THE KADETS 1/4 THEOREM FOR POLYNOMIALS

JORDI MARZO AND KRISTIAN SEIP

Abstract. We determine the maximal angular perturbation of the $(n+1)$th roots of unity permissible in the Marcinkiewicz–Zygmund theorem on L^p means of polynomials of degree at most n. For $p = 2$, the result is an analogue of the Kadets 1/4 theorem on perturbation of Riesz bases of complex exponentials.

1. Introduction

A classical theorem of J. Marcinkiewicz and A. Zygmund generalizes the elementary mean value formula

$$\frac{1}{n+1}\sum_{j=0}^{n}\left|P\left(e^{i\frac{2\pi j}{n+1}}\right)\right|^2 = \int_0^{2\pi} |P(e^{i\theta})|^2 \,\frac{d\theta}{2\pi}, \tag{1}$$

valid for holomorphic polynomials P of degree at most n, in the following way: For $1 < p < \infty$, there is a constant C_p independent of n such that

$$\frac{C_p^{-1}}{n+1}\sum_{j=0}^{n}\left|P\left(e^{i\frac{2\pi j}{n+1}}\right)\right|^p \le \int_0^{2\pi} |P(e^{i\theta})|^p \,\frac{d\theta}{2\pi} \le \frac{C_p}{n+1}\sum_{j=0}^{n}\left|P\left(e^{i\frac{2\pi j}{n+1}}\right)\right|^p \tag{2}$$

for every complex polynomial P of degree at most n. (See [8] or Theorem 7.5 in Chapter X of [15].) It is natural to ask if the norm equivalence expressed by (2) remains valid if we replace the $(n+1)$th roots of unity $\omega_{nj} = e^{i\frac{2\pi j}{n+1}}$ by $n+1$ points z_{nj} on the unit circle with a less regular distribution. C. K. Chui, X.-C. Shen, and L. Zhong [2] considered this problem and found that the norm equivalence is stable under small perturbations of the points ω_{nj}. We will prove the following sharp version of their result:

Theorem 1.1. *Suppose $1 < p < \infty$ and set $q = \max(p, p/(p-1))$. The following statement holds if and only if $\delta < 1/(2q)$: There is a constant C_p independent of n such that if $|\arg(z_{nj}\overline{\omega_{nj}})| \le 2\pi\delta/(n+1)$ for $0 \le j \le n$, then*

$$\frac{C_p^{-1}}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p \le \int_0^{2\pi} |P(e^{i\theta})|^p \,\frac{d\theta}{2\pi} \le \frac{C_p}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p \tag{3}$$

for every holomorphic polynomial P of degree at most n.

Date: December 20, 2007.
The first author is supported by projects 2005SGR00611 and MTM2005-08984-C02-02.
The second author is supported by the Research Council of Norway grant 160192/V30.

We will see that this theorem is a consequence of a general result of Chui and Zhong [3], characterizing the so-called L^p Marcinkiewicz–Zygmund families (to be defined below) in terms of Muckenhoupt (A_p) weights.

Readers familiar with Paley–Wiener spaces will see the analogy with the Kadets 1/4 theorem on perturbations of Riesz bases of complex exponentials in L^2 of an interval [4]. One may view polynomials as discrete versions of band-limited functions, with the degree of the polynomial being the counterpart to the notion of "bandwidth". The identity (1) is the discrete analogue of the Plancherel identity or—what amounts to the same—the Shannon formula for bandlimited functions. In the case when $p = 2$ and we require $\delta < 1/4$, our theorem corresponds precisely to the Kadets 1/4 theorem. The L^p version ($1 < p < \infty$) of the Kadets theorem, analogous to our theorem, can be found in [7].

It is interesting to note that our problem as well as that of the classical Kadets theorem fits into a general theory of unconditional bases in so-called model spaces. (See [13], [10], and [6] for original work and [11] or [14] for more recent expositions.) In particular, the theorem of Chui and Zhong to be used in this note can be obtained from a theorem given in [6]. We refer to [9] for the details of this link and to [12], where the connection between Marcinkiewicz–Zygmund inequalities and model spaces was first mentioned explicitly.

For $p = 2$, the proof to be given below is an adaption of S. Khrushchev's proof of the classical Kadets 1/4 theorem [5], and, for general p, we act in a similar way as was done in [7]. Khrushchev also showed how to obtain other perturbation results, such as a theorem of S. Avdonin [1]. We will confine ourselves to proving the theorem stated above and refer to [9] for the counterpart of Avdonin's theorem as well as other analogues of results for Paley-Wiener spaces and families of complex exponentials.

2. Preliminaries

Suppose that for each nonnegative integer n we are given a set $\mathcal{Z}(n) = \{z_{nj}\}_{j=0}^{n}$ of $n+1$ distinct points on the unit circle. We denote by $\mathcal{Z} = \{\mathcal{Z}(n)\}_{n\geq 0}$ the corresponding triangular family of points. The family $\mathcal{Z}$ is declared to be uniformly separated if there exists a positive number ε such that

$$\inf_{j\neq k} |z_{nj} - z_{nk}| \geq \frac{\varepsilon}{n+1}$$

for every $n \geq 0$.

We will say that $\mathcal{Z}$ is an L^p Marcinkiewicz–Zygmund family if there exists a constant $C_p > 0$ such that for every $n \geq 0$ and complex polynomial P of degree at most n, we have

$$\frac{C_p^{-1}}{n+1}\sum_{j=0}^{n} |P(z_{nj})|^p \leq \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \leq \frac{C_p}{n+1}\sum_{j=0}^{n} |P(z_{nj})|^p. \tag{4}$$

In order to describe such families, we associate with $\mathcal{Z}$ the following generating polynomials

$$F_n(z) = \prod_{j=0}^{n}\left(1 - \frac{n}{n+1}\overline{z_{jn}}z\right).$$

The theorem of Chui and Zhong reads as follows [3].

Theorem 2.1. *Suppose $1 < p < \infty$. The family $\mathcal{Z} = \{\mathcal{Z}(n)\}_{n\geq 0}$ of points on the unit circle is an L^p Marcinkiewicz–Zygmund family if and only if it is uniformly separated and there exists a constant K_p such that*

$$\left(\frac{1}{|I|}\int_I |F_n(e^{i\theta})|^p d\theta\right)^{1/p}\left(\frac{1}{|I|}\int_I |F_n(e^{i\theta})|^{-p/(p-1)} d\theta\right)^{(p-1)/p} \leq K_p \tag{5}$$

for every subarc I of the unit circle and every $n \geq 0$.

In other words, the sequence $|F_n|^p$ satisfies a uniform (A_p) condition.

In the proof of the positive part of the $p = 2$ case of our theorem, we will make use of the equivalence between the (A_2) and Helson–Szegő conditions. We will derive the result for $p \neq 2$ from the $p = 2$ case using the following estimate.

Lemma 2.2. *Let $\alpha, \kappa > 0$ be given, and set $\rho_{\kappa n} = \max(1/2, 1 - \kappa/(n+1))$. If a given triangular family of real numbers δ_{nj} satisfies $\sup_{nj} |\delta_{nj}| < 1/2$, then*

$$\left|\prod_{j=0}^{n}(z - \rho_{\kappa n} e^{\frac{2\pi i(j+\alpha\delta_{nj})}{n+1}})\right| = R_n(z)\left|\prod_{j=0}^{n}(z - \rho_{\kappa n} e^{\frac{2\pi i(j+\delta_{nj})}{n+1}})\right|^{\alpha},$$

where $R_n(z)$ is bounded from above and below by positive constants, independently of $z \in \mathbb{T}$ and $n \geq 0$.

Proof. Set

$$P_\beta(\theta) = \left|\prod_{j=0}^{n}(e^{i\theta} - \rho_{\kappa n} e^{i\lambda_j(\beta)})\right|, \quad \text{where} \quad \lambda_j(\beta) = \frac{2\pi j}{n+1} + \frac{2\pi\beta\delta_{nj}}{n+1}.$$

We have

$$\log P_\beta(\theta) - \log P_0(\theta) = \operatorname{Re}\sum_{j=0}^{n}\int_{\rho_{\kappa n}e^{i\lambda_j(0)}}^{\rho_{\kappa n}e^{i\lambda_j(\beta)}} \frac{d\xi}{\xi - e^{i\theta}} = \sum_{j=0}^{n}\int_{\lambda_j(0)}^{\lambda_j(\beta)} h(\theta - t)dt,$$

where

$$h(t) = \frac{\rho_{\kappa n}\sin t}{1 + \rho_{\kappa n}^2 - 2\rho_{\kappa n}\cos t}.$$

By the fundamental theorem of calculus,

$$\log P_\beta(\theta) - \log P_0(\theta) = \sum_{j=0}^{n}(\lambda_j(\beta) - \lambda_j(0))h(\theta - \lambda_j(0)) + \sum_{j=0}^{n}\int_{\lambda_j(0)}^{\lambda_j(\beta)}\int_{\lambda_j(0)}^{t} h'(\theta - \tau)d\tau dt.$$

We compute $h'(t)$ and find that the absolute value of the latter sum is bounded independently of θ and n. Therefore,

$$\begin{aligned}\log P_\alpha(\theta) - \log P_0(\theta) &= \alpha\sum_{j=0}^{n}\frac{2\pi\delta_{nj}}{n+1}h(\theta - \lambda_j(0)) + b_{n,\alpha}(z)\\ &= \alpha(\log P_1(\theta) - \log P_0(\theta) - b_{n,1}(z)) + b_{n,\alpha}(z)\end{aligned}$$

with uniform bounds on the L^∞ norms of $b_{n,\alpha}$. This gives the result because $P_0(\theta)$ is trivially bounded from above and below by positive constants, independently of $z \in \mathbb{T}$ and $n \geq 0$. $\square$

3. Proof of the theorem: Sufficiency

For each set $\mathcal{Z}(n)$, we define $C_p(\mathcal{Z}(n))$ as the minimum of all positive numbers C such that

$$\frac{C^{-1}}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p \leq \int_0^{2\pi}|P(e^{i\theta})|^p\,\frac{d\theta}{2\pi} \leq \frac{C}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p$$

for every complex polynomial P of degree at most n. Among all sets $\mathcal{Z}(n)$ satisfying $|\arg(z_{nj}\overline{\omega_{nj}})| \leq 2\pi\delta/(n+1)$ for $0 \leq j \leq n$, we may choose a set with maximal $C_p(\mathcal{Z}(n))$. From now on, we will assume that the points $z_{nj} = \omega_{nj}e^{\frac{2\pi i\delta_{nj}}{n+1}}$ constitute a set of points with this extremal property. It suffices to show that the corresponding triangular family is an L^p Marcinkiewicz–Zygmund family. Clearly, this family is uniformly separated when $\delta < 1/(2q)$.

When $p = 2$, condition (5) is equivalent to the following uniform Helson-Szegő condition: There exist sequences u_n and v_n of real functions in $L^\infty(\mathbb{T})$ such that

$$|F_n|^2 = e^{u_n+\widetilde{v_n}} \quad \text{with} \quad \sup_n \|u_n\|_\infty < \infty \quad \text{and} \quad \sup_n \|v_n\|_\infty < \pi/2. \tag{6}$$

Here $v \mapsto \widetilde{v}$ denotes the conjugation operator.

We need two steps in order to identify the appropriate functions u_n and v_n. In the first step, we "pull" the points z_{nj} more deeply into the unit disc. For $\kappa > 0$, we set $\rho_{\kappa n} = \max(1/2, 1 - \kappa/(n+1))$. We define

$$F_{\kappa n}(z) = \prod_{j=0}^{n}(1 - \rho_{\kappa n}\overline{z_{nj}}z).$$

For fixed $\kappa > 0$, we find that

$$|F_n(e^{it})|^2 = e^{u_{\kappa n}(e^{it})}|F_{\kappa n}(e^{it})|^2,$$

with $\sup_n \|u_{\kappa n}\|_\infty < \infty$.

We now move to the second step. Writing

$$B_{\kappa n}(z) = \prod_{j=0}^{n}\frac{z - \rho_{\kappa n}z_{nj}}{1 - \rho_{\kappa n}\overline{z_{nj}}z},$$

we get

$$B_{\kappa n}(z) = z^{n+1}\frac{\overline{F_{\kappa n}(z)}}{F_{\kappa n}(z)} = z^{n+1}\frac{|F_{\kappa n}(z)|^2}{F_{\kappa n}^2(z)}$$

for $|z|=1$. Since $F_{\kappa n}^2$ is an outer function with $F_{\kappa n}^2(0)=1$, this means that $F_{\kappa n}^2 = e^{\widetilde{v_{\kappa n}}}$, where

$$v_{\kappa n}(e^{i\theta}) = \int_0^\theta \sum_{j=0}^n \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n} z_{nj}|^2} d\eta - (n+1)\theta - c$$

and c is any suitable constant. If we set

$$c = \sum_{j=0}^n \int_{-2\pi\delta_j}^0 \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n}\omega_{nj}|^2} d\eta,$$

then we may write

$$v_{\kappa n}(e^{i\theta}) = \sum_{j=0}^n \int_0^{\theta-2\pi\delta_j} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n}\omega_{nj}|^2} d\eta - (n+1)\theta.$$

On the other hand, using that

$$\int_\theta^{\theta+2\pi/(n+1)} \sum_{j=0}^n \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n}\omega_{nj}|^2} d\eta = 2\pi \quad \text{and} \quad \left| \sum_{j=0}^n \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n}\omega_{nj}|^2} - (n+1) \right| \le \frac{C(n+1)}{\kappa},$$

we get

$$v_{\kappa n}(e^{i\theta}) = \sum_{j=0}^n \int_\theta^{\theta-2\pi\delta_j} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n}\omega_{nj}|^2} d\eta + O(\kappa^{-1})$$

when $\kappa\to\infty$. Consequently,

$$\|v_{\kappa n}\|_\infty \le \sup_\theta \int_\theta^{\theta+2\pi\delta/(n+1)} \sum_{j=0}^n \frac{1-\rho_{\kappa n}^2}{|e^{i\eta}-\rho_{\kappa n}\omega_{nj}|^2} d\eta + O(\kappa^{-1}) = 2\pi\delta + O(\kappa^{-1}).$$

Assuming $\delta < 1/4$, we now obtain (6) by choosing κ sufficiently large.

Finally, we consider the case $p \neq 2$. We introduce the triangular family given by the sets

$$\mathcal{Z}_{q/2}(n) = \{e^{i\lambda_{nj}(q/2)}\}_{j=0}^n \quad \text{with} \quad \lambda_{nj}(q/2) = \frac{2\pi j}{n+1} + \frac{\pi q \delta_{nj}}{n+1}.$$

If $\delta < 1/(2q)$, then the $p=2$ case applies. In other words, if we set

$$G_n(z) = \prod_{j=0}^n \left(1 - \rho_{\kappa n} e^{-i\lambda_{nj}(q)} z\right),$$

then the functions $|G_n|^2$ meet the uniform (A_2) condition. By Lemma 2.2 and Hölder's inequality, this implies that the functions $|F_n|^p$ satisfy the uniform (A_p) condition.

4. Proof of the theorem: Necessity

We will consider the sets

$$\mathcal{Z}(2n)=\left\{e^{2\pi ij/(2n+1)}\right\}_{j=0}^{n}\bigcup\left\{e^{-2\pi i(j-2\delta)/(2n+1)}\right\}_{j=1}^{n},$$

which can be viewed as perturbations of the rotated $(2n+1)$th roots of unity $e^{2\pi\delta/(2n+1)}\omega_{(2n)j}$. Let F_{2n} be the generating polynomial for $\mathcal{Z}(2n)$. We set $\phi_n(z)=F_{2n}(z)/(z^{2n+1}-\rho_{2n}^{2n+1})$ and observe that we may write

$$\phi_n(z)=\prod_{j=1}^{n}\frac{z-\rho_{2n}e^{\frac{-2\pi i(j-2\delta)}{2n+1}}}{z-\rho_{2n}e^{\frac{-2\pi ij}{2n+1}}}.$$

We have

$$\log|\phi_n(z)|=\Re(\log\phi_n(z))=\Re\sum_{j=1}^{n}\int_{\Gamma_{nj}}\frac{d\xi}{\xi-z},$$

where Γ_{nj} is the arc with the parametrization $\Gamma_{nj}(t)=\rho_{2n}e^{\frac{-2\pi ij}{2n+1}}e^{\frac{it}{2n+1}}$, $0\le t\le 4\delta\pi$. It follows that

$$|\phi_n(e^{it})|\longrightarrow\left|\frac{1-e^{it}}{1+e^{it}}\right|^{2\delta}$$

for $0<t<\pi$. By Fatou's lemma,

$$\left(\int_0^{\pi}\left|\frac{1-e^{it}}{1+e^{it}}\right|^{2\delta p}dt\right)\left(\int_0^{\pi}\left|\frac{1-e^{it}}{1+e^{it}}\right|^{-\frac{2\delta p}{p-1}}dt\right)^{p-1}\le\liminf_n\int_0^{\pi}|\phi_n|^p\left(\int_0^{\pi}|\phi_n|^{-\frac{p}{p-1}}\right)^{p-1}.$$

Hence, when $\delta=1/2q$, the weights $|\phi_n|^p$ do not meet the uniform (A_p) condition, and the same holds for the weights $|F_{2n}|^2$ since the polynomials $z^{2n+1}-\rho_{2n}^{2n+1}$ are uniformly bounded away from 0 for $|z|=1$.

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Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain
E-mail address: jmarzo@ub.edu

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway
E-mail address: seip@math.ntnu.no