

## The perfect lens on a finite bandwidth

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The resolution associated with the so-called perfect lens of thickness  $d$  is  $-2\pi d/\ln(|\chi+2|/2)$ . Here the susceptibility  $\chi$  is a Hermitian function in  $H^2$  of the upper half-plane, i.e., a  $H^2$  function satisfying  $\chi(-\omega)=\overline{\chi(\omega)}$ . An additional requirement is that the imaginary part of  $\chi$  be non-negative for non-negative arguments. Given an interval  $I$  on the positive half-axis, we compute the distance in  $L^\infty(I)$  from a negative constant to this class of functions. This result gives a surprisingly simple and explicit formula for the optimal resolution of the perfect lens on a finite bandwidth. © 2009 American Institute of Physics. [DOI: 10.1063/1.3068751]

### I. INTRODUCTION

The resolution associated with imaging in conventional optics is of the order of a wavelength. This is a severe limitation in a number of applications in nanoscience, e.g., in lithography, microscopy, and spectroscopy. A remedy to this impasse—a so-called perfect lens—was proposed by Pendry.<sup>1</sup> His idea was to use metamaterials that allow for tailoring the dielectric permittivity  $\epsilon$  and the magnetic permeability  $\mu$  by structuring the medium at a length scale much smaller than a wavelength.<sup>2-4</sup> This may lead to negative refraction and restoration of the near fields. The perfect lens is a negative refraction metamaterial slab of a certain thickness  $d$ . The metamaterial is considered linear, isotropic, homogeneous, and without spatial dispersion. These ideal limits may be approached by appropriate metamaterial designs. Note, however, that certain implementations may inherently violate some of these assumptions. For example, in a metal slab there is necessarily spatial dispersion (nonlocality) of the dielectric response; this limits the resolution to roughly 5 nm.<sup>5</sup>

As the permittivity and permeability of the perfect lens are negative, the material is necessarily dispersive.<sup>6</sup> Thus the perfect lens conditions  $\epsilon=-1$  and  $\mu=-1$  can only be approached at a single frequency. Since most practical applications involve a finite bandwidth, this fact limits the performance of the perfect lens.

In the present work, we will quantify the finite bandwidth behavior. The resolution at a single frequency is found to be  $r=-2\pi d/\ln(|\chi+2|/2)$ , where  $\chi=\epsilon-1$  or  $\chi=\mu-1$ , depending on the incident polarization. Thus the imaginary parts of  $\epsilon$  and  $\mu$  and the real parts' deviation from  $-1$  are both crucial for the resolution. The formula for  $r$  suggests an interesting mathematical problem: Given a physically realizable susceptibility and an interval  $I=[a,b]$  on the positive angular frequency half-axis, find the infimum of  $\|\chi+2\|_{L^\infty(I)}$ . Quite remarkably, this problem can be solved explicitly, and as a result we obtain a simple formula for the optimal resolution on a finite bandwidth.

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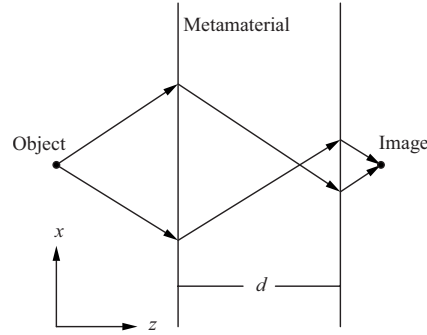


FIG. 1. The perfect lens: A metamaterial slab surrounded by vacuum. The distance from the object to the lens plus the distance from the lens to the image equals  $d$ . The arrows refer to the case when the metamaterial parameters  $\epsilon$  and  $\mu$  are close to  $-1$ .

## II. RESOLUTION AT A SINGLE FREQUENCY AND ON A FINITE BANDWIDTH

Let  $\omega$  denote the (real) angular frequency. The realizability criteria are causality, conjugate symmetry, and passivity. They can be expressed as follows:

$$\chi \in H^2, \quad (1a)$$

$$\chi(-\omega) = \overline{\chi(\omega)}, \quad (1b)$$

$$\text{Im } \chi(\omega) > 0 \quad \text{for } \omega > 0. \quad (1c)$$

The causality criterion (1a) stating that  $\chi$  belongs to the Hardy space  $H^2$  of the upper half-plane means that the real and imaginary parts of  $\chi$  form a Hilbert transform pair (Kramers–Kronig relations).<sup>7-9</sup> The conjugate symmetry (1b) is a result of the fact that the time-domain response function (inverse Fourier transform of  $\chi$ ) must be real.<sup>8</sup> Condition (1c) is valid for all passive media, i.e., media in thermodynamic equilibrium in the absence of the variable field.<sup>8</sup>

The resolution as a function of  $d$ ,  $\epsilon$ , and  $\mu$  can be found by solving Maxwell's equations.<sup>10,11</sup> First we assume that the object to be imaged is one dimensional. The slab has orientation orthogonal to the  $z$ -axis, the object varies along the  $x$ -direction, and the polarization of the magnetic field is taken to be along the  $y$ -axis, see Fig. 1. The object and the slab are surrounded by vacuum. For each spatial frequency  $k_x$  of the object, we define the transmission coefficient  $T$  as the ratio between the plane wave amplitudes at the image and the object. By matching tangential electric and magnetic fields at both surfaces, we find

$$T = \frac{4\epsilon k_z k'_z e^{i(k_z - k'_z)d}}{(\epsilon k_z + k'_z)^2 e^{-2ik'_z d} - (\epsilon k_z - k'_z)^2}, \quad (2)$$

where  $k_z^2 = \omega^2/c^2 - k_x^2$  and  $k'_z{}^2 = \epsilon\mu\omega^2/c^2 - k_x^2$ . The sign of  $k_z$  must be chosen such that  $\text{Im } k_z \geq 0$ , while the sign of  $k'_z$  does not matter in (2). We are now interested in what happens for large values of  $k_x$ , corresponding to near fields that decay exponentially in vacuum. We therefore assume that  $k_x \gg \omega/c$ . With the additional assumptions that  $|\epsilon+1| \ll 1$ ,  $|\mu|$  at most of the order of 1, and  $d$  at most of the order a vacuum wavelength ( $\omega d/c \lesssim 1$ ), we obtain

$$1/T \approx 1 - [(1 - \epsilon^2) - (\epsilon - \mu)(\omega/c k_x)^2]^2 \exp(2k_x d)/16, \quad (3)$$

where we have used that

$$\epsilon k_z + k'_z = (\epsilon^2 k_z^2 - k_z'^2) / (\epsilon k_z - k'_z) = [(1 - \epsilon^2) k_x^2 + \epsilon(\epsilon - \mu)(\omega/c)^2] / (\epsilon k_z - k'_z).$$

We take the resolution  $r$  to be the smallest value of  $2\pi/k_x$  such that the modulus of the second term on the right-hand side of (3) is equal to 1. This definition makes sense no matter what this term's phase happens to be, since the exponential factor  $\exp(2k_x d)$  will force  $|T|$  to decrease rapidly when  $2\pi/k_x$  gets smaller than  $r$ . In general, the resolution becomes a nontrivial function of  $\epsilon$  and  $\mu$ , but if  $|\epsilon - \mu|(\omega/c k_x)^2 \ll 2|\epsilon + 1|$ , then

$$r = -\frac{2\pi d}{\ln \frac{|\epsilon + 1|}{2}}. \quad (4)$$

By the assumption  $\omega/c k_x \ll 1$ , the requirement  $|\epsilon - \mu|(\omega/c k_x)^2 \ll 2|\epsilon + 1|$  can be rewritten as

$$|\epsilon - \mu|(\omega/c)d \lesssim -2|\epsilon + 1| \ln \frac{|\epsilon + 1|}{2}.$$

Thus (4) is always valid if  $\epsilon = \mu$ . Also, it is valid provided the lens is sufficiently thin. Note that in the latter case the resolution is independent of  $\mu$ . If we had chosen the opposite polarization (i.e., electric field along the  $y$ -axis), we would have arrived at exactly the same result only with the roles of  $\epsilon$  and  $\mu$  interchanged. In other words, for a one-dimensional object it is sufficient to have one of the parameters  $\epsilon$  and  $\mu$  close to  $-1$  with a small imaginary part.<sup>1</sup> For a two-dimensional object, both polarizations are necessarily present; thus both  $\epsilon$  and  $\mu$  should be close to  $-1$ .

The  $L^\infty$ -norm of the resolution restricted to the interval  $I$  measures the poorest resolution in the corresponding frequency band. In order to optimize the resolution, we should therefore minimize the  $L^\infty$  norm on  $I$ . In terms of the electric or magnetic susceptibility  $\chi = \epsilon - 1$  or  $\chi = \mu - 1$ , our task will therefore be to compute the following distance:

$$\inf \|\chi + 2\|_{L^\infty(I)}.$$

### III. THE MAIN RESULT

Before stating our main result, we introduce a few notational conventions.

For  $0 < p < \infty$ , the Hardy space  $H^p$  of the upper half-plane  $\{\zeta = \omega + i\eta; \eta > 0\}$  consists of those analytic functions  $f$  in this domain for which

$$\|f\|_{H^p}^p = \sup_{\eta > 0} \int_{-\infty}^{\infty} |f(\omega + i\eta)|^p d\omega < \infty;$$

$H^\infty$  is the space of bounded analytic functions. A function  $f$  in  $H^p$  has nontangential boundary limits at almost every point of the real axis, and the corresponding limit function, also denoted  $f$ , is in  $L^p = L^p(\mathbb{R})$ . Indeed, the  $L^p$  norm of the boundary limit function coincides with the  $H^p$  norm introduced above. Thus we may view  $H^p$  as a subspace of  $L^p$ .

As already noted [see (1)], we will require the following symmetry condition:  $f(-\omega) = \overline{f(\omega)}$ . Functions  $f$  satisfying this condition will be referred to as Hermitian functions. We observe that Hermitian functions have even real parts and odd imaginary parts.

The Hilbert transform of a function  $u$  in  $L^p$  ( $1 \leq p < \infty$ ) is defined as

$$\tilde{u}(\omega) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{\omega - t} dt.$$

It acts boundedly on  $L^p$  for  $1 < p < \infty$  and isometrically on  $L^2$ . If  $u$  is a real-valued function in  $L^p$  for  $1 < p < \infty$ , then  $u + i\tilde{u}$  is in  $H^p$ , and so the role of the Hilbert transform is to link the real and imaginary parts of functions in  $H^p$ . We will only work with Hermitian functions, and we will be

interested in computing real parts from imaginary parts. For this reason, it will be convenient for us to consider the following Hilbert operator:

$$\mathcal{H}v(\omega) = \text{p.v.} \frac{1}{\pi} \int_0^\infty v(t) \left( \frac{1}{t-\omega} + \frac{1}{t+\omega} \right) dt,$$

acting on functions in  $L^p(\mathbb{R}^+)$ . Provided  $1 < p < \infty$ , the function  $\mathcal{H}v + iv$  will then be in  $H^p$ , with the presumption that  $v$  is an odd function.

For a finite interval  $I=[a, b]$  ( $0 < a < b$ ) set

$$\Delta = \frac{b^2 - a^2}{b^2 + a^2}.$$

Our main theorem now reads as follows.

**Theorem 1:**

$$\inf\{\|\mathcal{H}v + iv + 2\|_{L^\infty(I)} : v \geq 0, v \in L^2(\mathbb{R}^+)\} = \frac{2\Delta}{1 + \sqrt{1 - \Delta^2}}.$$

It will become clear that the infimum is not a minimum, but we may extract from our proof explicit functions that bring us as close as we wish to the infimum.

#### IV. AUXILIARY RESULTS

The main result of Ref. 12 will play a central role in the proof of Theorem 1. To state it, we define for each real  $\alpha$  the family of functions

$$K_\alpha(I) = \{v \in L^2(\mathbb{R}^+) : v(\omega) \geq 0 \text{ for } \omega > 0, \mathcal{H}v(\omega) = \alpha \text{ for } \omega \in I\}.$$

(Here and elsewhere we suppress the obvious ‘‘almost everywhere’’ provisions needed when considering pointwise restrictions.) We think of functions in  $K_\alpha(I)$ , or more generally functions in  $L^2(\mathbb{R}^+)$ , as the imaginary parts of Hermitian functions, and we view them therefore as odd functions on  $\mathbb{R}$ .

We also need the function

$$\sigma(\zeta) = \frac{1}{\sqrt{\zeta^2 - b^2} \sqrt{\zeta^2 - a^2}}.$$

It is taken to be positive for real arguments  $w > b$  and is analytic in the slit plane  $\mathbb{C} \setminus ([-b, -a] \cup [a, b])$ . For real arguments  $a < |\zeta| < b$  we define  $\sigma(\zeta)$  by extending it continuously from the upper half-plane. Thus  $\sigma(\zeta)$  takes values on the negative imaginary half-axis when  $\zeta$  is in  $(a, b)$  and on the positive imaginary half-axis when  $-\zeta$  is in  $(a, b)$ , and otherwise it is real for real arguments.

Let us also associate with the interval  $I$  the following Hilbert operator:

$$\mathcal{H}_I v(\omega) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}^+ \setminus I} v(t) \left( \frac{1}{t-\omega} + \frac{1}{t+\omega} \right) dt.$$

The main result of Ref. 12 was the following parametrization of  $K_\alpha(I)$ .

**Theorem 2:** A nonnegative function  $v$  in  $L^2(\mathbb{R}^+)$  is in  $K_\alpha(I)$  if and only if the following three conditions hold:

$$\int_{\mathbb{R}^+ \setminus I} v(t) |\sigma(t)| dt < \infty, \tag{5}$$

$$\frac{2}{\pi} \int_{\mathbb{R}^+ \setminus I} tv(t)\sigma(t)dt = \alpha, \quad (6)$$

$$v(\omega) = \mathcal{H}_I(\sigma v)(\omega)/|\sigma(\omega)|, \quad \omega \in I. \quad (7)$$

The integrability condition (5) is merely a slight growth condition at the end points of  $I$ ; we may write it more succinctly as

$$\int_0^a [v(a-t) + v(b+t)] \frac{dt}{\sqrt{t}} < \infty.$$

This condition ensures that the integral in (6) and the Hilbert transform appearing in (7) are both well defined.

At first sight, the theorem may not seem to give an explicit parametrization of  $K_\alpha(I)$ . However, the Hilbert transform appearing in (7) is given by

$$\mathcal{H}_I(\sigma v)(\omega) = \frac{1}{\pi} \int_{\mathbb{R}^+ \setminus I} v(t)\sigma(t) \frac{2t}{t^2 - \omega^2} dt,$$

and we observe that the integrand on the right is non-negative whenever  $v(t)$  is non-negative. Hence  $v(\omega) \geq 0$  for  $\omega$  off  $I$  implies  $v(\omega) \geq 0$  for  $\omega$  in  $I$ . This small miracle implies that  $K_\alpha(I)$  is parametrized by those non-negative functions  $v$  in  $L^2(\mathbb{R}^+ \setminus I)$  for which (5) and (6) hold and such that

$$\int_I |\mathcal{H}_I(\sigma v)(\omega)|^2 |\sigma(\omega)|^{-2} d\omega < \infty.$$

By rephrasing this condition in more explicit terms, we arrive at the following corollary.<sup>12</sup>

**Corollary 1:** *A non-negative function  $v$  in  $L^2(\mathbb{R}^+ \setminus I)$  has an extension to a function in some class  $K_\alpha(I)$  if and only if the following condition holds:*

$$\int_0^a \int_0^a [v(a-t)v(a-\tau) + v(b+t)v(b+\tau)] \frac{|\log(t+\tau)|}{\sqrt{t\tau}} dt d\tau < \infty.$$

The difference between (5) and the condition above is the logarithmic factor, which means that the condition of the corollary is only a very slight strengthening of (5). It is clear that, for instance, boundedness of  $v$  near the end points of  $I$  is more than enough.

We note that the integrand in (6) is negative to the left of  $I$  and positive to the right of  $I$ . This means that if  $\alpha$  is negative, then

$$|\alpha| \leq \frac{2}{\pi} \int_0^a tv(t)|\sigma(t)|dt,$$

with equality holding if  $v$  vanishes to the right of  $I$ . It follows that

$$\mathcal{H}_I(\sigma v)(\omega) \geq \frac{1}{\pi} \int_0^a v(t)\sigma(t) \frac{2t}{t^2 - \omega^2} dt \geq \frac{|\alpha|}{\omega^2}, \quad (8)$$

we may come as close as we wish to this lower bound by choosing any suitable  $v$  supported by a small set sufficiently close to 0.

We remark that the results stated above can be proven by a method similar to that given in Sec. V, the key ingredient being a conformal map sending the “two-sided” segment  $[a^2, b^2]$  to the unit circle.

## V. PROOF OF THEOREM 1

In what follows,  $H^\infty(\Omega)$  denotes the space of bounded analytic functions on  $\Omega$  equipped with the supremum norm, and  $H^p(\mathbb{D})$  stands for the  $H^p$  spaces of the open unit disk  $\mathbb{D}$ .

Let  $v$  be an arbitrary function in  $L^2(\mathbb{R}^+)$  such that  $v \geq 0$  and

$$\|\mathcal{H}v + iv\|_{L^\infty(I)} < \infty.$$

We will also assume<sup>13</sup> that  $\nu = v|_{\mathbb{R}^+ \setminus I}$  satisfies the condition of Corollary 1. We know from Theorem 2 that the extension of  $\nu$  to a function in some class  $K_\alpha(I)$  is unique, and we may therefore use the notation  $\nu$  for this extension as well. If now  $\varphi$  denotes an arbitrary bounded function supported on  $I$  with  $\mathcal{H}\varphi$  also bounded, we get the inequality

$$\|\mathcal{H}v + iv + 2\|_{L^\infty(I)} \geq \inf_{\varphi} \|\mathcal{H}\varphi + i\varphi + i\nu + 2 + \alpha\|_{L^\infty(I)}.$$

Setting

$$\nu_0(t) = \begin{cases} \nu(t), & t \in I \\ 0, & t \in \mathbb{R}^+ \setminus I, \end{cases}$$

we may write

$$\|\mathcal{H}v + iv + 2\|_{L^\infty(I)} \geq \inf_{\varphi} \|\mathcal{H}\varphi + i\varphi - \frac{1}{2}(\mathcal{H}\nu_0 - i\nu_0) + 2 + \alpha\|_{L^\infty(I)}.$$

The function  $\mathcal{H}\varphi + i\varphi$  is the boundary limit function of a bounded analytic function in the upper half-plane. In fact, since the imaginary part has limit 0 for every point in  $\mathbb{R} \setminus ([-b, -a] \cup [a, b])$ , this function extends by Schwarz reflection to a bounded analytic function  $f$  in  $\mathbb{C}^* \setminus ([-b, -a] \cup [a, b])$ , where  $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ . This function  $f$  satisfies  $f(\bar{\zeta}) = \overline{f(\zeta)}$  and  $f(\infty) = 0$ . The function  $\mathcal{H}\nu_0 + i\nu_0$  extends in the same fashion to a function  $g$  satisfying  $g(\bar{\zeta}) = \overline{g(\zeta)}$  and  $g(\infty) = 0$ . Both these functions are, in fact, analytic in the variable  $\xi = \zeta^2$ ; we write

$$F(\xi) = f(\sqrt{\xi}) \quad \text{and} \quad G(\xi) = g(\sqrt{\xi}),$$

and then it follows that

$$\inf_{\varphi} \|\mathcal{H}\varphi + i\varphi - \frac{1}{2}(\mathcal{H}\nu_0 - i\nu_0) + 2 + \alpha\|_{L^\infty(I)} = \inf_{F: F(\infty)=0} \|F + \frac{1}{2}\bar{G} + 2 + \alpha\|_{H^\infty(\mathbb{C}^* \setminus [a^2, b^2])}.$$

The remaining computation is most easily done if we first map  $\mathbb{C}^* \setminus [a^2, b^2]$  conformally onto the open unit disk  $\mathbb{D}$ , say by the map

$$w = w(\xi) = \frac{2}{b^2 - a^2} \left( \xi - \frac{b^2 + a^2}{2} + \sqrt{\left( \xi - \frac{b^2 + a^2}{2} \right)^2 - \left( \frac{b^2 - a^2}{2} \right)^2} \right),$$

where the square root is positive for positive arguments. We write  $\Gamma(w) = G(\xi(w))$  and obtain

$$\inf_{\varphi} \|\mathcal{H}\varphi + i\varphi - \frac{1}{2}(\mathcal{H}\nu_0 - i\nu_0) + 2 + \alpha\|_{L^\infty(I)} = \inf_{F: F(0)=0} \|F + \frac{1}{2}\bar{\Gamma} + 2 + \alpha\|_{H^\infty(\mathbb{D})}.$$

Since  $\Gamma(0) = F(0) = 0$ , we have by orthogonality

$$\|F + \frac{1}{2}\bar{\Gamma} + 2 + \alpha\|_{H^\infty(\mathbb{D})}^2 \geq \|\Gamma\|_{H^2(\mathbb{D})}^2 + (2 + \alpha)^2.$$

We may assume  $\alpha \leq 0$  since for  $\alpha = 0$  we may choose  $\Gamma \equiv 0$ . By (7) and (8), the expression on the right-hand side is minimal if

$$\nu_0(\omega) = \frac{|\alpha| \sqrt{(b^2 - \omega^2)(\omega^2 - a^2)}}{\omega^2}.$$

For  $|w|=1$ , we therefore get

$$\operatorname{Im} \Gamma(w) = \frac{|\alpha| \operatorname{Im} w}{\Delta^{-1} + \operatorname{Re} w}.$$

Hence

$$\Gamma(w) = \frac{2\delta|\alpha|w}{1 + \delta w},$$

where  $\delta < 1$  is determined by the equation

$$\frac{2\delta}{1 + \delta^2} = \Delta,$$

or in other words,

$$\delta = \Delta^{-1} - \sqrt{\Delta^{-2} - 1} = \frac{\Delta}{1 + \sqrt{1 - \Delta^2}}.$$

It follows that

$$\left\| F + \frac{1}{2}\bar{\Gamma} + 2 + \alpha \right\|_{H^\infty(\mathbb{D})} \geq \left( (2 + \alpha)^2 + \alpha^2 \frac{\delta^2}{1 - \delta^2} \right)^{1/2} \geq 2\delta,$$

the minimum on the right is obtained when

$$\alpha = -2(1 - \delta^2).$$

It remains for us to prove the remarkable fact that this lower bound is, in fact, an infimum. To this end, observe that

$$2\delta^2 + 2(1 - \delta^2)\delta \frac{w}{1 + \delta w} = 2\delta \frac{w + \delta}{1 + \delta w}.$$

In other words, the minimum is achieved if we choose  $\alpha = -2(1 - \delta^2)$ ,  $\varphi \equiv 0$ , and

$$\Gamma(w) = \frac{2\delta(1 - \delta^2)w}{1 + \delta w}.$$

In view of (7) and (8), we see that we can get as close as we wish to the associated minimum by picking a function  $v \geq 0$  such that  $v|_{\mathbb{R}^+ \setminus V}$  is supported on a small set close to 0,

$$\frac{2}{\pi} \int_{\mathbb{R}^+ \setminus V} tv(t)\sigma(t)dt = -2(1 - \delta^2) \quad (9)$$

and

$$v(\omega) = \frac{1}{2} \mathcal{H}_I(\sigma v)(\omega) / |\sigma(\omega)|, \quad \omega \in I. \quad (10)$$

## VI. DISCUSSION AND CONCLUSION

Theorem 1 together with (4) gives the optimal resolution as a function of bandwidth,

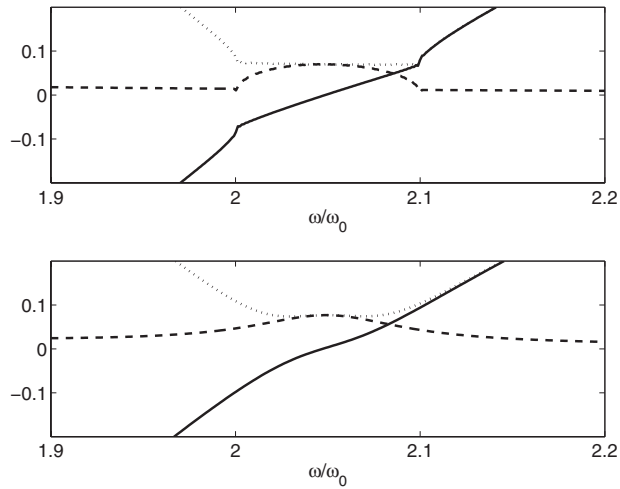


FIG. 2. The real parts (solid lines) and imaginary parts (dashed lines) of  $\chi(\omega)+2$  for a near-optimal case (upper plot) and a Lorentzian approximation (lower plot). Also shown is the associated modulus  $|\chi(\omega)+2|$  (dotted lines). Only the frequency range in the vicinity of  $I$  is shown; in addition there is a strong resonance at  $\omega_0$ . The Hilbert transforms are computed using fast Fourier transforms. The parameters used are  $a=2\omega_0$ ,  $b=2.1\omega_0$ , and  $\gamma=0.1\omega_0$ .

$$r = -\frac{2\pi d}{\ln \frac{\Delta}{1 + \sqrt{1 - \Delta^2}}} \approx -\frac{2\pi d}{\ln \frac{\Delta}{2}}, \quad (11)$$

where the approximation is valid for  $\Delta \ll 1$ . In this limit,  $\Delta \approx (b-a)/b$  is the relative bandwidth. This optimal resolution is approached when the susceptibility  $\chi(\omega) = u(\omega) + iv(\omega)$  contains a strong resonance at low frequencies and a weak resonance centered at the relevant bandwidth. As an example, let  $v(\omega)$  off  $I$  be the imaginary part of a Lorentzian resonance function, i.e.,  $v(\omega) = \text{Im } l(\omega)$  for  $\omega \in \mathbb{R}^+ \setminus I$ , where

$$l(\omega) = \frac{L\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\gamma}.$$

Here the resonance frequency  $\omega_0$  and the bandwidth  $\gamma$  are positive constants, while the strength  $L$  is found from the requirement (9). Computing  $v(\omega)$  for  $\omega \in I$  using (10), and computing  $u(\omega) = \mathcal{H}v(\omega)$ , we obtain the upper plot in Fig. 2. Note that with this procedure, one may get arbitrarily close to the bound (11) by choosing  $\omega_0$  and  $\gamma$  sufficiently small. Although realizable in principle, it might be difficult to fabricate a metamaterial with this near-optimal response. As an alternative, we can approximate the near-optimal susceptibility by letting  $\chi(\omega) = u(\omega) + iv(\omega)$  be the superposition of two Lorentzian functions; one centered at  $\omega_0$  and one centered at  $(a+b)/2$ . We let the first Lorentzian be equal to that in the former example. The bandwidth of the second Lorentzian is  $b-a$ , and we choose the strength such that  $du((a+b)/2)/d\omega$  coincides with the corresponding value for the near-optimal case. The result is given in the lower plot in Fig. 2.

Since the distance from the object to the lens plus the distance from the lens to the image equals  $d$ , there may be practical reasons for not reducing  $d$ . If so, the optimal resolution  $r$  can only be reduced by shrinking the operational bandwidth. Unfortunately, the logarithmic dependence of  $\Delta$  may require an unpractical, small bandwidth.

It has been suggested to use a multilayer stack of alternating negative index and positive index materials as the lens.<sup>14</sup> This effectively reduces  $d$  in (11); however, then the distance from the object to the lens plus the distance from the lens to the image equals the thickness of each layer. In the limit when the layer thicknesses approach zero, the resulting effective medium acts as a fiber-optic bundle, but one that even acts on the near field.<sup>14</sup> To optimize the resolution and

minimize aberrations, one again ends up with the problem of minimizing  $|\epsilon+1|$  and/or  $|\mu+1|$ . If aberrations can be tolerated, only  $\text{Im } \epsilon$  and/or  $\text{Im } \mu$  need to be minimized. This can be achieved on a finite bandwidth at the expense of some variation in  $\text{Re } \epsilon$  and/or  $\text{Re } \mu$ .<sup>8,15</sup>

By simple scaling, Theorem 1 gives the maximum operational bandwidth of all components with desired permittivity  $\epsilon_{\text{des}}$  and/or permeability  $\mu_{\text{des}}$  less than unity; with a certain tolerance of  $|\epsilon-\epsilon_{\text{des}}|$  and/or  $|\mu-\mu_{\text{des}}|$ . For components with permittivity and permeability larger than or equal to unity, there is no bandwidth limitation resulting from (11). Our result may, for instance, prove useful for establishing the maximum operational bandwidth of invisibility cloaks,<sup>16-18</sup> defining the bandwidth as the frequency range where a suitable figure-of-merit function is within a prescribed tolerance.

The resolution formula (11) has been derived for passive media. For active media, the situation is different; the susceptibilities can approach any desired square integrable function on a finite bandwidth.<sup>15,19</sup> More precisely, any function in  $L^2(a,b)$  can be approximated by a function satisfying the conditions of causality (1) and conjugate symmetry (1), with any precision in  $L^2$  norm. Thus, in principle, for active media the perfect lens condition  $\epsilon=\mu=-1$  can be approached with any precision, even on a finite bandwidth.

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- <sup>1</sup>J. B. Pendry, *Phys. Rev. Lett.* **85**, 3966 (2000).
- <sup>2</sup>J. B. Pendry, A. J. Holden, W. J. Stewart, and I. Youngs, *Phys. Rev. Lett.* **76**, 4773 (1996).
- <sup>3</sup>J. B. Pendry, A. J. Holden, D. J. Robbins, and W. J. Stewart, *IEEE Trans. Microwave Theory Tech.* **47**, 2075 (1999).
- <sup>4</sup>D. R. Smith, W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, *Phys. Rev. Lett.* **84**, 4184 (2000).
- <sup>5</sup>I. A. Larkin and M. I. Stockman, *Nano Lett.* **5**, 339 (2005).
- <sup>6</sup>V. G. Veselago, *Sov. Phys. Usp.* **10**, 509 (1968).
- <sup>7</sup>H. M. Nussenzveig, *Causality and dispersion relations* (Academic, New York, 1972), Chap. 1.
- <sup>8</sup>L. D. Landau and E. M. Lifshitz, *Electrodynamics of continuous media* (Pergamon, New York, 1960), Chap. 9.
- <sup>9</sup>If the medium is conducting at zero frequency, the electric susceptibility  $\epsilon-1$  is singular at  $\omega=0$ . Then (1) is still valid provided we set  $\chi=\epsilon-1-i\sigma/\omega$ , where  $\sigma>0$  is the zero frequency conductivity (Ref. 8). In other words, such a medium will have larger loss  $\text{Im } \epsilon$  for the same variation in  $\text{Re } \epsilon$ . With no loss of generality we can therefore exclude such media.
- <sup>10</sup>S. A. Ramakrishna, J. B. Pendry, D. Schurig, D. R. Smith, and S. Schultz, *J. Mod. Opt.* **49**, 1747 (2002).
- <sup>11</sup>M. Nieto-Vesperinas, *J. Opt. Soc. Am. A* **21**, 491 (2004).
- <sup>12</sup>K. Seip and J. Skaar, *Skr. K. Nor. Videns. Selsk.* **3**, 1 (2005); e-print arXiv:math.CV/0506620v1.
- <sup>13</sup>This assumption may seem unjustified. However, we may first restrict attention to the smaller interval  $I_\epsilon=[a+\epsilon, b-\epsilon]$  so that  $v$  is bounded near the endpoints of  $I_\epsilon$ . Letting  $\epsilon\rightarrow 0$ , we would then obtain the same lower bound as we do with our *a priori* assumption.
- <sup>14</sup>S. A. Ramakrishna, J. B. Pendry, M. C. K. Wiltshire, and W. J. Stewart, *J. Mod. Opt.* **50**, 1419 (2003).
- <sup>15</sup>J. Skaar and K. Seip, *J. Phys. D* **39**, 1226 (2006).
- <sup>16</sup>J. B. Pendry, D. Schurig, and D. R. Smith, *Science* **312**, 1780 (2006).
- <sup>17</sup>U. Leonhardt and T. G. Philbin, *New J. Phys.* **8**, 247 (2006).
- <sup>18</sup>T. Ochiai, U. Leonhardt, and J. C. Nacher, *J. Math. Phys.* **49**, 032903 (2008).
- <sup>19</sup>B. Nistad and J. Skaar, *Phys. Rev. E* **78**, 036603 (2008).