_2(\overline{U_1}) < \varepsilon/2$. Assume then that sets $U_1, \ldots U_n$ with the right properties have been found. Since we are dealing with finite unions of open squares, it holds that $m_2(\overline{\bigcup_{j=1}^n U_j}) \leq \sum_{j=1}^n m_2(U_j) < \varepsilon/2$ and hence we may apply the continuous version of Weyl's equi-distribution theorem for Kronecker flows in order to select $t_{n+1} \geq n + 1$ with

$$\nu\big((\overline{\cup_{j=1}^n U_j}) \cap [0, it_{n+1}]\big) < \varepsilon/2.$$

Then U_{n+1} is obtained by covering a sufficiently large portion of the set $[0, it_{n+1}] \setminus \overline{\bigcup_{j=1}^n U_j}$ by a union of open dyadic squares that has a positive distance to $\overline{\bigcup_{j=1}^n U_j}$ and satisfies $m_2(U_{n+1}) < \varepsilon - \sum_{j=1}^n m_2(U_j)$. This completes the induction.

Set $U = \bigcup_{k=1}^\infty U_{2k-1}$ and $V = \bigcup_{k=1}^\infty U_{2k}$. We run the Rudin construction exactly as in the proof of part (ii) corresponding to the lower semicontinuous boundary function $\chi_U + (\varepsilon/2)\chi_{U^c}$. Hence, we obtain a polyharmonic function h on $\mathbb{D}^2$ with (quasi-)radial boundary values 1 at almost every point of $U \cap L$ (respectively $\varepsilon/2$ at almost every point of V), and such that h is the real part of the analytic function H on $\mathbb{D}^2$. By property (1) of the sets $U_1, U_2, \ldots$, it is then evident that with sufficiently large k the function $f(s) := \exp(k(H(2^{-s}, 3^{-s}) - 1))$ satisfies $f \in \mathscr{H}^\infty$ and

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^2 dt \leq \varepsilon$$

as well as

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^2 dt = 1.$$

□

Question 1. Which are the functions ψ such that $\psi = |f(it)|$ almost everywhere for some function f in $\mathscr{H}^\infty$?

This is no doubt a difficult question, because we do not even have a description of the radial limits of $|F|$ for F in $H^\infty(\mathbb{D}^2)$. A loose restatement of the question is as follows: How much of the almost periodicity of $|f(\sigma + it)|$ on the vertical lines in $\mathbb{C}_+$ is carried to the boundary limit function $|f(it)|$?

3. The embedding problem for $\mathscr{H}^p$

We begin by recalling the definition of the spaces $\mathscr{H}^p$. We will use standard multi-index notation, which means that if $\beta = (\beta_1, \ldots, \beta_k, 0, 0, \ldots)$, then $z^\beta = z_1^{\beta_1} \cdot \ldots \cdot z_k^{\beta_k}$. If $p > 0$ and $F(z) = \sum b_\beta z^\beta$ is a finite polynomial in the variables $z_1, z_2, \ldots$, its H^p norm is

$$\|F\|_{H^p(\mathbb{D}^\infty)} := \left(\int_{\mathbb{T}^\infty} |F(\tau)|^p \, dm_\infty(\tau) \right)^{1/p},$$

where m_∞ is the Haar measure on the distinguished boundary $\mathbb{T}^\infty$. The space $H^p(\mathbb{D}^\infty)$ is obtained by taking the closure of the set of polynomials with respect to this norm (quasi-norm in case $p \in (0,1)$). For $p \geq 1$, $H^p(\mathbb{D}^\infty)$ consists of all analytic elements in $L^p(\mathbb{T}^\infty)$, i.e., all functions in L^p for which all Fourier coefficients with at least one negative index vanish. Obviously, $\|F\|^2_{H^2(\mathbb{D}^\infty)} = \sum_\beta |b_\beta|^2$.

Now let $f(s) = \sum_{n=1}^m a_n n^{-s}$ be a finite Dirichlet polynomial. By the Bohr correspondence, f lifts to the polynomial $F(z) = \sum b_\beta z^\beta$ on $\mathbb{D}^\infty$, where $b_\beta = a_n$, given that n has the prime factorization $n = p_1^{\beta_1} p_2^{\beta_2} \cdot \ldots \cdot p_k^{\beta_k}$; here $p_1 = 2, p_2 = 3, \ldots$ are the primes listed in increasing order. We define $\|f\|_{\mathscr{H}^p} := \|F\|_{H^p(\mathbb{D}^\infty)}$. The space $\mathscr{H}^p$ is obtained by taking the closure of the Dirichlet polynomials with respect to this norm (see [1]). Consequently, the spaces $H^p(\mathbb{D}^\infty)$ and $\mathscr{H}^p$ are isometrically isomorphic via the Bohr correspondence.

When $p = 2$, the definition given above coincides with the original one for the Hilbert space $\mathscr{H}^2$, and the Bohr correspondence $f \leftrightarrow F$ carries over to the case $p = \infty$. In fact, also Carlson's theorem (with $\sigma = 0$) can be used to define the $\mathscr{H}^p$ norm: for every Dirichlet polynomial f and $p > 0$ we have

$$\|f\|^p_{\mathscr{H}^p} = \lim_{T \to \infty} \frac{1}{T} \int_0^T |f(it)|^p \, dt. \tag{16}$$

The equality follows by an application of the ergodic theorem, since f is continuous. However, let us also sketch a more elementary proof. The identity (16) holds trivially for $p = 2, 4, 6, \ldots$. We then obtain the result for general p by applying Weierstrass's approximation theorem to the continuous function $x \mapsto x^{p/2}$ for x in the closed interval $[0, \|f\|_\infty]$.

Estimates obtained by B. Cole and T. Gamelin [6, Theorem 6.1] verify that point evaluations $f \mapsto f(s)$ are bounded in $\mathscr{H}^p$ if and only if s is in the half-plane $\mathbb{C}^+_{1/2} = \{s = s + it : \sigma > 1/2\}$. The norm of the functional of point evaluation is of order $(\sigma - 1/2)^{-1/p}$, just as it is for functions in $H^p(\mathbb{C}^+_{1/2})$. Hence elements of $\mathscr{H}^p$ are analytic in $\mathbb{C}^+_{1/2}$ with uniformly converging Dirichlet series in any half-plane $\sigma \geq 1/2 + \varepsilon$ We refer to [1] for additional information about the spaces $\mathscr{H}^p$.

A question of primary importance concerning $\mathscr{H}^p$, first considered by Bayart [1], is whether the analogue of the embedding (4) holds for $p \neq 2$. It suffices to formulate the question only for polynomials, since existence of non-tangential boundary values almost everywhere would be an immediate consequence of a positive answer, and the inequality could then be stated for all elements in $\mathscr{H}^p$.

Question 2 (The embedding problem)**.** Fix an exponent $p > 0$, that is not an even integer. Does there exist a constant $C_p < \infty$ such that

$$\int_0^1 \left| f\left(\frac{1}{2}+it\right)\right|^p dt \le C_p \|f\|^p_{\mathscr{H}^p} \tag{17}$$

for every Dirichlet polynomial f?

We have excluded the case $p = 2k$ with $k \in \mathbb{N}$ because then the answer is trivially positive: Just apply the case $p = 2$ to the function f^k in $\mathscr{H}^2$. This observation[1] provides evidence in favor of a positive answer. The growth estimates for functions in $\mathscr{H}^p$ mentioned above point in the same direction. An answer to Question 2 seems to be a prerequisite for a further development of the theory of the spaces $\mathscr{H}^p$.

Let us now point at some properties of the spaces $\mathscr{H}^p$—no doubt known to specialists—indicating that Question 2 is deep and most probably very difficult. First of all, it is easily seen that for $p > 1$ the isometric subspace $\mathscr{H}^p(\mathbb{D}^\infty) \subset L^p(\mathbb{T}^\infty)$ is not complemented in $L^p(\mathbb{T}^\infty)$ unless $p = 2$. Namely, if there were a bounded projection, one could easily apply the Rudin averaging technique to show that the L^2-orthogonal projection is bounded in L^p. In other words, the infinite product of one-dimensional Riesz projections would be bounded in L^p. A fortiori, the only possibility is that the norm of the dimensional projection is one (simply consider products of functions each depending on one variable only), i.e. $p = 2$. This fact makes it difficult to apply interpolation between the already known values $p = 2, 4, 6, \ldots$. Moreover, similar arguments show that, in the natural duality, we have $\mathscr{H}^{p'} \subset (\mathscr{H}^p)'$, but the inclusion is strict whenever $p \neq 2$. In fact, for $p \in (1, 2)$ one has

$$\text{if} \quad p \in (1,2), \quad \text{then} \quad (\mathscr{H}^p)' \subset \mathscr{H}^q \quad \text{if and only if} \quad q \le 2. \tag{18}$$

There are some famous unresolved conjectures in analytic number theory, due to H. Montgomery, that deal with norm inequalities for Dirichlet polynomials (see [12, pp. 129, 146] or [10, pp. 232–235]). One of Montgomery's conjectures states that for every $\varepsilon > 0$ and $p \in (2, 4)$ there exists $C = C(\varepsilon)$ such that for all finite Dirichlet polynomials $f = \sum_{n=1}^N a_n n^{-s}$ with $|a_n| \le 1$ one has

$$\int_0^T |f(it)|^p \, dt \le C N^{p/2+\varepsilon}(T + N^{p/2}) \quad \text{for } T > 1. \tag{19}$$

If true, this inequality would imply the density hypothesis for the zeros of the Riemann zeta function. It is quite interesting to note that (19) is also known to be true for $p = 2, 4$ (or any even integer). The similarities suggest for a possible connection between Montgomery's conjectures and our embedding problem. Although it appears to be difficult to give a precise link, both problems can be understood as dealing with the "degree of flatness" of Dirichlet polynomials.

As a first step towards a solution of the embedding problem, one could ask for a weaker partial result:

[1] In [1], Bayart proclaimed a positive answer to Question 2 for $p > 2$. Unfortunately, his proof, based on this observation and an interpolation argument, contains a mistake.

Question 3. Assume that $2 < q < p < 4$. Is it true that

$$\left(\int_0^1 \left|f\left(\frac{1}{2}+it\right)\right|^q dt\right)^{1/q} \leq C_q \|f\|_{\mathscr{H}^p}?$$

Is this true for at least one such pair of exponents?

Let us denote by A the adjoint operator of the natural embedding operator. Thus for functions g on $[0,1]$ one has

$$Ag(s) = \sum_{n=1}^{\infty} \big(n^{-1/2}\widehat{g}(\log n)\big)n^{-s},$$

where $\widehat{g}$ is the Fourier transform of g. Observe that, due to (18), the existence of the embedding (17) does not imply that $A : L^{p'}(0,1) \to \mathscr{H}^{p'}$. However, a positive answer to the following question would, by interpolation, imply that for each $p > 2$ there is $q > 2$ such that the embedding operator acts boundedly from $\mathscr{H}^p$ to $L^q(0,1)$.

Question 4. Is there an exponent $r \in (1,2)$ such that $A : L^r \to H^1(\mathbb{D}^\infty)$ is bounded?

We mention without proof that we have been able to verify that such an r cannot be smaller than $4/3$.

4. Fatou theorems for $\mathscr{H}^p$

We will now in some sense return to what appeared as a difficulty in the proof of Theorem 1, namely that the imaginary axis has measure zero when viewed as a subset of $\mathbb{T}^2$. Thus, a priori, it makes no sense to speak about the restriction to the imaginary axis of a function in $L^p(\mathbb{T}^\infty)$. We will now show that, for functions in $H^p(\mathbb{D}^\infty)$, we can find a meaningful connection to the boundary limits of the corresponding Dirichlet series.

We consider a special type of boundary approach by setting for each $\tau = (\tau_1, \tau_2, \dots) \in \mathbb{T}^\infty$ and $\theta \geq 0$

$$b_\theta(\tau) := (p_1^{-\theta}\tau_1, p_2^{-\theta}\tau_1, \dots).$$

We also recall that the Kronecker flow on $\overline{\mathbb{D}}^\infty$ is defined by setting

$$T_t((z_1, z_2, \dots)) := (p_1^{-it}z_1, p_2^{-it}z_2, \dots).$$

For an arbitrary $z \in \overline{\mathbb{D}}^\infty$, we denote by $T(z)$ the image of z under this flow, i.e., $T(z)$ is the one-dimensional complex variety $T(z) := \{T_t(z) : t \in \mathbb{R}\}$. We equip $T(z)$ with the natural linear measure, which is just Lebesgue measure on the real t-line. Moreover, for $\sigma > 0$, we set $\mathbb{T}^\infty_\sigma := b_\sigma(\mathbb{T}^\infty)$. The natural Haar measure $m_{\infty,\sigma}$ on $\mathbb{T}^\infty_\sigma$ is obtained as the pushforward of m_∞ under the map b_θ. The set $\mathbb{T}^\infty_{1/2}$ is of special interest, since in a sense it serves as a natural boundary for the set $\mathbb{D}^\infty \cap \ell^2$, where point evaluations are bounded for the space $H^p(\mathbb{D}^\infty)$ with $p \in (0,\infty)$.

Our version of Fatou's theorem for H^∞ reads as follows.

Theorem 2. *Let F be a function in $H^\infty(\mathbb{D}^\infty)$. Then we may pick a representative $\widetilde{F}$ for the boundary function of F on the distinguished boundary $\mathbb{T}^\infty$ such that $\widetilde{F}(\tau) = \lim_{\theta\to 0^+} F(b_\theta(\tau))$ for almost every $\tau \in \mathbb{T}^\infty$. In fact, for every $\tau \in \mathbb{T}^\infty$, we have $\widetilde{F}(\tau') = \lim_{\theta\to 0^+} F(b_\theta(\tau'))$ for almost every $\tau' \in T(\tau)$.*

Proof. Recall that by [6] the values of $H^\infty(\mathbb{D}^\infty)$-functions are well-defined in $\mathbb{D}^\infty$ at points z with coordinates tending to zero, i.e. for $z \in c_0$. We simply define the desired representative $\widetilde{F}$ for the the boundary values by setting $\widetilde{F}(\tau) = \lim_{\theta\to 0^+} F(\theta\circ\tau)$ whenever this limit exists and otherwise $\widetilde{F}(\tau) = 0$. The Borel measurability of $\widetilde{F}$ is clear. The second statement follows immediately by considering for each $\tau \in \mathbb{T}^\infty$ the analytic function $f_\tau : f_\tau(\theta + it) = F(T_t b_\theta(\tau))$ and observing that for each $\tau \in \mathbb{T}^\infty$ we have $f_\tau \in \mathscr{H}^\infty$. Now the classical Fatou theorem applies to f_τ. The fact that the set $\{\tau \in \mathbb{T}^\infty : \lim_{\theta\to 0^+} F(b_\theta(\tau)) \text{ exists}\}$ has full measure is an immediate consequence of the ergodicity of the Kronecker flow $\{T_t\}_{t\geq 0}$ and the second statement. Finally, we observe that it is easy to check the formula

$$\widehat{F}(\beta) = p_1^{\beta_1\sigma}\cdot\ldots\cdot p_k^{\beta_k\sigma}\int_{\mathbb{T}^\infty} F(b_\theta(\tau))\overline{\tau}^\beta m_\infty(d\tau)$$

for the Fourier coefficients of an $H^\infty(\mathbb{D}^\infty)$-function. Lebesgue's dominated convergence theorem now yields $\widehat{\widetilde{F}} = \widehat{F}$, whence $\widetilde{F} = F$ almost surely, and this finishes the proof of the first statement. □

To arrive at a similar result for $\mathscr{H}^p$, we need to make sense of the restriction $F \mapsto F|_{\mathbb{T}^\infty_{1/2}}$ as a map from $H^p(\mathbb{D}^\infty)$ to $L^p(\mathbb{T}^\infty_{1/2}, m_{\infty,1/2})$. When F is a polynomial, we must have

$$F|_{\mathbb{T}^\infty_{1/2}}(\tau) = F(b_{1/2}(\tau)).$$

Since this formula can be written as a Poisson integral and the polynomials are dense in $H^p(\mathbb{D}^\infty)$, this leads to a definition of $F|_{\mathbb{T}^\infty_{1/2}}$ for general F. Indeed, by using elementary properties of Poisson kernels for finite polydiscs, we get that $F \mapsto F|_{\mathbb{T}^\infty_{1/2}}$ is a contraction from $H^p(\mathbb{D}^\infty)$ to $L^p(\mathbb{T}^\infty_{1/2}, m_{\infty,1/2})$.

Theorem 3. *Let F be a function $H^p(\mathbb{D}^\infty)$ for $p \geq 2$. Then we may pick a representative $\widetilde{F}_{1/2}$ for the restriction $F|_{\mathbb{T}^\infty_{1/2}}$ on the distinguished boundary $\mathbb{T}^\infty$ such that $\widetilde{F}(\tau) = \lim_{\theta\to 1/2^+} F(b_\theta(\tau))$ for almost every $\tau \in \mathbb{T}^\infty$. In fact, for every $\tau \in \mathbb{T}^\infty$, we have $\widetilde{F}_{1/2}(\tau') = \lim_{\theta\to 1/2^+} F(b_\theta(\tau'))$ for almost every $\tau' \in T(\tau)$.*

Proof. The existence of the boundary values is obtained just as in the proof of Theorem 2. This time one applies the known embedding for $p = 2$ to define $\widetilde{F}_{1/2}$. □

We may now observe that if F is in $H^p(\mathbb{D}^\infty)$ ($p \geq 2$) and the embedding (17) holds, then we have for every $\tau \in \mathbb{T}^\infty_{1/2}$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\widetilde{F}(T_t\tau)|^p\,dt = \|\widetilde{F}_{1/2}\|^p_{L^p(\mathbb{T}^\infty)}. \tag{20}$$

Indeed, (20) holds for polynomials. Hence, employing (17) and the fact that polynomials are dense in $H^p(\mathbb{D}^\infty)$, we obtain (20).

On the other hand, if (20) is true, then by the closed graph theorem (fix $T = 1$), the embedding (17) follows. We have therefore made sense of the statement that the "p-Carlson identity" (20) is equivalent to the embedding (17).

It is rather puzzling that (20), which may be understood as a strengthened variant of the Birkhoff–Khinchin ergodic theorem for functions in $H^p(\mathbb{D}^\infty)$, is known to hold only when $p = 2, 4, 6, \ldots$.

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