

PHASE SPACE DISTRIBUTION OF GABOR EXPANSIONS

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ABSTRACT. We present an example of a complete and minimal Gabor system consisting of time-frequency shifts of a Gaussian, localized at the coordinate axes in the time-frequency plane (phase space). Asymptotically, the number of time-frequency shifts contained in a disk centered at the origin is only $2/\pi$ times the number of points from the von Neumann lattice found in the same disk. Requiring a certain regular distribution in phase space, we show that our system has minimal density among all complete and minimal systems of time-frequency shifts of a Gaussian.

1. INTRODUCTION

For a function g in $L^2(\mathbb{R})$ and two real numbers x and y , we define

$$(1) \quad \rho_{x,y}g(t) = e^{2i\pi yt}g(t-x).$$

We refer to $\rho_{x,y}g$ as the time-frequency shift of g with respect to the point (x, y) in phase space. Given g in $L^2(\mathbb{R})$ and a sequence Λ of distinct points in \mathbb{R}^2 , we are interested in the spanning properties of the Gabor system

$$(2) \quad \mathcal{G}_{g,\Lambda} = \{\rho_{x,y}g : (x, y) \in \Lambda\}.$$

It is well known (see [9], [15], [3]) that for $\mathcal{G}_{g,\Lambda}$ to be a frame for $L^2(\mathbb{R})$, the lower Beurling–Landau density of Λ has to be at least 1. This means that the minimal number of points from Λ to be found in a disk of radius r must be at least $\pi r^2 + o(r^2)$ when $r \rightarrow \infty$. The canonical case is when Λ is a lattice $a\mathbb{Z} \times b\mathbb{Z}$, the lower Beurling–Landau density of which is $1/(ab)$ for positive lattice constants a and b . When g is a Gaussian, the lattice structure of Λ is inessential and we have the following complete description: Assuming Λ is a separated sequence, $\mathcal{G}_{g,\Lambda}$ is a frame for $L^2(\mathbb{R})$ if and only if the Beurling–Landau density of Λ exceeds one [12, 17, 18].

It is easy to construct examples of complete Gabor systems $\mathcal{G}_{g,\Lambda}$ for which the Beurling–Landau density of Λ is 0. However, when $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, the density condition $ab \leq 1$ is still necessary for $\mathcal{G}_{g,\Lambda}$ to be complete in $L^2(\mathbb{R})$ (see [16]), and it seems to be commonly thought that Λ should somehow be uniformly localized throughout the entire phase space \mathbb{R}^2 for there to be nice expansions associated with $\mathcal{G}_{g,\Lambda}$. The main point of the present note is to show that the phase space distribution of Λ may be dramatically different from

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that of a lattice if we require $\mathcal{G}_{g,\Lambda}$ to be a complete and minimal system in $L^2(\mathbb{R})$. Indeed, we have the following example when g is a Gaussian:

Theorem 1. *Set $g(t) = \exp(-\pi t^2)$ and*

$$\Lambda = \{(-1, 0), (1, 0)\} \cup \left\{ (0, \pm\sqrt{2n}) : n = 1, 2, 3, \dots \right\} \cup \left\{ (\pm\sqrt{2n}, 0) : n = 1, 2, 3, \dots \right\}.$$

Then $\mathcal{G}_{g,\Lambda}$ is a complete and minimal system in $L^2(\mathbb{R})$.

We see that the sequence Λ of this example is localized at the two coordinate axes in \mathbb{R}^2 . In view of the density results mentioned above, it may seem surprising that there exist arbitrarily large disks containing no points from Λ . It is also quite striking that the number of points $\lambda = (\xi, \eta)$ from Λ satisfying $\xi^2 + \eta^2 \leq r^2$ is $2r^2 + O(1)$ when $r \rightarrow \infty$. This is asymptotically only $2/\pi$ times the number of points from a lattice of Beurling–Landau density 1 to be found in any disk of radius r . It is well-known that when Λ is a lattice of Beurling–Landau density 1 with one point omitted, $\mathcal{G}_{g,\Lambda}$ is also a complete and minimal system, see e.g. [13]. Thus the phase space distribution of complete and minimal Gabor systems may differ in a rather fundamental way.

Our second theorem shows that we have in fact identified two extreme configurations among all complete and minimal systems of time-frequency shifts of a Gaussian, assuming the time-frequency shifts to be regularly distributed as defined in the theory of entire functions. To make this statement precise, we switch to complex notation, i.e., we view our sequence Λ as a subset of the complex plane¹ via the map $(\xi, \eta) \mapsto \xi + i\eta$. Given $r > 0$ and $0 \leq \theta < \vartheta \leq 2\pi$, we write $n_\Lambda(r, \theta, \vartheta) = \#\{\lambda \in \Lambda; |\lambda| < r, \theta < \arg \lambda \leq \vartheta\}$, and we say that Λ has angular density if the limit

$$\Delta_\Lambda(\theta, \vartheta) = \lim_{r \rightarrow \infty} \frac{n_\Lambda(r, \theta, \vartheta)}{\pi r^2}$$

exists for all θ and ϑ , except possibly for a countable set of such pairs of numbers. From this definition we see that if Λ has angular density, then the limit $\Delta_\Lambda(0, 2\pi)$ exists. We set $\Delta_\Lambda = \Delta_\Lambda(0, 2\pi)$ and refer to this number as the density of Λ . If Λ has angular density and, in addition, the limit

$$\lim_{r \rightarrow \infty} \sum_{\lambda \in \Lambda, |\lambda| < r} \lambda^{-2}$$

exists, we say that Λ is a regularly distributed set. We refer to Chapter II of [10] for more refined versions of these definitions (depending on the order of the functions in question) and their significance in the theory of entire functions of completely regular growth.

It is clear that the sequence Λ of Theorem 1 as well as any lattice $a\mathbb{Z} \times b\mathbb{Z}$ is a regularly distributed set. The following theorem therefore places our examples in context.

Theorem 2. *Set $g(t) = \exp(-\pi t^2)$ and let Λ be a regularly distributed set. If $\mathcal{G}_{g,\Lambda}$ is a complete and minimal system in $L^2(\mathbb{R})$, then $2/\pi \leq \Delta_\Lambda \leq 1$.*

¹In what follows, we will allow ourselves to view the sequence Λ alternately as a subset of \mathbb{R}^2 and of \mathbb{C} , with the tacit choice of viewpoint depending on whatever fits the context.

There are obvious ways in which our two theorems can be generalized: The example of Theorem 1 can be extended to much more general sets, and using the notion of proximate order as defined in Chapter 1 of [10], one may relax the assumption about regular distribution of Λ in Theorem 2. We decided not to pursue such elaborations; our aim is to present the essence of an interesting qualitative phenomenon, avoiding as far as possible obscuring technicalities.

The assumption that $\mathcal{G}_{g,\Lambda}$ be complete and minimal ensures the existence of a unique biorthogonal system in $L^2(\mathbb{R})$. Thus we may write down meaningful Fourier expansions for functions in $L^2(\mathbb{R})$ with respect to the system $\mathcal{G}_{g,\Lambda}$. However, typically this expansion is not convergent, so that one may have to find a special procedure to recover a function from its Fourier coefficients. In [13], a simple summation method was shown to do the job for certain sequences similar to lattices. It would be interesting to identify those functions for which the Gabor expansion exhibited in Theorem 1 converges, and to find an appropriate summation method for arbitrary functions in $L^2(\mathbb{R})$.

We can find examples of complete sets $\mathcal{G}_{g,\Lambda}$ with Λ localized at a single line [14, 19]. It has been proved that there are no Schauder bases of Gabor systems with such phase space localization when $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, but we still do not know if there exists such a system $\mathcal{G}_{g,\Lambda}$ that is complete and minimal in $L^2(\mathbb{R})$. We refer to [4, 8] for this result as well as other results related to Gabor Schauder bases. In the case of Gaussian functions, the corresponding sequence Λ would have to be a regularly distributed set, but then its density would be 0, and this is incompatible with Theorem 2. It has also been proved that there are no complete and minimal systems of time-frequency shifts of the Poisson function [2, 20].

The contrast with what is known about Paley–Wiener spaces and families of complex exponentials in L^2 of an interval is worth noting. In this case, we have the following precise density condition: If the family $\{\exp(i\lambda t)\}_{\lambda \in \Lambda}$ is a complete and minimal system in $L^2(-a, a)$, then the number of points from Λ to be found in the disk $|z| \leq r$ behaves asymptotically as $2ar/\pi$ when $r \rightarrow \infty$; see Theorem 1 in Lecture 17 of [11].

In the next brief section, we will make a transition to the Fock space of entire functions, and then the remaining two sections are devoted to proving our two theorems.

2. TRANSITION TO THE FOCK SPACE

For the rest of the paper, g denotes the normalized Gaussian: $g(t) = 2^{1/4}e^{-\pi t^2}$. We define the Bargmann transform in the following way:

$$(3) \quad \mathcal{B}f(z) = e^{-i\pi xy} e^{\frac{\pi}{2}|z|^2} \int_{-\infty}^{\infty} f(t) \overline{(\rho_{x,-y}g)(t)} dt = 2^{\frac{1}{4}} \int_{-\infty}^{\infty} f(t) e^{-\pi t^2} e^{2\pi tz} e^{-\frac{\pi}{2}z^2} dt,$$

where we have put $z = x + iy$. The Bargmann transform maps $L^2(\mathbb{R})$ isometrically onto the Fock space

$$\mathcal{F} = \left\{ F : F \text{ is entire and } \|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < \infty \right\};$$

here dm denotes planar Lebesgue measure. See [6, 7] for this fact and other basic properties of the Bargmann transform. Employing methods from the theory of entire functions, we may prove much stronger results for a Gaussian than what can be obtained for arbitrary functions in $L^2(\mathbb{R})$.

We recall that a sequence of distinct points Λ in \mathbb{C} is a set of uniqueness for \mathcal{F} if the only function in \mathcal{F} vanishing at every point from Λ is the zero function. A set of uniqueness Λ has zero excess if it fails to be a set of uniqueness for \mathcal{F} on the removal of any one of the points from Λ . Note that since a function F in \mathcal{F} remains in \mathcal{F} when we change the position of a single zero of F , it does not matter which particular point we remove from Λ . A standard duality argument shows that (3) yields the following translation of our problem.

Lemma 1. *The system $\mathcal{G}_{g,\Lambda}$ is complete and minimal in $L^2(\mathbb{R})$ if and only if Λ is a set of uniqueness of zero excess for \mathcal{F} .*

Proof. We set $\bar{\Lambda} = \{(\xi_j, -\eta_j)\}$ and note that, by symmetry, $\mathcal{G}_{g,\Lambda}$ is a complete and minimal system in $L^2(\mathbb{R})$ if and only if $\mathcal{G}_{g,\bar{\Lambda}}$ is a complete and minimal system in $L^2(\mathbb{R})$. Since every function in \mathcal{F} can be expressed as the Bargmann transform of some function in $L^2(\mathbb{R})$ and the factor $e^{-\pi ixy} e^{\frac{\pi}{2}|z|^2} \neq 0$ at every point z , the result follows by duality. \square

3. PROOF OF THEOREM 1

This proof will use some standard notions and results from the theory of entire functions of exponential type, such as indicator diagrams, indicator functions, and the Phragmén–Lindelöf principle. Good references for this material are Lectures 6 and 9 in [11], Chapter 1 of [10], and Chapter 5 of [1].

In this proof, the meaning of the notation $A(w) \asymp B(w)$ for w in some set W is that the ratio between the two positive functions $A(w)$ and $B(w)$ is bounded from below and above by positive constants independent of w in W .

To prove that Λ is a set of uniqueness for \mathcal{F} , we will argue by contradiction. Thus we begin by assuming that there does indeed exist a nontrivial function Φ in \mathcal{F} vanishing at Λ . We may assume that Φ is an even function. Were it not, we could replace it by $\Phi(z) + \Phi(-z)$, which is again a function in \mathcal{F} vanishing on Λ ; admittedly, this function is identically 0 if Φ is an odd function, but were this the case, we could replace Φ by $\Phi(z)/z$.

We note that Λ is the zero set of the function

$$s(z) = (z^2 - 1)z^{-2} \sin \frac{\pi z^2}{2}.$$

It follows that the function $\Psi(z) = \Phi(z)s(z)^{-1}$ is also an even entire function of order at most 2.

We will use the following elementary estimate for the sine function:

$$|\sin \pi z| \asymp \exp(\pi|\Im z|), \quad \text{dist}(z, \mathbb{Z}) > \varepsilon,$$

where ε is an arbitrary positive number. It follows from this estimate that

$$(4) \quad |s(re^{i\theta})| \asymp e^{\frac{\pi}{2}r^2|\sin 2\theta|}, \quad \text{dist}(re^{i\theta}, \Lambda) > \varepsilon r^{-\frac{1}{2}}.$$

In particular,

$$|s(re^{i(\frac{\pi}{4}+k\frac{\pi}{2})})| \asymp e^{\frac{\pi}{2}r^2}, \quad k = 0, 1, 2, 3 \quad \text{and} \quad r > 0.$$

Combining the latter estimate with the fact that every function F in \mathcal{F} satisfies

$$(5) \quad |F(z)| = o(e^{\frac{\pi}{2}|z|^2}), \quad z \rightarrow \infty$$

(see Lemma 3 in [13] for a stronger statement), we obtain

$$(6) \quad |\Psi(re^{i(\frac{\pi}{4}+k\frac{\pi}{2})})| = o(1) \quad \text{as} \quad r \rightarrow \infty, \quad k = 0, 1, 2, 3.$$

Since Ψ is an even function, $\Omega(z) = \Psi(e^{i\frac{\pi}{4}}z^{\frac{1}{2}})$ is an entire function of exponential type. We observe that (6) yields

$$(7) \quad |\Omega(x)| = o(1)$$

when $x \rightarrow \pm\infty$. Hence the indicator diagram of Ω is the segment $[-i\tau_-, i\tau_+]$ on the imaginary axis, and its indicator function is

$$(8) \quad h_{\Omega}(\theta) = \limsup_{r \rightarrow \infty} r^{-1} \log |\Omega(re^{i\theta})| = \begin{cases} \tau_+ \sin \theta & \text{if } \sin \theta > 0, \\ -\tau_- \sin \theta & \text{if } \sin \theta < 0. \end{cases}$$

If $\max\{\tau_+, \tau_-\} = 0$, then the Phragmén–Lindelöf principle and (7) show that Ω is a bounded entire function and therefore a constant, by Liouville's theorem. Thus $\Omega \equiv 0$ and, consequently, $\Phi \equiv 0$, which is the desired contradiction.

It remains to consider the possibility of $\max\{\tau_+, \tau_-\} > 0$. By symmetry, we may confine ourselves to the case $\tau_+ > 0$. We assume that $\sin \theta > 0$ and rewrite (8) as

$$h_{\Psi}(\theta) = \limsup_{r \rightarrow \infty} r^{-2} \log |\Psi(re^{i\theta})| = \tau_+ |\sin(2\theta + \pi/2)| = \tau_+ |\cos(2\theta)|.$$

Combining this with (4) and using that $\Psi(z) = \Phi(z)s(z)^{-1}$, we get

$$(9) \quad \limsup_{r \rightarrow \infty} r^{-2} \log |\Phi(re^{i\theta})| = \frac{\pi}{2} |\sin 2\theta| + \tau_+ |\cos(2\theta)|.$$

On the other hand, (5) with $F = \Phi$ yields

$$(10) \quad \limsup_{r \rightarrow \infty} r^{-2} \log |\Phi(re^{i\theta})| \leq \frac{\pi}{2}.$$

The two relations (9) and (10) are incompatible because an elementary calculus argument shows that, for each $\tau_+ > 0$, there exists $\delta > 0$ such that

$$\frac{\pi}{2} |\sin 2\theta| + \tau_+ |\cos(2\theta)| > \frac{\pi}{2}$$

for $0 < |\theta - \pi/4| < \delta$. Thus we cannot have $\max\{\tau_+, \tau_-\} > 0$ and conclude that Λ is a set of uniqueness for \mathcal{F} .

Direct estimates based on inequality (4) show that $(z-1)^{-1}s(z)$ is in \mathcal{F} . In other words, $\Lambda \setminus \{(1, 0)\}$ is not a set of uniqueness for \mathcal{F} , and consequently Λ has zero excess.

4. PROOF OF THEOREM 2

We will now assume that Λ is a set of uniqueness of zero excess for \mathcal{F} . We fix a point λ' in Λ and let Φ be a function \mathcal{F} with only simple zeros and whose zero set coincides with $\Lambda \setminus \{\lambda'\}$. The densities of Λ and $\Lambda \setminus \{\lambda'\}$ are of course the same.

We begin by noting that the right inequality $\Delta_\Lambda \leq 1$ is a simple consequence of Jensen's formula; in particular, we only need the required regular distribution of Λ to define the density Δ_Λ . Set $n_\Lambda(r) = n_\Lambda(r, 0, 2\pi)$ and assume that $\Phi(0) \neq 0$. Then Jensen's formula gives

$$\int_0^r \frac{n_\Lambda(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |\Phi(re^{i\theta})| d\theta - \log |\Phi(0)|.$$

Using again (5), we obtain

$$\liminf_{r \rightarrow \infty} \frac{n_\Lambda(r)}{\pi r^2} \leq 1,$$

where the left-hand side equals the density Δ_Λ whenever the latter quantity exists.

The left inequality $2/\pi \leq \Delta_\Lambda$ is an immediate consequence of the following two lemmas on the indicator function

$$h_\Phi(\theta) = \limsup_{r \rightarrow \infty} r^{-2} \log |\Phi(re^{i\theta})|, \quad \theta \in [0, 2\pi].$$

Lemma 2. *Let Φ be a nontrivial entire function whose zero set Λ is a regularly distributed set. Then*

$$\Delta_\Lambda = \frac{1}{\pi^2} \int_0^{2\pi} h_\Phi(\theta) d\theta.$$

Proof. The lemma is a special case of Theorem 3 in Chapter IV of [10]. We note in passing that this is the point where the regular distribution of Λ is crucial. \square

Lemma 3. *Let Φ be a function in \mathcal{F} such that whenever G is a nontrivial entire function with nonempty zero set, the function $G\Phi$ does not belong to \mathcal{F} . Then*

$$\int_0^{2\pi} h_\Phi(\theta) d\theta \geq 2\pi.$$

Proof. On the assumption of the lemma, we will first prove the following basic relation:

$$(11) \quad \sup_{\theta \in [\theta_0 - \pi/4, \theta_0 + \pi/4]} h_\Phi(\theta) = \frac{\pi}{2}$$

for every θ_0 in $[0, 2\pi)$. We argue again by contradiction and assume that (11) fails, say for $\theta_0 = 0$. In other words, we assume that

$$(12) \quad \epsilon = \frac{\pi}{2} - \sup_{|\theta| \leq \pi/4} h_\Phi(\theta) > 0.$$

We will construct a function Ψ in \mathcal{F} whose zero set is larger than $\Lambda \setminus \{\lambda'\}$. The existence of Ψ will contradict the assumption of the lemma.

Consider the Mittag-Leffler function

$$(13) \quad E_{\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k/2)}.$$

We refer to Section 18.1 of [5] for a detailed exposition of the properties of this function. What we need, is that $E_{\frac{1}{2}}$ is an entire function of order 2 with infinitely many zeros, and also that

$$(14) \quad E_{\frac{1}{2}}(z) = \begin{cases} 2e^{z^2} + O(|z|^{-1}), & |\arg z| \leq \frac{\pi}{4} \\ O(|z|^{-1}), & \frac{\pi}{4} \leq |\arg z| \leq \pi \end{cases}$$

when $|z| \rightarrow \infty$; see relations (7) and (9) in Section 18.1 of [5].

We now introduce the following function:

$$\Psi(z) = \Phi(z)E_{\frac{1}{2}}(z(\epsilon/2)^{1/2}).$$

It follows from (14) that

$$|\Psi(z)| \leq C|\Phi(z)|, \quad \frac{\pi}{4} \leq |\arg z| \leq \pi,$$

for some constant C , and since $\Phi \in \mathcal{F}$, we obtain

$$\int_{\frac{\pi}{4} \leq |\arg z| \leq \pi} |\Psi(z)|^2 e^{-\pi|z|^2} dm(z) < \infty.$$

On the other hand, using (14) and an estimate based on the Phragmén–Lindelöf principle (see Theorem 28 in Chapter I of [10]), we get

$$|\Psi(z)| \leq Ce^{(\frac{\pi}{2} - \frac{\epsilon}{3})|z|^2}, \quad |\arg z| \leq \frac{\pi}{4},$$

with C a positive constant. We conclude that Ψ belongs to \mathcal{F} , which is the desired contradiction.

To obtain the conclusion of the lemma from (11), we will use that the indicator function $h_{\Phi}(\theta)$ is a 2-trigonometrically convex function. A consequence of this property is that if h_{Φ} has a local maximum at θ_0 , then

$$(15) \quad h_{\Phi}(\theta) \geq h_{\Phi}(\theta_0) \cos 2(\theta - \theta_0), \quad |\theta - \theta_0| \leq \frac{\pi}{4}.$$

We refer to Chapter 1 of [10] or Lecture 8 in [11] for definitions and proofs.

To finish the proof, we need the following auxiliary construction. For each collection of directions $\theta_1, \theta_2, \dots, \theta_n, \theta_{n+1}$ such that $0 \leq \theta_1 < \theta_2 < \dots < \theta_n < 2\pi \leq \theta_{n+1} = \theta_1 + 2\pi$, we set $\theta'_j = (\theta_{j+1} - \theta_j)/2$ for $j = 1, 2, \dots, n$ and $\theta'_{n+1} = \theta'_1 + 2\pi$, and we then define the function $H(\theta; \theta_1, \theta_2, \dots, \theta_n)$ by declaring that

$$H(\theta; \theta_1, \theta_2, \dots, \theta_n) = \frac{\pi}{2} \cos 2(\theta_{j+1} - \theta), \quad \theta'_j \leq \theta < \theta'_{j+1}$$

for $j = 1, 2, \dots, n$. If we also require that $\theta_{j+1} - \theta_j \leq \pi/2$, then

$$(16) \quad \int_0^{2\pi} H(\theta; \theta_1, \theta_2, \dots, \theta_n) d\theta \geq \int_0^{2\pi} H(\theta; 0, \pi/2, \pi, 3\pi/2) d\theta,$$

as can be verified by a comparison of the level sets of the two integrands. It follows from (11) that $\max h_\Phi(\theta) = \pi/2$ and also that there is a finite set $\{\theta_j\}_1^n$ of global maxima of h_Φ such that $|\theta_j - \theta_{j+1}| \leq \pi/2$ for $j = 1, 2, \dots, n$; here $\theta_{n+1} = \theta_1 + 2\pi$ as before. Using (15) and (16), we get

$$\int_0^{2\pi} h_\Phi(\theta) d\theta \geq \int_0^{2\pi} H(\theta; \theta_1, \theta_2, \dots, \theta_n) d\theta \geq \int_0^{2\pi} H(\theta; 0, \pi/2, \pi, 3\pi/2) d\theta = 2\pi,$$

where the last equality follows by a simple calculation. \square

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