

AN EXTREMAL PROBLEM RELATED TO NEGATIVE REFRACTION

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ABSTRACT. We solve an extremal problem that arises in the study of the refractive indices of passive metamaterials. The problem concerns Hermitian functions in H^2 of the upper half-plane, i.e., H^2 functions satisfying $f(-x) = \overline{f(x)}$. An additional requirement is that the imaginary part of f be nonnegative for nonnegative arguments. We parameterize the class of such functions whose real part is constant on an interval, and solve the problem of minimizing the imaginary part on the interval on which the function's real part takes a given constant value.

1. INTRODUCTION

We consider in this note an extremal problem that arose in investigations of certain electromagnetic parameters of artificial materials (metamaterials). The physical interpretation of our solution in terms of bounds for refractive indices and theoretical limitations for the design of metamaterials is described elsewhere [6]; the purpose of the present work is to give an account of the underlying mathematical problem, which seems to be of some independent interest.

We will be dealing with Hardy spaces H^p of the upper half-plane $\{z = x + iy : y > 0\}$. We are primarily interested in H^2 , but it is convenient to have at our disposal the whole range of spaces corresponding to $0 < p \leq \infty$. For $0 < p < \infty$, H^p consists of those analytic functions f in the upper half-plane for which

$$\|f\|_p^p = \sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty;$$

H^∞ is the space of bounded analytic functions. A function f in H^p has a nontangential boundary limit at almost every point of the real axis, and the corresponding limit function, also denoted f , is in $L^p = L^p(\mathbb{R})$. Indeed, the L^p norm of the boundary limit function coincides with the H^p norm introduced above. Thus we may view H^p as a subspace of L^p . We refer to [1] for these and other basic facts about H^p , as well as the twin theory of H^p of the unit disk. (We will make a reference to the disk setting at one point.)

The Hilbert space H^2 is the image of $L^2(\mathbb{R}^+)$ under the Fourier transform. In practice, it is quite common that one considers functions in H^2 that are Fourier transforms of real-valued functions in L^2 . This leads to the following symmetry condition: $f(-x) = \overline{f(x)}$. Functions f satisfying this condition will be referred to as Hermitian functions. Thus Hermitian functions have even real parts and odd imaginary parts.

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The Hilbert transform of a function u in L^p ($1 \leq p < \infty$) is defined as

$$\tilde{u}(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{x-t} dt.$$

It acts boundedly on L^p for $1 < p < \infty$ and isometrically on L^2 . If u is a real-valued function in L^p for $1 < p < \infty$, then $u + i\tilde{u}$ is in H^p , and so the role of the Hilbert transform is to link the real and imaginary parts of functions in H^p . We will only work with Hermitian functions, and we will be interested in computing real parts from imaginary parts. For this reason, it will be convenient for us to consider the following Hilbert operator:

$$\mathcal{H}v(x) = \text{p.v.} \frac{1}{\pi} \int_0^{\infty} v(t) \left(\frac{1}{t-x} + \frac{1}{t+x} \right) dt,$$

acting on functions in $L^p(\mathbb{R}^+)$. Provided $1 < p < \infty$, the function $\mathcal{H}v + iv$ will then be in H^p , with the presumption that v is an odd function.

A natural type of problem in H^2 is that of approximating a given Hermitian function supported on two symmetric intervals. This means that one specifies a desired behavior in a certain frequency band and attempts to find, in some appropriate sense, an optimal approximation in H^2 . Without further restrictions, such a problem makes little sense, because it is easy to see that approximations can be made with arbitrary precision in L^2 norm. It may be reasonable to prescribe bounds for the norm of the approximating function; see for instance the work of M. G. Kreĭn and P. Ya. Nudel'man [2], [3] for interesting results along such lines. In the present note, we take a different route. We shall require the imaginary part of the function to be nonnegative for nonnegative arguments.¹ We are interested in a specific problem of this general kind; it turns out to have an explicit and remarkably simple solution.

2. RESULTS

We turn to the statement of the problem. For a finite interval $I = [a, b]$ ($0 < a < b$) and every real number α we define the family of functions

$$K_\alpha(I) = \{v \in L^2(\mathbb{R}^+) : v(t) \geq 0 \text{ for } t > 0, \mathcal{H}v(t) = \alpha \text{ for } t \in I\}.$$

(Here and elsewhere we suppress the obvious ‘‘almost everywhere’’ provisions needed when considering pointwise restrictions.) We think of functions in $K_\alpha(I)$, or more generally functions in $L^2(\mathbb{R}^+)$, as the imaginary parts of Hermitian functions, and we view them therefore as odd functions on \mathbb{R} .

Our purpose is to give a parametrization of $K_\alpha(I)$ and to solve the extremal problem

$$\lambda = \inf_{v \in K_{-1}(I)} \|\chi_I v\|_\infty,$$

¹This reflects the passivity condition for our electromagnetic medium.

where χ_I denotes the characteristic function of I . We will show that the extremal problem has the following explicit solution:

$$\lambda = \frac{b^2 - a^2}{2ab}.$$

We note that the quantity on the right is invariant under dilations $sI = [sa, sb]$. This is as it should be since $v(t)$ is in $K_{-1}(I)$ if and only if $v(t/s)$ is in $K_{-1}(sI)$ for $s > 0$. It will become clear that the extremal value λ is not attained by any function in $K_{-1}(I)$. We will also see that the problem is insensitive to which L^p norm we choose to minimize.

Clearly, the corresponding extremal problem for $K_\alpha(I)$ has solution $|\alpha|\lambda$ when $\alpha < 0$. However, if $\alpha \geq 0$, the extremal problem is uninteresting and has solution 0. Thus the sign in the relation $\mathcal{H}v(t) = \alpha$ for $t \in I$ matters in a decisive way.²

The following lemma is basic for our parametrization of $K_\alpha(I)$.

Lemma 1. *A real-valued function in $L^2(\mathbb{R}^+ \setminus I)$ is the restriction to $\mathbb{R}^+ \setminus I$ of at most one real-valued function v in $L^2(\mathbb{R}^+)$ such that $\mathcal{H}v(t)$ is constant on I .*

Proof. We assume two real-valued functions v_1 and v_2 in $L^2(\mathbb{R}^+)$ coincide off I and are such that both $u_1(t) = \mathcal{H}v_1(t)$ and $u_2(t) = \mathcal{H}v_2(t)$ are constant on I . If we set $c = u_2(t) - u_1(t)$ for t in I , then the function $h = [(u_1 - u_2 + i(v_1 - v_2) + c)^2]$ will be real for real arguments. A change of variables argument shows that then $h(i(1+z)/(1-z))$ belongs to H^1 of the unit disk. But a function in H^1 can be real only if it is a constant. Clearly, h can be a constant only if $u_1 - u_2 + i(v_1 - v_2) = 0$. ■

We note that the assumption that v is in L^2 is essential for this lemma; the proof would break down if we assumed, say, that v belonged to some L^p for $p < 2$.

The following function will play an essential role in what follows:

$$\sigma(z) = \frac{1}{\sqrt{z^2 - b^2}\sqrt{z^2 - a^2}}.$$

This function, which is taken to be positive for real arguments $x > b$, is analytic in the slit plane $\mathbb{C} \setminus ([-b, -a] \cup [a, b])$. For real arguments $a < |x| < b$ we define $\sigma(x)$ by extending it continuously from the upper half-plane. Thus $\sigma(x)$ takes values on the negative imaginary half-axis when x is in (a, b) and on the positive imaginary half-axis when $-x$ is in (a, b) , and otherwise it is real for real arguments. The key point, besides the symmetry $\sigma(-x) = \overline{\sigma(x)}$, is that σ provides a means for switching between real and imaginary when switching off and on I .

The following is our main result.

²The case $\alpha < -1$ corresponds to the interesting physical phenomenon of negative refraction, which has received considerable attention in recent years. Artificial, negatively refracting materials, called metamaterials, have been realized in the microwave range [7], building on previous theoretical ideas [8, 5, 4]. As explained in [6], the solution to our extremal problem provides a bound for the loss of negatively refracting materials when the real part of the refractive index is constant in a finite bandwidth.

Theorem. *A nonnegative function v in $L^2(\mathbb{R}^+)$ is in $K_\alpha(I)$ if and only if the following three conditions hold:*

$$(1) \quad \int_{\mathbb{R}^+ \setminus I} v(t)|\sigma(t)|dt < \infty$$

$$(2) \quad \frac{2}{\pi} \int_{\mathbb{R}^+ \setminus I} tv(t)\sigma(t)dt = \alpha$$

$$(3) \quad v(x) = \mathcal{H}((1 - \chi_I)\sigma v)(x)/|\sigma(x)|, \quad x \in I.$$

Some remarks are in order before we give the proof of the theorem.

The integrability condition (1) is merely a slight growth condition at the endpoints of I ; we may write it more succinctly as

$$\int_0^a [v(a-t) + v(b+t)] \frac{dt}{\sqrt{t}} < \infty.$$

This condition ensures that the integral in (2) and the Hilbert transform appearing in (3) are both well-defined.

At first sight, the theorem may not seem to give an explicit parametrization of $K_\alpha(I)$. However, the Hilbert transform appearing in (3) is given by

$$\mathcal{H}((1 - \chi_I)\sigma v)(x) = \frac{1}{\pi} \int_{\mathbb{R}^+ \setminus I} v(t)\sigma(t) \frac{2t}{t^2 - x^2} dt,$$

and we observe that the integrand on the right is nonnegative whenever $v(t)$ is nonnegative. Hence $v(x) \geq 0$ for x off I implies $v(x) \geq 0$ for x in I . This small miracle implies that $K_\alpha(I)$ is parameterized by those nonnegative functions v in $L^2(\mathbb{R}^+ \setminus I)$ for which (1) and (2) hold and such that

$$\int_I |\mathcal{H}((1 - \chi_I)\sigma v)(x)|^2 |\sigma(x)|^{-2} dx < \infty.$$

By rephrasing this condition in more explicit terms (see Lemma 3 below), we arrive at the following corollary.

Corollary. *A nonnegative function ν in $L^2(\mathbb{R}^+ \setminus I)$ has an extension to a function in some class $K_\alpha(I)$ if and only if the following condition holds:*

$$\int_0^a \int_0^a [\nu(a-t)\nu(a-\tau) + \nu(b+t)\nu(b+\tau)] \frac{|\log(t+\tau)|}{\sqrt{t\tau}} dt d\tau < \infty.$$

The difference between (1) and the condition above is the logarithmic factor, which means that the condition of the corollary is only a very slight strengthening of (1). It is clear that for instance boundedness of v near the endpoints of I is more than enough.

We note that the integrand in (2) is negative to the left of I and positive to the right of I . This means that if α is negative, then

$$|\alpha| \leq \frac{2}{\pi} \int_0^a tv(t)|\sigma(t)|dt,$$

with equality holding if v vanishes to the right of I . It follows that

$$\mathcal{H}((1 - \chi_I)\sigma v)(x) \geq \frac{1}{\pi} \int_0^a v(t)\sigma(t) \frac{2t}{t^2 - x^2} dt \geq \frac{|\alpha|}{x^2};$$

we may come as close as we wish to this lower bound by choosing any suitable v supported on a small set sufficiently close to 0. Hence our extremal problem (corresponding to $\alpha = -1$) has solution

$$\lambda = \max_{x \in I} \frac{\sqrt{(b^2 - x^2)(x^2 - a^2)}}{x^2} = \frac{b^2 - a^2}{2ab},$$

as proclaimed above. We also observe that the same function $1/(x^2|\sigma(x)|)$ would give the infimum for the L^p norm over I for any other value of $p > 0$.

If, on the other hand, α is positive, we have instead

$$\alpha \leq \frac{2}{\pi} \int_b^\infty tv(t)|\sigma(t)|dt,$$

with equality holding if v vanishes to the left of I . In this case, arguing in the same fashion as above, we find that we can get $\mathcal{H}((1 - \chi_I)\sigma v)(x)$ as small as we please by letting v be supported on a set sufficiently far to the right of I .

3. PROOFS

We now turn to the proof of the theorem and its corollary. We will rely on Lemma 1 and two additional lemmas.

Lemma 2. *For every t in $(0, a) \cup (b, \infty)$ the function*

$$f_t(x) = \frac{2t}{t^2 - x^2} \left(1 - \frac{\sigma(t)}{\sigma(x)} \right) - 2t\sigma(t)$$

is in H^p for $p > 1/2$, and the following estimates hold

$$\|f_t\|_1 \leq \frac{C_1}{\sqrt{|(a-t)(b-t)|}}, \quad t < 2b,$$

$$\int_{2b}^\infty |f_t(x)|^2 dt \leq \frac{C_2}{|x| + 1},$$

where the constants C_1 and C_2 only depend on a and b .

Proof. It is immediate that f_t belongs to H^p for $p > 1/2$ because the isolated singularities $\pm t$ are removable and $f_t(z) = O(z^{-2})$ when $z \rightarrow \infty$. The norm estimates follow from elementary calculations. ■

Lemma 3. *A nonnegative function ν in $L^2(\mathbb{R}^+ \setminus I)$ satisfies*

$$\int_I |\mathcal{H}((1 - \chi_I)\sigma\nu)(x)|^2 |\sigma(x)|^{-2} dx < \infty$$

if and only if the following condition holds:

$$\int_0^a \int_0^a [\nu(a-t)\nu(a-\tau) + \nu(b+t)\nu(b+\tau)] \frac{|\log(t+\tau)|}{\sqrt{t\tau}} dt d\tau < \infty.$$

Proof. The necessary and sufficient condition for square-integrability at the left end-point of I is that

$$\int_0^a \left(\int_0^a \frac{\nu(a-t)}{\sqrt{t}(t+x)} dt \right)^2 x dx < \infty.$$

By Fubini's theorem, we may interchange the order of integration so that this condition becomes

$$\int_0^a \int_0^a \nu(a-t)\nu(a-\tau) \frac{|\log(t+\tau)|}{\sqrt{t\tau}} dt d\tau < \infty.$$

Combining this with the corresponding condition at the right end-point of I , we arrive at the condition of the lemma. \blacksquare

The theorem is now proved in the following way. We assume first that we are given a nonnegative function v in $L^2(\mathbb{R}^+)$ satisfying (1), (2), and (3). We claim that the function

$$f_1(x) = \frac{1}{\pi} \int_{(0,a) \cup (b,2b)} v(t) f_t(x) dt$$

is in H^1 . Indeed, by Lemma 2 and Fubini's theorem,

$$\|f_1\|_1 \leq \frac{C_1}{\pi} \int_{(0,a) \cup (b,2b)} \frac{v(t)}{\sqrt{|(a-t)(b-t)|}} dt,$$

and the integral on the right is bounded thanks to (1). On the other hand,

$$f_2(x) = \frac{1}{\pi} \int_{2b}^{\infty} v(t) f_t(x) dt$$

is in H^p for $p > 2$, because by the Cauchy–Schwarz inequality and Lemma 2 we have

$$\int_{-\infty}^{\infty} |f_2(x)|^p dx \leq \frac{C_2^{p/2}}{\pi^p} \|v\|_2^p \int_{-\infty}^{\infty} \frac{1}{(|x|+1)^{p/2}} dx.$$

The imaginary part of $f_1 + f_2$ is supported by I and equals $-v$ there, in view of (3). Since v is assumed to be in L^2 , it follows that $f_1 + f_2$ is in fact in H^2 .

We set $f = \mathcal{H}((1 - \chi_I)v) + i(1 - \chi_I)v$, which is a function in H^2 . We observe that the imaginary part of $f - (f_1 + f_2)$ equals v and that its real part equals α on I , when taking into account (2). So we have proved that the given v is indeed in $K_\alpha(I)$.

We now prove the necessity of the three conditions of the theorem. So assume we are given a nonnegative function v in $L^2(\mathbb{R}^+)$ belonging to some class $K_\alpha(I)$. Setting

$I_\varepsilon = [a + \varepsilon, b - \varepsilon]$, we see that v also belongs to $K_\alpha(I_\varepsilon)$ whenever $0 < \varepsilon < (b - a)/2$. But since the Hilbert transform of v is constant near the endpoints of I_ε , it follows that

$$\int_{\mathbb{R}^+ \setminus I_\varepsilon} v(t) |\sigma_\varepsilon(t)| dt < \infty,$$

where now

$$\sigma_\varepsilon(t) = \frac{1}{\sqrt{t^2 - (b - \varepsilon)^2} \sqrt{t^2 - (a + \varepsilon)^2}}.$$

We claim that this means that

$$(4) \quad v(x) = \mathcal{H}((1 - \chi_{I_\varepsilon})\sigma_\varepsilon v)(x) / |\sigma_\varepsilon(x)|,$$

provided x is in $(a + \varepsilon, b - \varepsilon)$. Indeed, by Lemma 1, it is enough to verify that

$$(1 - \chi_{I_\varepsilon}(x))v(x) + \chi_{I_\varepsilon}(x)\mathcal{H}((1 - \chi_{I_\varepsilon})\sigma_\varepsilon v)(x) / |\sigma_\varepsilon(x)|$$

is in $K_\alpha(I_\varepsilon)$ for some α . Since, in view of Lemma 3, the function on the right-hand side of (4) is square-integrable on I_ε , the claim follows by repeating the argument in the first part of the proof.

We may view $\mathcal{H}((1 - \chi_{I_\varepsilon})\sigma_\varepsilon v)(x)$ as the L^1 norm of the function

$$h_{x,\varepsilon}(t) = \frac{1}{\pi} ((1 - \chi_{I_\varepsilon}(t))\sigma_\varepsilon(t)v(t)) \frac{2t}{t^2 - x^2}.$$

Then (4) says that $\|h_{x,\varepsilon}\|_1 \rightarrow v(x)|\sigma(x)|$ when $\varepsilon \rightarrow 0$. Since we also have that

$$h_{x,\varepsilon}(t) \rightarrow \frac{1}{\pi} ((1 - \chi_I(t))\sigma(t)v(t)) \frac{2t}{t^2 - x^2}$$

for every t , we obtain

$$v(x) = \mathcal{H}((1 - \chi_I)\sigma v)(x) / |\sigma(x)|$$

for every x in (a, b) . By a similar argument, we find that

$$\alpha = \lim_{\varepsilon \rightarrow 0} \frac{2}{\pi} \int_{\mathbb{R}^+ \setminus I_\varepsilon} tv(t)\sigma_\varepsilon(t)dt = \frac{2}{\pi} \int_{\mathbb{R}^+ \setminus I} tv(t)\sigma(t)dt.$$

The necessity of (1) has already been observed; without it we would reach the contradictory conclusion that $v(x) = \infty$ for almost every $x \in (a, b)$.

We finally note that the corollary is an immediate consequence of the theorem and lemmas 1 and 3.

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