SURJECTIVE TOEPLITZ OPERATORS

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ABSTRACT. We obtain a necessary and sufficient condition for a noninjective Toeplitz operator on H^2 of the unit disk to be surjective. The condition involves the extremal function for the kernel of the operator. The canonical right inverse of a surjective Toeplitz operator is shown to be a product of three Toeplitz operators.

1. INTRODUCTION

We obtain in this note a characterization of surjectivity for noninjective Toeplitz operators on the Hardy space H^2 of the unit disk. The characterization is in terms of the extremal function for the kernel of the operator in question, and it is closely related to the invertibility criterion for Toeplitz operators of A. Devinatz and H. Widom, to be stated below. In order to formulate our result, and to place it in context, we first need to review some basic properties of Toeplitz operators and related matters.

Our notations are standard. We denote the open unit disk by $\mathbb{D}$, the unit circle by $\mathbb{T}$, and normalized Lebesgue measure on $\mathbb{T}$ by m. For φ in L^∞ $(= L^\infty(m))$, the Toeplitz operator T_φ on H^2 with symbol φ is given by $T_\varphi f = P_+(\varphi f)$, where P_+ is orthogonal projection in L^2 with range H^2. The shift operator on H^2, which is the Toeplitz operator whose symbol is the coordinate function, is denoted by S.

As usual in this subject, outer functions and inner functions play prominent roles. It will at times be convenient to allow such functions to be unnormalized. (An outer function is normalized if it is positive at the origin, an inner function if its first nonvanishing power series coefficient at the origin is positive.)

The following basic results enable us to limit our attention, as far as our main result goes, to Toeplitz operators with unimodular symbols. (The first result is immediate and well known to anyone who works with Toeplitz operators; see, e.g., [BöSi90, p. 65] for a proof of the second result.)

Product Lemma. *If φ and ψ are in L^∞ and at least one of them is in H^∞, then $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$.*

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Hartman-Wintner Theorem. *If φ is in L^∞ but not invertible in L^∞, then T_φ is not bounded below.*

If T_φ is surjective, then $T_{\overline{\varphi}}$ is left-invertible, hence bounded below. By the Hartman–Wintner theorem, φ is then invertible in L^∞. Assuming that to be the case, we let h be the outer function (normalized, for definiteness) whose modulus on $\mathbb{T}$ equals $|\varphi|$, and set $\psi = \varphi/\overline{h}$. Thus ψ is unimodular, and the product lemma gives $T_\varphi = T_{\overline{h}}T_\psi$. Since $T_{\overline{h}}$ is invertible, the operator T_φ is surjective if and only if T_ψ is, and the two operators have the same kernel. Because of this, we shall assume henceforth that our symbol φ is unimodular.

The Devinatz–Widom theorem, which characterizes the invertibility of T_φ, is closely related to a theorem in prediction theory of H. Helson and G. Szegő [HeSz60]. We recall that a Helson–Szegő weight is a function on $\mathbb{T}$ of the form $e^{u+\tilde{v}}$, where u and v are real functions in L^∞, $\|v\|_\infty < \pi/2$, and $\tilde{v}$ denotes the conjugate function of v.

Devinatz–Widom Theorem. (See, e.g., [BöSi90, pp. 58–59] or [Ni02, Vol. 1, p. 250].) *For a unimodular function φ the following conditions are equivalent.*

(i) T_φ *is invertible.*
(ii) $\operatorname{dist}(\varphi, H^\infty) < 1$ *and* $\operatorname{dist}(\overline{\varphi}, H^\infty) < 1$.
(iii) *There is an outer function h in H^∞ such that* $\|\varphi - h\|_\infty < 1$.
(iv) *There is an outer function G in H^2 such that $\varphi = \overline{G}/G$ and $|G|^2$ is a Helson–Szegő weight.*

The equivalence of (i) and (ii) has a variant for left invertibility, also due to Devinatz and Widom: if φ is unimodular, then T_φ is left-invertible if and only if $\operatorname{dist}(\varphi, H^\infty) < 1$ [BöSi90, pp. 58–59]. T. Nakazi [Na93, p. 444] pointed out the following consequence: if φ is unimodular, then T_φ is left-invertible if and only if there is an inner function J such that $T_{\overline{J}\varphi}$ is invertible. In fact, the sufficiency of the preceding condition for left invertibility is immediate, because $T_\varphi = T_{\overline{J}\varphi}T_J$. Conversely, if T_φ is left-invertible, then by the Devinatz–Widom criterion for left invertibility, there is a function h in H^∞ such that $\|\varphi - h\|_\infty < 1$. If $h = Jh_0$ is the inner-outer factorization of h, then $\|\overline{J}\varphi - h_0\|_\infty < 1$, so the equivalence of (i) and (iii) in the Devinatz–Widom theorem tells us that $T_{\overline{J}\varphi}$ is invertible. Of course, if T_φ is actually invertible, then J must be a constant, but otherwise the inner function J is highly nonunique, for one can replace h in the preceding argument by small perturbations of itself.

Nakazi's observation can be translated into a criterion for surjectivity, since surjectivity and right invertibility are equivalent: T_φ is surjective if and only if there is an inner function J such that $T_{J\varphi}$ is invertible. Using condition (iv) in the Devinatz–Widom theorem, we can restate this as follows: if φ is unimodular, then T_φ is surjective if and only if there are an inner function J and an outer function G such that $|G|^2$ is a Helson–Szegő weight and $\varphi = \overline{JG}/G$. But again, if T_φ is not invertible, the case of interest to us, J and G are highly nonunique, leaving this surjectivity criterion rather amorphous.

We shall now state an alternative characterization of surjectivity, our main result, after a few more preliminaries. Let φ be a unimodular function such that T_φ has a nontrivial kernel. Our analysis will be based on work of E. Hayashi [Ha85], [Ha86], [Ha90] on kernels of Toeplitz operators, and on work of D. Hitt [Hi88] on nearly S^*-invariant subspaces, of which kernels of Toeplitz operators form a subclass. A subspace M of H^2 is said to be nearly invariant under the backward shift S^* if it contains S^*f whenever it contains f and $f(0) = 0$. The extremal function of such a subspace M, assumed to be nontrivial, is the function g in it of unit norm for which $\operatorname{Re} g(0)$ is maximum; equivalently, it is the unique function in $M \ominus (M \cap H_0^2)$ that has unit norm and is positive at the origin. Hitt proved that $M' = \{f/g : \ f \in M\}$ is an S^*-invariant subspace of H^2, and, moreover, multiplication by g maps M' isometrically onto M.

When $M = \ker T_\varphi$, the case of interest here, the extremal function g is easily seen to be an outer function, and M' must be proper, hence of the form $H^2 \ominus IH^2$ for an inner function I. The function I vanishes at the origin since M' contains the constant function 1, and if I is properly scaled, one has $\varphi = \overline{I}\overline{g}/g$. (See [Sa94b, p. 161] for the simple argument.) In addition, the H^1-function g^2 is a rigid function, meaning the only other functions in H^1 having the same argument as g^2 almost everywhere on $\mathbb{T}$ are the positive scalar multiples of g^2. In fact, from the equality $\varphi = \overline{I}\overline{g}/g$ and the form of $\ker T_\varphi$ one sees that $T_{\overline{g}/g}$ has a trivial kernel, a condition known to be equivalent to the rigidity of g^2 (see [Sa94a, p.70]). These results about kernels of Toeplitz operators were obtained, independently of Hitt, by Hayashi [Ha85], [Ha86]. We note that a sufficient (but not a necessary) condition for an H^1-function to be rigid is that its reciprocal be in H^1 (see [Sa94a, pp. 70–71]), a condition that holds, in particular, if the function is outer and its modulus is a Helson–Szegő weight. This means that the function G of condition (iv) in the Devinatz–Widom theorem is unique up to multiplication by nonzero real numbers, a fact that will be used without explicit mention in the sequel.

Associated with the extremal function g is a pair of functions a and b in the unit ball of H^∞, in terms of which g has the expression

$$g = \frac{a}{1-b}.$$

The function a is outer, the function b is given in terms of the Herglotz integral of $|g|^2$ by

$$\frac{1+b(z)}{1-b(z)} = \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, |g(\zeta)|^2\, dm(\zeta),$$

and the equality $|a|^2 + |b|^2 = 1$ holds almost everywhere on $\mathbb{T}$. More on this can be found in the paper [Sa88], which presents an alternative approach to and an extension of Hitt's characterization of nearly S^*-invariant subspaces. In particular, it is proved there that the inner function I divides the function b, so that we can write $b = Ib_0$ with b_0 in the unit ball of H^∞. The function

$$g_0 = \frac{a}{1-b_0}$$

is in H^2, and Hayashi proved in [Ha90] (see also [Sa94b]) that g_0^2 is rigid and $(1-|b_0|^2)/|1-b_0|^2$ is the Poisson integral of $|g_0|^2$ (so that $(1+b_0)/(1-b_0)$ is, to within addition of an imaginary constant, the Herglotz integral of $|g_0|^2$). This is one direction in his characterization of the kernels of Toeplitz operators.

The full theorem of Hayashi provides a recipe for constructing the most general nontrivial kernel of a Toeplitz operator. One starts with functions a, b_0, I such that (i) a and b_0 are in H^∞, a is outer, $a(0) > 0$, and $|a|^2 + |b_0|^2 = 1$ on $\mathbb{T}$; (ii) g_0^2 is rigid, where $g_0 = a/(1-b_0)$; (iii) the measure having Poisson integral $(1-|b_0|^2)/|1-b_0|^2$ is absolutely continuous; (iv) I is inner, with $I(0) = 0$. The function $g = a/(1-b)$, where $b = Ib_0$, is then the extremal function for the nearly invariant subspace $g(H^2 \ominus IH^2)$, and that subspace is the kernel of T_φ, where $\varphi = \overline{I}\overline{g}/g$. Hayashi's recipe underlies the examples we shall give at the ends of Sections 3 and 4.

Retaining the preceding notations, we can now state our main result.

Theorem 1. *The operator T_φ is surjective if and only if $|g_0|^2$ is a Helson–Szegő weight.*

As we shall show below, the surjectivity of T_φ does not imply that $|g|^2$ itself is a Helson–Szegő weight. If T_φ is surjective and $|g|^2$ is not a Helson–Szegő weight, the inner function I associated with $\ker T_\varphi$ cannot serve as one of the inner functions J appearing in the surjectivity criterion discussed earlier coming from Nakazi's observation.

A proof of Theorem 1 is given in Section 3; it involves de Branges–Rovnyak spaces. Those spaces were the basis of the analysis in [Sa88], and also in [Sa94b], which contains an alternative approach to Hayashi's characterization of kernels of Toeplitz operators. The needed properties of de Branges–Rovnyak spaces are reviewed in Section 2. In Section 4 we make some explicit computations of inverse images under T_φ of reproducing kernel functions. This will enable us, in case T_φ is surjective, to express in terms of Toeplitz operators the right inverse of T_φ whose range is orthogonal to $\ker T_\varphi$. It will also lead to an alternative proof of Theorem 1.

To close this introduction we mention that the projection operator P_+ can be applied to L^1-functions as well as to L^2-functions. For f in L^1 we interpret P_+f as the Cauchy integral of f, the function in $\mathbb{D}$ given by

$$(P_+f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\overline{\zeta}z}\, dm(\zeta).$$

Of course, P_+ is unbounded as an operator on L^1, but it is continuous relative to the norm topology in L^1 and the topology of locally uniform convergence in $H(\mathbb{D})$, the space of holomorphic functions in $\mathbb{D}$.

A Toeplitz operator with an unbounded symbol in L^2 can be regarded, according to convenience, either as an unbounded operator on H^2 (with domain containing H^∞) or, via the preceding interpretation of P_+, as a continuous operator from H^2 to $H(\mathbb{D})$. Below we shall have occasion to form products of such operators. This will present no technical problems worth mentioning, and basic properties of bounded Toeplitz operators, such as the product lemma, are easily seen to carry over. It is not uncommon for the product of two unbounded Toeplitz operators to be bounded. For example, if our operator T_φ is invertible, and if G is as in part (iv) of the Devinatz–Widom theorem, then $T_\varphi^{-1} = T_G T_{1/\overline{G}}$. (Note that $1/\overline{G}$ is in L^2 because $|G|^2$ is in this case a Helson–Szegő weight.)

2. de Branges–Rovnyak spaces

We introduce just those properties of de Branges–Rovnyak spaces relevant for present purposes. The spaces were first studied in [dBrRo66]; a more recent account is [Sa94a].

We assume, as in the second half of Section 1, that φ is a unimodular function such that T_φ has a nontrivial kernel. We retain the notations g, I, a, b, b_0, g_0 introduced earlier. The space $\mathcal{H}(b)$ is by definition the range of the operator $(1-T_bT_{\overline{b}})^{1/2}$ with the range norm, i.e., the norm that makes $(1-T_bT_{\overline{b}})^{1/2}$ a coisometry of H^2 onto $\mathcal{H}(b)$. The norm on $\mathcal{H}(b)$ will be denoted by $\|\cdot\|_b$. The spaces $\mathcal{H}(b_0)$ and $\mathcal{H}(I)$ are defined analogously. In particular, $\mathcal{H}(I)$ is just the range of the projection $1-T_IT_{\overline{I}}$, in other words, it is the S^*-invariant subspace $H^2 \ominus IH^2$. Thus $\ker T_\varphi = T_g\mathcal{H}(I)$.

The space $\mathcal{M}(a)$ is by definition the range of the operator T_a, with the range norm. We note the inclusion $\mathcal{M}(a) \subset \mathcal{H}(b)$ (see [Sa94a, p. 24]). The norm in $\mathcal{M}(a)$ will not play a role here.

Proposition A. (See [Sa94a, p. 10].) *The space $\mathcal{H}(b)$ is the orthogonal direct sum of $\mathcal{H}(I)$ and $T_I\mathcal{H}(b_0)$. The operator T_I acts as an isometry of $\mathcal{H}(b_0)$ into $\mathcal{H}(b)$.*

Proposition B. (See [Sa94a, p. 30].) *The operator $T_{1-b}T_{\overline{g}}$ is an isometry of H^2 onto $\mathcal{H}(b)$, and the operator $T_{1-b_0}T_{\overline{g_0}}$ is an isometry of H^2 onto $\mathcal{H}(b_0)$.*

Proposition C. (See [Sa94b, p. 161].) *The operator $T_{1-b}T_{\overline{g}}$ maps $\ker T_\varphi$ onto $\mathcal{H}(I)$. It acts on $\ker T_\varphi$ as division by g.*

By a corona pair is meant a pair of functions in H^∞ that satisfies the hypothesis of the corona theorem, namely, the maximum of the moduli of the functions is bounded away from 0 in $\mathbb{D}$.

Proposition D. (See [Sa94a, p. 66].) *The following conditions are equivalent.*

(i) $\mathcal{H}(b) = \mathcal{M}(a)$
(ii) $T_{\overline{g}/g}$ *is invertible*
(iii) $T_{\overline{a}/a}$ *is invertible, and the functions a and b form a corona pair.*

3. Proof of Theorem 1

As before, we assume φ is a unimodular function such that T_φ has a nontrivial kernel. In addition to the extremal function g and the associated functions I, a, b, b_0, g_0 introduced earlier, we let $\varphi_0 = \overline{g_0}/g_0$. Theorem 1 follows quickly once the following lemma has been established.

Lemma 1. $T_{\varphi_0}T_{\overline{\varphi_0}} = T_\varphi T_{\overline{\varphi}}$.

Proof. It will suffice to show that $\|T_{\overline{\varphi_0}}f\| = \|T_{\overline{\varphi}}f\|$ for all f in H^2. By Proposition B,

$$\|T_{\overline{\varphi}}f\| = \|T_{1-b}T_{\overline{g}}T_{\overline{\varphi}}f\|_b.$$

Applying the product lemma, we get

$$T_{1-b}T_{\overline{g}}T_{\overline{\varphi}} = T_{1-b}T_{\overline{g}}T_{Ig/\overline{g}} = T_{1-b}T_{Ig} = T_{Ia}.$$

Thus $\|T_{\overline{\varphi}}f\| = \|Iaf\|_b$, which by Proposition A equals $\|af\|_{b_0}$. Exactly the same reasoning, with g_0 and b_0 in place of g and b, gives $\|T_{\overline{\varphi_0}}f\| = \|af\|_{b_0}$. ■

Proof of Theorem 1. After the elaborate preparations for Theorem 1, its proof seems almost an afterthought. By Lemma 1, the operators T_φ and T_{φ_0} are surjective or not surjective together. But, by Hayashi's theorem, the function g_0^2 is rigid, so that $T_{\overline{g_0}/g_0} = T_{\varphi_0}$ has a trivial kernel. Hence T_{φ_0} is surjective if and only if it is invertible, in other words, if and only if $|g_0|^2$ is a Helson–Szegő weight. ■

If $|g|^2$ is a Helson–Szegő weight, then $T_{\overline{g}/g}$ is invertible, and it follows that $T_\varphi = T_{\overline{I}}T_{\overline{g}/g}$ is surjective. The next result shows that the reverse implication fails.

Theorem 2. *Let T_φ be surjective. Then $|g|^2$ is a Helson–Szegő weight if and only if the functions a and I form a corona pair.*

Proof. By Theorem 1 and Proposition D, the functions a and b_0 form a corona pair, while a and b do if and only if $|g|^2$ is a Helson–Szegő weight. But a and b form a corona pair if and only if a and I do. ■

To obtain an example in which T_φ is surjective but $|g|^2$ is not a Helson–Szegő weight, we start with a normalized outer function a such that $\|a\|_\infty < 1$, $|a|^2$ is a Helson–Szegő weight, but a is not invertible in H^∞. Let b_0 be the normalized outer function such that $|b_0|^2 = 1 - |a|^2$ on $\mathbb{T}$, and let $g_0 = a/(1-b_0)$. The function b_0 is invertible in H^∞, so a and b_0 form a corona pair in a trivial way. Since $|a|/2 \le |g_0| \le 2/|a|$, the functions g_0 and $1/g_0$ are both in H^2, hence g_0^2 is rigid. Moreover, $(1+b_0)/(1-b_0) = (1+b_0)g_0/a$ is in H^1, guaranteeing that its real part, $(1-|b_0|^2)/|1-b_0|^2$, is the Poisson integral of its boundary function, i.e., of an absolutely continuous measure. Because a is not invertible in H^∞, it is easy to find an inner function I with $I(0) = 0$ such that a and I do not form a corona pair. We have all the ingredients needed in Hayashi's recipe. Letting $b = Ib_0$, $g = a/(1-b)$, and $\varphi = \overline{I}\overline{g}/g$, we obtain by Hayashi's theorem that g is the extremal function for $\ker T_\varphi = T_g\mathcal{H}(I)$. Because of the properties we arranged for a and b_0 to possess, it follows from Proposition D, the Devinatz–Widom theorem, and Theorem 1 that T_φ is surjective. Finally, we conclude by Theorem 2 that $|g|^2$ is not a Helson–Szegő weight.

4. Inverse images

We continue to let φ be a unimodular function such that T_φ has a nontrivial kernel, and we retain the notations $g, I, a, b, b_0, g_0, \varphi_0$ introduced earlier. Also, for λ in $\mathbb{D}$, we let k_λ denote the kernel function in H^2 for the functional of evaluation at λ: $k_\lambda(z) = (1-\overline{\lambda}z)^{-1}$.

Our first objective in this section is to determine the inverse image of k_λ under T_φ that is orthogonal to $\ker T_\varphi$. This will enable us, in case T_φ is surjective, to determine the right inverse of T_φ whose range is orthogonal to $\ker T_\varphi$. It will also form the basis for an alternative proof of Theorem 1.

It is easy to find at least one function in $T_\varphi^{-1}(k_\lambda)$. Formally, disregarding concerns about boundedness, we can factor T_φ as $T_\varphi = T_{\overline{I}\overline{g}}T_{1/g}$. Hence, formally, the operator $T_{Ig}T_{1/\overline{g}}$

is a right inverse of T_φ, suggesting that the function $T_{Ig}T_{1/\overline{g}}k_\lambda = Igk_\lambda/\overline{g(\lambda)}$ should be in $T_\varphi^{-1}(k_\lambda)$. That this is indeed the case can be verified directly:

$$T_\varphi(Igk_\lambda) = P_+(\overline{g}k_\lambda) = \overline{g(\lambda)}k_\lambda.$$

To find the function in $T_\varphi^{-1}(k_\lambda)$ orthogonal to $\ker T_\varphi$, we need to project the function $Igk_\lambda/\overline{g(\lambda)}$ onto $(\ker T_\varphi)^\perp$. The first part of the following lemma will help.

Lemma 2. (i) $P_+(|g|^2 Ik_\lambda) = \dfrac{Ik_\lambda}{1-b} + \dfrac{\overline{b_0(\lambda)}k_\lambda}{1-\overline{b(\lambda)}}$.

(ii) $P_+(|g|^2 k_\lambda) = \dfrac{k_\lambda}{1-b} + \dfrac{\overline{b(\lambda)}k_\lambda}{1-\overline{b(\lambda)}}$.

Proof. We shall prove (i); the proof of (ii) is entirely analogous. On $\mathbb{T}$ we have

$$|g|^2 = \frac{1-|b|^2}{|1-b|^2} = \frac{1}{1-b} + \frac{\overline{b}}{1-\overline{b}}.$$

Since $b = Ib_0$, we obtain, again on $\mathbb{T}$,

$$|g|^2 Ik_\lambda = \frac{Ik_\lambda}{1-b} + \frac{\overline{b_0}k_\lambda}{1-\overline{b}}.$$

Now suppose $\|b\|_\infty < 1$. Then both summands on the right side of the preceding equality are in L^2, and application of P_+ gives

$$P_+(|g|^2 Ik_\lambda) = \frac{Ik_\lambda}{1-b} + \frac{\overline{b_0(\lambda)}k_\lambda}{1-\overline{b(\lambda)}},$$

the desired equality. To pass to the case $\|b\|_\infty = 1$, it suffices to apply what was just done with rb in place of b ($0 < r < 1$) and let $r \to 1$. Indeed, since the L^1 norm of $(1-|rb|^2)/|1-rb|^2$ coincides with that of $|g|^2$ and $(1-|rb|^2)/|1-rb|^2$ tends to $|g|^2$ almost everywhere, it follows that $(1-|rb|^2)/|1-rb|^2$ tends to $|g|^2$ in L^1. Thus we obtain the desired conclusion by using the continuity of P_+ as a map from L^1 to $H(\mathbb{D})$. ■

We can now identify the function in $T_\varphi^{-1}(k_\lambda)$ belonging to $(\ker T_\varphi)^\perp$.

Lemma 3. *The function $d_\lambda = g(I - \overline{b_0(\lambda)})k_\lambda/\overline{a(\lambda)}$ lies in $(\ker T_\varphi)^\perp$ and is mapped by T_φ to k_λ.*

Proof. We first obtain an expression for the orthogonal projection in H^2 with range $\ker T_\varphi$. We let $P_I = 1 - T_I T_{\overline{I}}$, the orthogonal projection in H^2 with range $\mathcal{H}(I)$. By Propositions B and C, the operator $T_{1-b}T_{\overline{g}}$ is an isometry of H^2 onto $\mathcal{H}(b)$, and it sends $\ker T_\varphi$ onto $\mathcal{H}(I)$, acting on $\ker T_\varphi$ as division by g. By Proposition A, the orthogonal projection in $\mathcal{H}(b)$ with range $\mathcal{H}(I)$ coincides with the restriction of P_I to $\mathcal{H}(b)$. The orthogonal projection in H^2 with range $\ker T_\varphi$ therefore equals $T_g P_I T_{1-b} T_{\overline{g}}$.

We apply the preceding operator to the function Igk_λ and use Lemma 2 (i):

$$\begin{aligned} T_g P_I T_{1-b} T_{\overline{g}}(Igk_\lambda) &= T_g P_I T_{1-b} P_+(I|g|^2 k_\lambda) = T_g P_I T_{1-b}\left[\frac{Ik_\lambda}{1-b} + \frac{\overline{b_0(\lambda)}k_\lambda}{1-\overline{b(\lambda)}}\right] \\ &= T_g P_I(Ik_\lambda) + \frac{\overline{b_0(\lambda)}}{1-\overline{b(\lambda)}} T_g P_I T_{1-b} k_\lambda. \end{aligned}$$

The first summand on the right side is 0. Moreover, because I divides b, we have $P_I T_b = 0$, so $P_I T_{1-b} = P_I$. Since $P_I k_\lambda = (1 - \overline{I(\lambda)} I)k_\lambda$, we obtain

$$T_g P_I T_{1-b} T_{\overline{g}}(Igk_\lambda) = \frac{\overline{b_0(\lambda)}}{1-\overline{b(\lambda)}}(1 - \overline{I(\lambda)} I) g k_\lambda.$$

Finally, we divide the last function by $\overline{g(\lambda)}$ and subtract the result from $Igk_\lambda/\overline{g(\lambda)}$ to get the function in $(\ker T_\varphi)^\perp$ sent by T_φ to k_λ. After a little algebra one finds that the desired function is d_λ. ∎

We are now able to identify the canonical right inverse of T_φ, assuming T_φ is surjective.

Theorem 3. *Let T_φ be surjective. Then the right inverse of T_φ whose range is orthogonal to* $\ker T_\varphi$ *equals $T_g T_{I-\overline{b_0}} T_{1/\overline{a}}$.*

Proof. Let A be the right inverse in question, and let $B = T_g T_{I-\overline{b_0}} T_{1/\overline{a}}$. We first note that, by Theorem 1, the operator T_{φ_0} is invertible. An application of Proposition D, with g_0 replacing g, then shows that the operator $T_{\overline{a}/a}$ is invertible, implying that $|a|^2$ is a Helson–Szegő weight. In particular, $1/\overline{a}$ is in L^2, so, as noted at the end of Section 1, the operator $T_{1/\overline{a}}$ is continuous as an operator from H^2 to $H(\mathbb{D})$. The same is true of $T_{\overline{b_0}/\overline{a}}$. Combining these observations, one sees that B is continuous as an operator from H^2 to $H(\mathbb{D})$.

By Lemma 3, we must have $Ak_\lambda = d_\lambda$. A direct computation shows that $Bk_\lambda = d_\lambda$ (in fact, B was selected with this in mind). The operators A and B thus agree on a dense subset of H^2, namely, the linear span of the kernel functions. Continuity now guarantees that $A = B$. ∎

Corollary. *If T_φ is surjective, then the left inverse of $T_{\overline{\varphi}}$ with the same kernel as T_φ equals $T_{1/a} T_{\overline{I}-b_0} T_{\overline{g}}$.*

The next lemma will be used in the alternative proof of Theorem 1.

Lemma 4. *If f is in H^2 and $T_{\overline{\varphi}} f = d_\lambda$, then f is the function $e_\lambda = (1 - \overline{b_0(\lambda)} b_0)k_\lambda/(\overline{a(\lambda)} a)$. Conversely, if e_λ is in H^2, then $T_{\overline{\varphi}} e_\lambda = d_\lambda$.*

Proof. Assuming $T_{\overline{\varphi}} f = d_\lambda$, we apply $T_{\overline{g}}$ to the equality. Since $T_{\overline{g}} T_{\overline{\varphi}} = T_{Ig}$, we get

$$Igf = T_{\overline{g}}\left(\frac{g(I - \overline{b_0(\lambda)})k_\lambda}{\overline{a(\lambda)}}\right) = \frac{1}{\overline{a(\lambda)}} P_+\big(|g|^2 (I - \overline{b_0(\lambda)})k_\lambda\big).$$

Applying Lemma 2 to the right side, one finds after some algebra that the equality can be rewritten as

$$Igf = \frac{k_\lambda}{\overline{a(\lambda)}}\left(\frac{I - \overline{b_0(\lambda)}b}{1-b}\right).$$

Dividing through by Ig, one obtains the desired equality. A reversal of the reasoning gives the converse. ∎

Lemmas 3 and 4 have versions with φ_0 in place of φ.

Lemma 5. (i) *The function* $d_\lambda^0 = g_0 k_\lambda/\overline{g_0(\lambda)}$ *is mapped by* T_{φ_0} *to* k_λ*.*

(ii) *If* f *is in* H^2 *and* $T_{\overline{\varphi_0}} f = d_\lambda^0$*, then* $f = e_\lambda$*. Conversely, if* e_λ *is in* H^2*, then* $T_{\overline{\varphi_0}} e_\lambda = d_\lambda^0$*.*

We omit the details of the proof. Part (i) can be verified directly. The proof of (ii) uses the same idea as the proof of Lemma 4, but is less complicated. We note that the functions e_λ lie in H^2 if and only if the function $1/a$ does.

Alternative proof of Theorem 1. The proof will use Lemmas 3–5 and Hitt's theorem. The rigidity of g_0^2, which is given by Hayashi's theorem, will emerge from other considerations in the argument to follow.

Assume T_φ is surjective. We prove that $|g_0|^2$ is then a Helson–Szegő weight. Since $T_\varphi T_{\overline{\varphi}}$ is invertible, there is a function f in H^2 such that $T_\varphi T_{\overline{\varphi}} f = k_\lambda$. By Lemmas 3 and 4, we have $f = e_\lambda$ and so e_λ is in H^2. In fact, again by the invertibility of $T_\varphi T_{\overline{\varphi}}$, the functions e_λ ($\lambda \in \mathbb{D}$) span H^2 because the kernel functions k_λ ($\lambda \in \mathbb{D}$) do. Since the functions e_λ are in H^2, so is $1/a$, and hence so is $1/g_0 = (1-b_0)/a$. Now it is a simple matter to show that if f is an outer function in H^2, then the operator $T_{f/\overline{f}}$ has a trivial kernel. Applying this with $f = 1/g_0$, we conclude that T_{φ_0} has a trivial kernel. Also, we can then infer by Lemma 5 that $T_{\varphi_0} T_{\overline{\varphi_0}} e_\lambda = k_\lambda$. Since $T_\varphi T_{\overline{\varphi}}$ and $T_{\varphi_0} T_{\overline{\varphi_0}}$ coincide on a dense subset of H^2, we conclude that $T_{\varphi_0} T_{\overline{\varphi_0}} = T_\varphi T_{\overline{\varphi}}$, and therefore T_{φ_0} is surjective, hence invertible. The Devinatz–Widom theorem now tells us that $|g_0|^2$ is a Helson–Szegő weight, as desired.

Assume, conversely, that $|g_0|^2$ is a Helson–Szegő weight. We prove that T_φ is surjective. By the Devinatz–Widom theorem, T_{φ_0} is invertible. Using Lemma 5, we infer that the functions e_λ are in H^2 and that $T_{\varphi_0} T_{\overline{\varphi_0}} e_\lambda = k_\lambda$. From Lemmas 3 and 4 we get $T_\varphi T_{\overline{\varphi}} e_\lambda = k_\lambda$, so arguing as above $T_\varphi T_{\overline{\varphi}} = T_{\varphi_0} T_{\overline{\varphi_0}}$, giving the surjectivity of T_φ. ∎

We close with a few comments. First, the inclusion of $1/a$ in H^2 is not automatic. In fact, as shown in [Sa89, pp. 491–492], if $a(z) = (1+z)/2$ and $b_0(z) = z(1-z)J(z)/2$, where J is a Blaschke product with zeros on $(-1, 0)$ and tending to -1, then the measure having Poisson integral $(1-|b_0|^2)/|1-b_0|^2$ is absolutely continuous and $g_0^2 = (a/(1-b_0))^2$ is rigid. Let I be any inner function vanishing at 0, $b = b_0 I$, $g = a/(1-b)$, and $\varphi = \overline{I}\overline{g}/g$. By Hayashi's theorem, g is the extremal function for $\ker T_\varphi = T_g \mathcal{H}(I)$. For this particular φ, then, the corresponding function $1/a$ fails to be in H^2.

Second, even if $1/a$ is in H^2, the operator T_φ need not be surjective. In fact, take a normalized outer function a such that $\|a\|_\infty < 1$ and $1/a$ is in H^2 but not in $H^{2+\varepsilon}$ for any $\varepsilon > 0$. Let b_0 be the outer function whose modulus on $\mathbb{T}$ equals $(1-|a|^2)^{1/2}$, and let

$g_0 = a/(1 - b_0)$. As in the example at the end of Section 3, the measure having Poisson integral $(1 - |b_0|^2)/|1 - b_0|^2$ is then absolutely continuous because $(1 + b_0)/(1 - b_0)$ is in H^1. Since $1/g_0$ is in H^2, the function g_0^2 is rigid. As above, take any inner function I vanishing at 0 and let $b = b_0 I$, $g = a/(1 - b)$, $\varphi = \overline{I}\overline{g}/g$. By Hayashi's theorem, g is the extremal function for $\ker T_\varphi = T_g\mathcal{H}(I)$. However, the function $|a|^2$ is not a Helson–Szegő weight, because a Helson–Szegő weight has the property that its reciprocal is in $L^{1+\varepsilon}$ for sufficiently small ε. By Theorem 1 and Proposition D, the operator T_φ is not surjective.

Finally, we wish to remark that, while we believe our characterization of surjectivity is interesting from a theoretical perspective, it is unclear how useful it might be in analyzing specific Toeplitz operators or classes of Toeplitz operators. One obstacle to its use, possibly, is that it seems extremely rare to be able to describe concretely, in terms of the symbol φ, the extremal function g of $\ker T_\varphi$ and its associated inner function I. This matter awaits further investigation.

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