LOCAL WELL-POSEDNESS BELOW THE CHARGE NORM FOR THE DIRAC-KLEIN-GORDON SYSTEM IN TWO SPACE DIMENSIONS

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ABSTRACT. We prove that the Cauchy problem for the Dirac-Klein-Gordon equations in two space dimensions is locally well-posed in a range of Sobolev spaces of negative index for the Dirac spinor, and an associated range of spaces of positive index for the meson field. In particular, we can go below the charge norm, that is, the L^2 norm of the spinor. We hope that this can have implications for the global existence problem, since the charge is conserved. Our result relies on the null structure of the system, and bilinear space-time estimates for the homogeneous wave equation.

1. INTRODUCTION

We study the coupled Dirac-Klein-Gordon system of equations (DKG), which reads

$$(1)\qquad \begin{cases} (-i\gamma^\mu\partial_\mu + M)\,\psi = \phi\psi & (M \ge 0)\,, \\ \left(-\Box + m^2\right)\phi = \psi^\dagger\gamma^0\psi & \left(\Box = -\partial_t^2 + \Delta,\ m \ge 0\right), \end{cases}$$

where $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ represents a meson field and $\psi : \mathbb{R}^{1+n} \to \mathbb{C}^N$ is the Dirac spinor field, regarded as a column vector in $\mathbb{C}^N$; the dimension N of the spin space depends on the space dimension n. Points in the Minkowski space-time $\mathbb{R}^{1+n}$ are written (t,x), where $x = (x^1,\dots,x^n)$; we also denote $t = x^0$ when convenient. We write $\partial_\mu = \partial/\partial x^\mu$ and $\partial_t = \partial_0$. Greek indices $\mu,\nu,\dots$ range over $0,1,\dots,n$, roman indices $j,k,\dots$ over $1,\dots,n$, and repeated upper and lower indices are implicitly summed over these ranges. The γ^μ's are $N\times N$ matrices which should satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I, \qquad (\gamma^0)^\dagger = \gamma^0 \qquad \text{and} \qquad (\gamma^j)^\dagger = -\gamma^j,$$

where $g^{\mu\nu} = \operatorname{diag}(1,-1,\dots,-1)$. The superscript $\dagger$ denotes conjugate transpose.

We shall refer to the cases $n = 1$, 2 and 3 as 1d, 2d and 3d, respectively. In 3d, the smallest possible dimension of the spin space, i.e., the smallest N for which a realization of the Dirac matrices γ^μ can be found, is $N = 4$, whereas in 2d and 1d, $N = 2$. In this article, we are interested in the 2d case.

Concerning the Cauchy problem, global existence in 1d was proved by Chadam [5], but remains open in space dimension two and higher. Motivated by the work of Klainerman and Machedon [12, 14] on the Maxwell-Klein-Gordon and Yang-Mills equations, we want to attack the global existence problem by improving the local well-posedness theory and then make use of conserved quantities, but the snag is

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that the energy density of DKG does not have a definite sign (see [10]). However, a partial replacement may be the charge conservation:

$$\int |\psi(t,x)|^2 \; dx = \text{const.},$$

which could prove useful for the global problem, provided one has local well-posedness with L^2 data for the Dirac spinor. In fact, Bournaveas [3] used the charge conservation to give a new proof of Chadam's global result in 1d, by proving a low regularity local well-posedness theorem with L^2 data for the spinor.

These considerations motivate our interest in the local well-posedness of the Cauchy problem for DKG with initial data

$$\psi|_{t=0} = \psi_0 \in H^s, \qquad \phi|_{t=0} = \phi_0 \in H^r, \qquad \partial_t \phi|_{t=0} = \phi_1 \in H^{r-1}, \tag{2}$$

for minimal $s, r \in \mathbb{R}$. Here $H^s = H^s(\mathbb{R}^n)$ is the standard L^2-based Sobolev space. The corresponding homogeneous space will be denoted $\dot{H}^s$. To get an idea of the minimal regularity required for local well-posedness, note that in the massless case $M = m = 0$, DKG is invariant under the rescaling

$$\psi(t,x) \longrightarrow \lambda^{3/2}\psi(\lambda t, \lambda x), \qquad \phi(t,x) \longrightarrow \lambda\phi(\lambda t, \lambda x).$$

The scale invariant data space is therefore

$$(\psi_0, \phi_0, \phi_1) \in \dot{H}^{(n-3)/2} \times \dot{H}^{(n-2)/2} \times \dot{H}^{(n-4)/2},$$

and one does not expect well-posedness with any less regularity than this. Since the charge corresponds to the L^2 norm of the spinor, we may say that DKG is charge-critical in 3d and charge-subcritical in 2d and 1d.

On the other hand, DKG is a system of nonlinear wave equations with quadratic nonlinearities, and it is well-known (see [18, 21]) that for such equations there is in general a gap between the regularity predicted by scaling and the minimal regularity at which one has local well-posedness, and this gap increases as the space dimension decreases. This is due to buildup effects in the product terms, but if the quadratic nonlinearities satisfy Klainerman's null condition, the worst interactions of products of waves are cancelled, and less regularity is required for local well-posedness. This idea first appeared in [11].

For DKG, the complete null structure was established by the authors in [6], building on earlier work by Klainerman and Machedon [13] and Beals and Bézard [1]. The new idea developed in [6] is that the quadratic form $\psi^\dagger\gamma^0\psi$, which appears in the Klein-Gordon part of DKG, and which was already known to be a null form (see [13, 1]), appears also in the Dirac part of DKG, not directly but via a duality argument.

In [6] we used the null structure, combined with bilinear space-time estimates of Klainerman-Machedon type, to prove local well-posedness of DKG in 3d for data (2) with $s > 0$ and $r = s + 1/2$. Thus, we get arbitrarily close to the scaling regularity. For earlier work on the 3d problem, see [2, 8], and concerning the 1d problem see [5, 3, 7] and more recently [20, 19].

In the present work we study the 2d problem. Here it is harder to get close to the scaling regularity, since the range of space-time estimates narrows as the dimension decreases. On the other hand, DKG is charge-subcritical in 2d, so we stand a better chance of exploiting the charge conservation than in the charge-critical 3d case. In [4], Bournaveas proved local well-posedness of DKG in 2d for data (2) with $s > 1/4$ and $r = s + 1/2$ by using linear Strichartz type estimates (see also [21]), and also with $s > 1/8$ and $r = s + 5/8$ by using the null structure reported in [13]. Here we prove the following:

Theorem 1. *Suppose $(s, r) \in \mathbb{R}^2$ belongs to the convex region described by (see Figure 1)*

$$s > -\frac{1}{5}, \qquad \max\left(\frac{1}{4} + \frac{|s|}{2}, s\right) < r < \min\left(\frac{3}{4} + 2s, 1 + s\right).$$

Then the DKG system in 2d is locally well-posed for data (2).

See Section 3 for a more precise statement. The proof relies on the null structure of the system, and some bilinear space-time estimates for the wave equation, to set up a contraction in Bourgain-Klainerman-Machedon spaces.

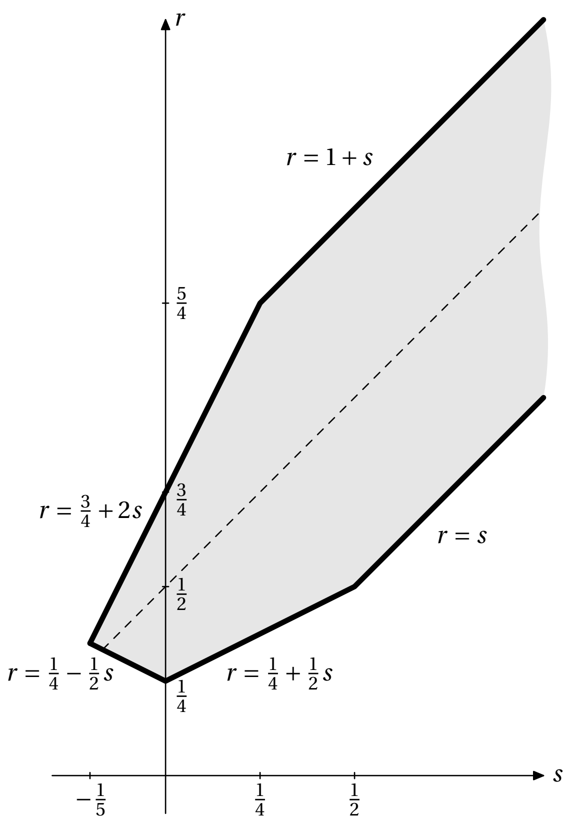

FIGURE 1. Local well-posedness holds in the interior of the shaded region, extending to the right. The dashed line $r = 1/2 + s$ represents the regularity predicted by scaling.

The critical regularity for the spinor is $s = -1/2$, so one may ask what happens in the interval $-1/2 \leq s \leq -1/5$, which is not covered by our theorem. Partial answers to this question can be obtained by studying in more detail the iterates of the problem. We have some results for the first and second nontrivial iterates, which will appear in a forthcoming paper. These results suggest, in particular, that there is no well-posedness for $s < -1/4$. This gap phenomenon is typical of 2d (and 1d) problems.

In connection with the iterates, we remark that the regularity of the first iterate for ϕ was studied by Zheng [23], who proved that if $\psi_0 \in L^2$, and $\psi^{(0)}$ is the homogeneous part of ψ, then the modified iterate, defined by (see the end of this section for notation)

$$\left(-\Box + m^2\right) \Phi^{(1)} = g\left\langle\, \beta\psi^{(0)}, \psi^{(0)} \,\right\rangle, \qquad \Phi^{(1)}(0,x) = \partial_t \Phi^{(1)}(0,x) = 0,$$

satisfies $\Phi^{(1)}(t) \in L^2$ for $t > 0$, provided $g = g(t)$ is C^1 and $g(0) = 0$, the point being that energy and Sobolev estimates are enough to show this if $\psi_0 \in H^\varepsilon$ for some $\varepsilon > 0$, but they fail to give the result for $\varepsilon = 0$. However, our result here shows that the regularity is in fact far better than L^2, namely $\Phi^{(1)}(t) \in H^{3/4-\varepsilon}$ for all $\varepsilon > 0$, if $\psi_0 \in L^2$; this follows from Lemma 2 in Section 4.

Throughout the rest of this article, the space dimension is understood to be $n = 2$. As a matter of convenience, we consider only the massless case $M = m = 0$, but the discussion can easily be modified to include the linear mass terms, since we deal with local-in-time theory in a contraction mapping setup.

For convenience we rewrite the system (1) in the form

$$\begin{cases} i\big(\partial_t + \alpha\cdot\nabla\big)\psi = -\phi\beta\psi, \\ \Box\phi = -\left\langle\, \beta\psi, \psi \,\right\rangle, \end{cases} \tag{3}$$

where $\alpha^j = \gamma^0\gamma^j$, $\beta = \gamma^0$, $\psi = [\psi_1, \psi_2]^T$, $\langle\, z, w \,\rangle = w^\dagger z$ for column vectors $z, w \in \mathbb{C}^2$, $\nabla = (\partial_1, \partial_2)$ and $\alpha\cdot\nabla = \alpha^1\partial_1 + \alpha^2\partial_2$. The Dirac matrices α^j, β should satisfy

$$\begin{gathered} \beta^\dagger = \beta, \qquad (\alpha^j)^\dagger = \alpha^j, \qquad \beta^2 = (\alpha^j)^2 = I, \\ \alpha^j\beta + \beta\alpha^j = 0, \qquad \alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta^{jk} I. \end{gathered} \tag{4}$$

A particular representation in 2d is

$$\alpha^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha^2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \beta = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the σ's are the Pauli matrices. Thus, $\psi^\dagger\beta\psi = \langle\, \beta\psi, \psi \,\rangle = |\psi_1|^2 - |\psi_2|^2$.

The rest of this paper is organized as follows: In Section 2 we fix the notation, and define the function spaces in which we iterate. We also review the splitting of the Dirac equation into positive and negative energy parts. In Section 3, we reduce Theorem 1 to proving two bilinear $X^{s,b}$ estimates, we review the crucial null structure of the bilinear forms involved, and we discuss two key ingredients that will be used to prove the $X^{s,b}$ estimates: a bilinear generalization of the Strichartz estimate for free waves in 2d, and a variation on the Klainerman-Machedon estimate in 2d. In Section 4 we begin the proof of the main theorem, reducing it to two lemmas which are then proved in the next two sections. In Section 7 we study the optimality of our estimates, and in Section 8 we prove the bilinear Strichartz type estimate.

2. Notation

In estimates, we use C to denote a large, positive constant which can change from line to line. If C is absolute, or only depends on parameters which are considered fixed, then we often write $\lesssim$, which means $\le$ up to multiplication by C. If X, Y are nonnegative quantities, $X \sim Y$ means $C^{-1}Y \le X \le CY$ for some absolute constant $C \gg 1$. Naturally, we then define $X \ll Y$ to mean $X \le C^{-1}Y$, and $X \gg Y$ to mean $X \ge CY$. Throughout we use the notation $\langle\cdot\rangle = 1 + |\cdot|$. The characteristic function of a set A is denoted χ_A.

All L^p norms are with respect to Lebesgue measure on $\mathbb{R}^2$, $\mathbb{R}^{1+2}$ or $\mathbb{R}$, unless stated otherwise. Often we indicate by a subscript which variable or variables the norm is taken over, as in L^p_x, $L^p_{t,x}$ or L^p_t. The norm on $L^q_t\big(\mathbb{R}; L^r_x(\mathbb{R}^2)\big)$ is denoted

$$\|u\|_{L^q_t L^r_x} = \Big\| \|u(t,x)\|_{L^r_x} \Big\|_{L^q_t} = \left(\int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^2} |u(t,x)|^r \, dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modification if q or r equals ∞.

The Fourier transforms in space and space-time are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx, \qquad \widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{1+2}} e^{-i(t\tau + x\cdot\xi)} u(t,x)\, dt\, dx.$$

Then Plancherel's theorem comes out somewhat awkwardly as $\big\|\widehat{f}\,\big\|_{L^2} = 2\pi \|f\|_{L^2}$ and $\|\widetilde{u}\|_{L^2} = (2\pi)^{3/2}\|u\|_{L^2}$. To avoid having to keep track of irrelevant normalization factors, we use $\simeq$ to mean equality up to multiplication by some fixed, positive factor. Given $u(t,x)$, we denote by $\lfloor u \rfloor$ the function whose space-time Fourier transform is $|\widetilde{u}|$.

We write $D = \nabla/i$, where $\nabla = (\partial_1, \partial_2)$. Then $(Df)\widehat{\ }(\xi) = \xi \widehat{f}(\xi)$, which explains the notation $\phi(D)$ for the multiplier defined by

$$\widehat{\phi(D) f}(\xi) = \phi(\xi)\widehat{f}(\xi),$$

for a given symbol ϕ. The multipliers $|D|^s$ and $\langle D\rangle^s = (1+|D|)^s$ are used to define $\dot{H}^s$ and H^s as the completions of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norms

$$\|f\|_{\dot{H}^s} = \||D|^s f\|_{L^2_x} \simeq \big\||\xi|^s \widehat{f}\,\big\|_{L^2_\xi}, \qquad \|f\|_{H^s} = \|\langle D\rangle^s f\|_{L^2_x} \simeq \big\|\langle\xi\rangle^s \widehat{f}\,\big\|_{L^2_\xi}.$$

Note that for $\dot{H}^s$ we need $s > -1$, since the space dimension is $n = 2$.

The operator $-i\alpha\cdot\nabla$ appearing in the Dirac equation is awkward to deal with, since it mixes the components of the spinor it acts on. To simplify, we decompose the spinor along an eigenbasis of the operator. Specifically, the matrix-valued symbol of $-i\alpha\cdot\nabla$ is $\xi\cdot\alpha = \xi_1\alpha^1 + \xi_2\alpha^2$, which is hermitian and satisfies $(\xi\cdot\alpha)^2 = |\xi|^2 I$, on account of (4). Thus, the eigenvalues are $\pm|\xi|$, and the corresponding projections onto the one-dimensional eigenspaces are

$$\Pi_\pm(\xi) = \frac{1}{2}\left(I \pm \frac{\xi}{|\xi|}\cdot\alpha \right) = \frac{1}{2}\begin{pmatrix} 1 & \pm(\hat{\xi}_1 - i\hat{\xi}_2) \\ \pm(\hat{\xi}_1 + i\hat{\xi}_2) & 1 \end{pmatrix}, \quad \text{where } \hat{\xi} \equiv \frac{\xi}{|\xi|}. \tag{5}$$

Then $-i\alpha\cdot\nabla = |D|\,\Pi_+(D) - |D|\,\Pi_-(D)$, so the solution of the linear Cauchy problem

$$-i\big(\partial_t + \alpha\cdot\nabla\big)\psi = F, \qquad \psi(0,x) = \psi_0(x), \tag{6}$$

splits into $\psi = \psi_+ + \psi_-$, where $\psi_\pm = \Pi_\pm(D)\psi$ satisfy

$$\begin{cases} \big(-i\partial_t \pm |D|\big)\psi_\pm = F_\pm & \big(\psi_\pm = \Pi_\pm(D)\psi,\ F_\pm = \Pi_\pm(D)F\big), \\ \psi_\pm(0,x) = \psi_0^\pm(x) & \big(\psi_0^\pm = \Pi_\pm(D)\psi_0\big). \end{cases} \tag{7}$$

Note that in a physical interpretation, at least for the free case $F = 0$, the spinors ψ_+ and ψ_- correspond to positive and negative energies, respectively. The free propagator for $-i\partial_t \pm |D|$ is the multiplier $S_\pm(t) = e^{\mp it|D|}$ with symbol $e^{\mp it|\xi|}$. Note that $S_\pm(t)$ acts componentwise on spinors.

To prove Theorem 1 we shall iterate $\psi_\pm$ and ϕ in $X^{s,b}$ type spaces associated to the operators $-i\partial_t \pm |D|$ and $\square$, whose symbols are $\tau \pm |\xi|$ and $\tau^2 - |\xi|^2$, respectively. See [6] for more details about the following spaces. Let $D_\pm$ be the multipliers with symbols $|\tau| \pm |\xi|$. For $s, b \in \mathbb{R}$, we define the Bourgain-Klainerman-Machedon spaces $X_\pm^{s,b}$, $H^{s,b}$ and $\mathcal{H}^{s,b}$ to be the completions of the Schwartz space $\mathcal{S}(\mathbb{R}^{1+2})$ with respect to the norms

$$\begin{aligned}
\|u\|_{X_\pm^{s,b}} &= \left\|\langle D\rangle^s \langle -i\partial_t \pm |D|\rangle^b u\right\|_{L^2_{t,x}} \simeq \left\|\langle \xi\rangle^s \langle \tau \pm |\xi|\rangle^b \widetilde{u}(\tau,\xi)\right\|_{L^2_{\tau,\xi}}, \\
\|u\|_{H^{s,b}} &= \left\|\langle D\rangle^s \langle D_-\rangle^b u\right\|_{L^2_{t,x}} \simeq \left\|\langle \xi\rangle^s \langle |\tau| - |\xi|\rangle^b \widetilde{u}(\tau,\xi)\right\|_{L^2_{(\tau,\xi)}}, \\
\|u\|_{\mathcal{H}^{s,b}} &= \|u\|_{H^{s,b}} + \|\partial_t u\|_{H^{s-1,b}} \sim \left\|\langle D\rangle^{s-1} \langle D_+\rangle \langle D_-\rangle^b u\right\|_{L^2_{t,x}},
\end{aligned}$$

where $\langle\cdot\rangle = 1 + |\cdot|$. We also need the restrictions to a time slab $S_T = (0,T) \times \mathbb{R}^2$, since we study local solutions. The restriction $X_\pm^{s,b}(S_T)$ is a Banach space with norm

$$\|u\|_{X_\pm^{s,b}(S_T)} = \inf\left\{ \|v\|_{X_\pm^{s,b}} : v \in X_\pm^{s,b} \text{ and } v = u \text{ on } S_T \right\}.$$

Completeness holds since $X_\pm^{s,b}(S_T) = X_\pm^{s,b}/\mathcal{M}_\pm$, where the set $\mathcal{M}_\pm = \{v \in X_\pm^{s,b} : v = 0 \text{ on } S_T\}$ is closed. The restrictions $H^{s,b}(S_T)$ and $\mathcal{H}^{s,b}(S_T)$ are defined in the same way.

3. Null structure and bilinear estimates

The complete null structure of DKG, found recently in [6], rests on the cancellation properties of the matrix-valued symbol

$$A_{\pm_1,\pm_2}(\eta,\zeta) = \Pi_{\pm_2}(\zeta)\beta\Pi_{\pm_1}(\eta) = \beta\Pi_{\mp_2}(\zeta)\Pi_{\pm_1}(\eta),$$

where to get the last equality we used $\beta\Pi_\pm(\xi) = \Pi_\mp(\xi)\beta$, which follows from (4). By orthogonality, $\Pi_{\mp_2}(\zeta)\Pi_{\pm_1}(\eta)$ vanishes when the vectors $\pm_1\eta$ and $\pm_2\zeta$ line up in the same direction. The following lemma, proved in [6], quantifies this cancellation. We shall use the notation $\measuredangle(\eta,\zeta)$ for the angle between vectors $\eta, \zeta \in \mathbb{R}^2$.

Lemma 1. $A_{\pm_1,\pm_2}(\eta,\zeta) = O\left(\measuredangle(\pm_1\eta, \pm_2\zeta)\right)$.

Note the following consequence: If $\psi, \psi' : \mathbb{R}^{1+2} \to \mathbb{C}^2$ are Schwartz functions,

$$\begin{aligned}
&\left|\langle \beta\Pi_{\pm_1}(D)\psi, \Pi_{\pm_2}(D)\psi' \rangle^{\sim}(\tau,\xi)\right| \\
&\qquad \le \int_{\mathbb{R}^{1+2}} \left| \left\langle \beta\Pi_{\pm_1}(\eta)\widetilde{\psi}(\lambda,\eta), \Pi_{\pm_2}(\eta-\xi)\widetilde{\psi}'(\lambda-\tau,\eta-\xi) \right\rangle \right| \, d\lambda \, d\eta \\
&\qquad \lesssim \int_{\mathbb{R}^{1+2}} \theta_{\pm_1,\pm_2} \left|\widetilde{\psi}(\lambda,\eta)\right| \left|\widetilde{\psi}'(\lambda-\tau,\eta-\xi)\right| \, d\lambda \, d\eta,
\end{aligned} \tag{8}$$

where $\theta_{\pm_1,\pm_2} = \measuredangle\big(\pm_1\eta, \pm_2(\eta-\xi)\big)$. Here we used the self-adjointness of the projections to move $\Pi_{\pm_2}(\eta-\xi)$ in front of β, and then we applied the lemma.

Let us now restate our main theorem in a more precise form.

Theorem 1. *Suppose $(s,r) \in \mathbb{R}^2$ belongs to the region (see Figure 1)*

$$s > -\frac{1}{5}, \qquad \max\left(\frac{1}{4} + \frac{|s|}{2}, s\right) < r < \min\left(\frac{3}{4} + 2s, 1+s\right). \tag{9}$$

Then for any data $\psi|_{t=0} = \psi_0 \in H^s$, $\phi|_{t=0} = \phi_0 \in H^r$ *and* $\partial_t\phi|_{t=0} = \phi_1 \in H^{r-1}$, *there exist a time* $T > 0$, *depending continuously on the* $H^s \times H^r \times H^{r-1}$ *norm of the data, and a solution*

$$\psi \in C([0,T];H^s), \qquad \phi \in C([0,T];H^r) \cap C^1([0,T];H^{r-1}),$$

of the DKG system (3) *on* $(0,T)\times\mathbb{R}^2$, *satisfying the initial condition above. Furthermore, writing* $\psi = \psi_+ + \psi_-$, *where* $\psi_\pm = \Pi_\pm(D)\psi$, *the solution has the regularity*

$$\psi_\pm \in X_\pm^{s,\sigma}(S_T), \qquad \phi \in \mathcal{H}^{r,\rho}(S_T), \tag{10}$$

for some choice of $1/2 < \sigma, \rho < 1$, *depending on* (s,r). *Moreover, the solution is unique in this class, and depends continuously on the data.*

The first step in the proof is to use (6) and (7) to reformulate (3) as

$$\begin{cases} \big(-i\partial_t + |D|\big)\psi_+ = -\Pi_+(D)\,(\phi\beta\psi)\,, \\ \big(-i\partial_t - |D|\big)\psi_- = -\Pi_-(D)\,(\phi\beta\psi)\,, \\ \Box\phi = -\left\langle\, \beta\psi, \psi \,\right\rangle. \end{cases}$$

We iterate in the space (10). Then by a standard argument, see [6] for more details, the theorem reduces to proving the following bilinear estimates, for some $\varepsilon = \varepsilon(r,s) > 0$ sufficiently small:

$$\left\| \Pi_{\pm_2}(D)\,(\phi\beta\Pi_{\pm_1}(D)\psi) \right\|_{X_{\pm_2}^{s,\sigma-1+\varepsilon}(S_T)} \lesssim \|\phi\|_{H^{r,\rho}(S_T)} \|\psi\|_{X_{\pm_1}^{s,\sigma}(S_T)}\,, \tag{11}$$

$$\left\| \langle\, \beta\Pi_{\pm_1}(D)\psi, \Pi_{\pm_2}(D)\psi' \,\rangle \right\|_{H^{r-1,\rho-1+\varepsilon}(S_T)} \lesssim \|\psi\|_{X_{\pm_1}^{s,\sigma}(S_T)} \|\psi'\|_{X_{\pm_2}^{s,\sigma}(S_T)}\,, \tag{12}$$

where $\pm_1, \pm_2$ denote independent signs, and $S_T = (0,T)\times\mathbb{R}^2$, with $0 < T \le 1$. The introduction of the small parameter $\varepsilon > 0$ is just a technical detail needed in the linear estimates; see [6, Lemmas 5 and 6]. Here and in the rest of the paper, it is understood that implicit constants may depend on s and r, but not on T.

In addition to the null form estimate (8), the key tools needed to prove (11) and (12) are some bilinear spacetime estimates for 2d free waves, which we now discuss. Recall that $S_\pm(t) = e^{\mp it|D|}$ is the free propagator for $-i\partial_t \pm |D|$. In the following discussion we let $f, g \in \mathcal{S}(\mathbb{R}^2)$ and write

$$u(t) = u_\pm(t) = S_\pm(t)f, \qquad v(t) = v_\pm(t) = S_\pm(t)g.$$

In estimates where the signs do not matter, we skip the subscript indicating the sign.

We begin with a generalization of the Strichartz estimate for free waves. The following extends (in the 2d case) an estimate due to Klainerman and Tataru [17].

Theorem 2. *The estimate*

$$\left\| |D|^{-s_3}\,(uv) \right\|_{L_t^q L_x^2} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}}$$

holds if

$$\begin{cases} 4 \le q \le \infty, \\ s_1 + s_2 + s_3 = 1 - 1/q, \\ s_1, s_2 < 1 - 1/q, \\ s_1 + s_2 > 1/q \quad (\iff s_3 < 1 - 2/q). \end{cases}$$

The case where $q = 4$, $s_1 = s_2$ and $s_3 \leq 0$ was proved in [17]. The above theorem is sharp up to endpoint cases, in view of [9, Proposition 14.15]. In particular, there are no estimates of this form with $q < 4$.

Note that (11) and (12) are L^2 in both space and time, so to apply the above theorem we need to use the finite support in time. For example, if we restrict to $S_T = (0, T) \times \mathbb{R}^2$, then by Hölder's inequality in time and Theorem 2 with $q = 4$,

$$\left\| |D|^{-s_3} (uv) \right\|_{L^2(S_T)} \lesssim T^{1/4} \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}} \quad \text{if} \quad \begin{cases} s_1 + s_2 + s_3 = 3/4, \\ s_1, s_2 < 3/4, \\ s_1 + s_2 > 1/4. \end{cases} \tag{13}$$

Theorem 2 suffices to prove Theorem 1 with the exception of one particularly delicate case (see subsection 5.2.1), where we need to use a variant of the Klainerman-Machedon estimate, which we now discuss.

For $q < 4$ there are no estimates of the type considered in Theorem 2, but Klainerman and Machedon [15] proved that if the product uv is replaced by the null form $D_-^\gamma(uv)$, where $\gamma \geq 1/4$, and D_- denotes the multiplier with symbol $\big||\tau| - |\xi|\big|$, then one can obtain a range of estimates with $q = 2$.

Theorem 3. [15, 9]. *The estimate*

$$\left\| |D|^{-s_3} D_-^{1/4}(uv) \right\|_{L^2_{t,x}} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}}$$

holds if and only if

$$\begin{cases} s_1 + s_2 + s_3 = 3/4, \\ s_1, s_2 < 3/4, \\ s_1 + s_2 > 1/2 \quad (\iff s_3 < 1/4). \end{cases}$$

The fact that $\gamma \geq 1/4$ is required in order to have an L^2 space-time estimate for $D_-^\gamma(uv)$, is related to the gap phenomenon in 2d (there is a gap between the regularity predicted by scaling and the regularity needed to have local well-posedness). Indeed, the null forms in DKG correspond to $\gamma = 1/2$, and in principle the null symbol may then be completely cancelled against the weight of the $X^{s,-1/2}$ norm in which the null form is estimated, resulting in a corresponding gain of "elliptic" derivatives. But in reality we can only cancel "half" of the symbol, due to the restriction $\gamma \geq 1/4$ in the L^2 estimate. This corresponds to a gap of a quarter of a derivative down to the scaling regularity. This is exactly the loss we incur if instead we cancel the full null symbol and then, assuming finite support in time, apply Hölder's inequality in time to replace L_t^2 by L_t^4, at which point we may apply Theorem 2. The latter approach is always better, however: comparing the last theorem with (13), we see that the conditions on the s_i are the same, except that in (13) one only needs $s_1 + s_2 > 1/4$, as opposed to $s_1 + s_2 > 1/2$ in Theorem 3.

As it stands, Theorem 3 is not useful in the present context, in view of the preceding remarks. However, as observed in [22, Theorem 6(b)] (or see [9, Theorem 12.1]), the condition $s_1 + s_2 > 1/2$ in Theorem 3 can be relaxed to $s_1 + s_2 > 1/4$ for products of type $(+,+)$ and $(-,-)$, i.e., if uv is u_+v_+ or u_-v_-. In fact,

$$\left\| |D|^{-s_3} D_-^{1/4}(u_+v_+) \right\|_{L^2_{t,x}} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}} \qquad \text{if} \quad \begin{cases} s_1 + s_2 + s_3 = 3/4, \\ s_1, s_2 < 3/4, \\ s_1 + s_2 > 1/4. \end{cases} \tag{14}$$

We are not aware of any previous application of this improved estimate. Here we shall use it to get a decisive improvement of (13), in the case where the product is of type $(+,+)$ or $(-,-)$ with high frequencies interacting to give output at low frequency. To make this precise, we introduce a bilinear operator $(\cdot,\cdot)_{\mathrm{HH}\to\mathrm{L}}$ which isolates this interaction:

$$[(f,g)_{\mathrm{HH}\to\mathrm{L}}]\,\widehat{}\,(\xi) = \int_{\mathbb{R}^2} \mathbf{1}_{|\xi|\ll|\eta|+|\xi-\eta|} \widehat{f}(\eta)\widehat{g}(\xi-\eta)\,d\eta.$$

Here $\mathbf{1}_{|\xi|\ll|\eta|+|\xi-\eta|}$ is the characteristic function of the set $\{\eta : |\xi| \ll |\eta| + |\xi-\eta|\}$. Then for free waves $u_+(t) = S_+(t)f$ and $v_+(t) = S_+(t)g$, using the fact that $\widetilde{u}_+(\tau,\xi) = \delta(\tau+|\xi|)\widehat{f}(\xi)$ and $\widetilde{v}_+(\tau,\xi) = \delta(\tau+|\xi|)\widehat{g}(\xi)$, we have

$$\begin{aligned}
&\left[D_-^{1/4}(u_+,v_+)_{\mathrm{HH}\to\mathrm{L}}\right]\widetilde{}\,(\tau,\xi)\\
&\quad= \int_{\mathbb{R}^{1+2}} \big||\tau|-|\xi|\big|^{1/4} \mathbf{1}_{|\xi|\ll|\eta|+|\xi-\eta|} \widehat{f}(\eta)\widehat{g}(\xi-\eta)\delta(\lambda+|\eta|)\delta(\tau-\lambda+|\xi-\eta|)\,d\lambda\,d\eta\\
&\quad= \int_{\mathbb{R}^2} (|\eta|+|\xi-\eta|-|\xi|)^{1/4}\, \mathbf{1}_{|\xi|\ll|\eta|+|\xi-\eta|} \widehat{f}(\eta)\widehat{g}(\xi-\eta)\delta(\tau+|\eta|+|\xi-\eta|)\,d\eta,
\end{aligned}$$

hence

$$\left\| |D|^{-s_3}\,(u_+,v_+)_{\mathrm{HH}\to\mathrm{L}} \right\|_{L^2_{t,x}} \sim \left\| |D|^{-s_3}\, D_-^{1/4}(|D|^{-1/8}\,u_+, |D|^{-1/8}\,v_+)_{\mathrm{HH}\to\mathrm{L}} \right\|_{L^2_{t,x}},$$

and from (14) we then obtain:

Theorem 4. *We have*

$$\left\| |D|^{-s_3}\,(u_+,v_+)_{HH\to L} \right\|_{L^2_{t,x}} \lesssim \|f\|_{\dot H^{s_1}}\,\|g\|_{\dot H^{s_2}} \qquad \text{if} \quad \begin{cases} s_1+s_2+s_3 = 1/2,\\ s_1,s_2 < 5/8,\\ s_1+s_2 > 0. \end{cases}$$

Note that this is a tremendous improvement over (13), for this particular interaction. This estimate will be used to handle a particularly delicate case occurring in the proof of Theorem 1 (see subsection 5.2.1), where Theorem 2 fails hopelessly.

To end this section, let us state the $X^{s,b}$ versions of the free wave estimates discussed above. It is a general principle (see, e.g., [6, Lemma 4]) that Strichartz type estimates for a free propagator imply corresponding estimates for the $X^{s,b}$ space associated to the propagator. Thus, Theorems 2 and 4 imply, respectively:

Corollary 1. *Suppose $\varepsilon > 0$ and q, s_1, s_2, s_3 satisfy the hypotheses in Theorem 2. Then*

$$\left\| |D|^{-s_3}\,(uv) \right\|_{L^q_t L^2_x} \le C_{q,s_1,s_2,\varepsilon}\, \left\| |D|^{s_1}\,u \right\|_{X_\pm^{0,1/2+\varepsilon}} \left\| |D|^{s_2}\,v \right\|_{X_\pm^{0,1/2+\varepsilon}}$$

for $u, v \in \mathcal{S}(\mathbb{R}^{1+2})$ and for all combinations of signs.

Corollary 2. *Suppose $\varepsilon > 0$ and s_1, s_2, s_3 satisfy the hypotheses in Theorem 4. Then*

$$\left\| |D|^{-s_3}\,(u,v)_{HH\to L} \right\|_{L^2_{t,x}} \lesssim \left\| |D|^{s_1}\,u \right\|_{X_+^{0,1/2+\varepsilon}} \left\| |D|^{s_2}\,v \right\|_{X_+^{0,1/2+\varepsilon}}$$

for $u, v \in \mathcal{S}(\mathbb{R}^{1+2})$. Note carefully the equality of the signs in the norms on the right.

A final ingredient needed in the proof is the following product law for the spaces $H^{s,b}$, here stated for two space dimensions.

Theorem 5. ([16]) *Suppose* $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$ *satisfy*

$$a+b+c>1, \qquad a+b\geq 0, \qquad a+c\geq 0, \qquad b+c\geq 0,$$
$$\alpha+\beta+\gamma>\frac{1}{2}, \qquad \alpha,\beta,\gamma\geq 0.$$

Then the product estimate (in 1+2 dimensions)

$$\|uv\|_{H^{-c,-\gamma}} \lesssim \|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}}$$

holds.

Proof. Note that at most one of a, b, c can be negative. But by the triangle inequality in Fourier space (i.e., Leibniz's rule), we can always reduce to the case $a, b, c \geq 0$, which was proved in [16, Proposition A.1] □

We remark that this estimate is the analogue of the product estimate for the standard Sobolev spaces, which in 2d reads

$$\|fg\|_{H^{-c}} \lesssim \|f\|_{H^a} \|g\|_{H^b},$$

for a, b, c as in the above theorem (here the conditions are sharp up to endpoints).

4. Proof of Theorem 1

We first reduce (11) and (12) to the global-in-time (i.e., unrestricted) estimates

$$\|\Pi_{\pm_2}(D)\,(\phi\beta\Pi_{\pm_1}(D)\psi)\|_{X^{s,\sigma-1+\varepsilon}_{\pm_2}} \lesssim \|\phi\|_{H^{r,\rho}} \|\psi\|_{X^{s,\sigma}_{\pm_1}} \tag{15}$$

$$\|\langle\,\beta\Pi_{\pm_1}(D)\psi, \Pi_{\pm_2}(D)\psi'\,\rangle\|_{H^{r-1,\rho-1+\varepsilon}} \lesssim \|\psi\|_{X^{s,\sigma}_{\pm_1}} \|\psi'\|_{X^{s,\sigma}_{\pm_2}}, \tag{16}$$

for all $\phi, \psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+2})$ such that ψ is supported in $[-2,2]\times\mathbb{R}^2$. Here it is understood that ϕ is real-valued and ψ, ψ' are $\mathbb{C}^2$-valued.

Let us show that (16) implies (12); a similar argument shows that (15) implies (11). Fix a smooth cutoff $\chi(t)$ such that $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$. Let $\psi \in X^{s,\sigma}_{\pm_1}(S_T)$ and $\psi' \in X^{s,\sigma}_{\pm_2}(S_T)$. Viewing $X^{s,\sigma}_{\pm}(S_T)$ as a space of equivalence classes, the equivalence relation being equality on S_T, we let $\Psi \in X^{s,\sigma}_{\pm_1}$ and $\Psi' \in X^{s,\sigma}_{\pm_2}$ denote arbitrary representatives of ψ and ψ', respectively. Recalling the assumption $0 < T \leq 1$, we observe that (16) implies

$$\begin{aligned}
\|\langle\,\beta\Pi_{\pm_1}(D)\psi, \Pi_{\pm_2}(D)\psi'\,\rangle\|_{H^{r-1,\rho-1+\varepsilon}(S_T)} \leq&\\
&\leq \|\langle\,\beta\Pi_{\pm_1}(D)(\chi\Psi), \Pi_{\pm_2}(D)\Psi'\,\rangle\|_{H^{r-1,\rho-1+\varepsilon}}\\
&\lesssim \|\chi\Psi\|_{X^{s,\sigma}_{\pm_1}} \|\Psi'\|_{X^{s,\sigma}_{\pm_2}}\\
&\lesssim \|\Psi\|_{X^{s,\sigma}_{\pm_1}} \|\Psi'\|_{X^{s,\sigma}_{\pm_2}},
\end{aligned}$$

and taking the infimum over all representatives Ψ, Ψ' yields (12). In the last step we used the easily proved estimate

$$\|\chi u\|_{X^{s,\sigma}_{\pm}} \leq C_{\chi,\sigma} \|u\|_{X^{s,\sigma}_{\pm}}, \tag{17}$$

valid for $\sigma \geq 0$.

Thus, Theorem 1 has been reduced to proving (15) and (16). However, in [6] it was shown that (15) is equivalent, by duality, to an estimate similar to (16), namely

$$\|\langle\,\beta\Pi_{\pm_1}(D)\psi, \Pi_{\pm_2}(D)\psi'\,\rangle\|_{H^{-r,-\rho}} \lesssim \|\psi\|_{X^{s,\sigma}_{\pm_1}} \|\psi'\|_{X^{-s,1-\sigma-\varepsilon}_{\pm_2}}, \tag{15$'$}$$

which must hold for all $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+2})$ such that ψ is supported in $[-2,2] \times \mathbb{R}^2$. Note the advantage of this formulation, in that the null form appears again.

We shall prove the following lemmas, which together imply Theorem 1.

Lemma 2. *Suppose*

$$s > -\frac{1}{4}, \qquad r < \min\left(\frac{3}{4} + 2s, 1 + s\right). \tag{18}$$

Then there exist $1/2 < \sigma, \rho < 1$ *and* $\varepsilon > 0$ *such that* (16) *holds for all* $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+2})$ *such that* ψ *is supported in* $[-2,2] \times \mathbb{R}^2$*. More precisely, we can take*

$$\rho = \frac{1}{2} + \varepsilon, \tag{19}$$

$$\sigma = \begin{cases} 1 - \varepsilon & \text{if } 0 < s < \frac{1}{2} \text{ and } r \geq \frac{3}{4} + \frac{3s}{2}, \\ \text{any number in } (1/2, 1) & \text{otherwise,} \end{cases} \tag{20}$$

with $\varepsilon > 0$ *sufficiently small.*

Lemma 3. *Suppose*

$$s \in \mathbb{R}, \qquad r > \max\left(\frac{1}{4} + \frac{|s|}{2}, |s|\right). \tag{21}$$

Then there exist $1/2 < \sigma, \rho < 1$ *and* $\varepsilon > 0$ *such that* (15′) *holds for all* $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+2})$ *such that* ψ *is supported in* $[-2,2] \times \mathbb{R}^2$*. More precisely, we can take*

$$\rho = \frac{1}{2} + \varepsilon, \tag{22}$$

$$\sigma = \begin{cases} \text{any number in } (1/2, 1) & \text{if } s > 0 \text{ and } r \geq \frac{3}{4} + \frac{3s}{2}, \\ \frac{1}{2} + \varepsilon & \text{otherwise,} \end{cases} \tag{23}$$

with $\varepsilon > 0$ *sufficiently small.*

Note that the conditions on σ in the two lemmas are compatible, for (s, r) in the region of Theorem 1 (the intersection of the domains in the two lemmas).

5. Proof of Lemma 2

Without loss of generality we take $\pm_1 = +$ and write $\pm_2 = \pm$. We assume (18), and choose ρ and σ according to (19) and (20). In the sequel, whenever we say that some condition involving ε holds, we mean that it holds for all $\varepsilon > 0$ small enough. We assume that $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+2})$ and ψ is supported in $[-2,2] \times \mathbb{R}^2$. The notation $\lfloor \psi \rceil$ is used for the function whose Fourier transform is $|\widetilde{\psi}|$.

In view of the null form estimate (8), we can reduce (16) to

$$I^{\pm} \lesssim \|\psi\|_{X_+^{s,\sigma}} \|\psi'\|_{X_\pm^{s,\sigma}},$$

where

$$I^{\pm} = \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle \xi \rangle^{r-1} \theta_\pm}{\langle |\tau| - |\xi| \rangle^{1/2 - 2\varepsilon}} \left| \widetilde{\psi}(\lambda, \eta) \right| \left| \widetilde{\psi}'(\lambda - \tau, \eta - \xi) \right| d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$\theta_\pm = \measuredangle\big(\eta, \pm(\eta - \xi)\big).$$

Let us right away dispose of the low-frequency case, where $\min(|\eta|, |\eta-\xi|) \le 1$ in $I^\pm$. Then $\langle\xi\rangle \sim \langle \max(|\eta|, |\eta-\xi|)\rangle$, so assuming $|\eta| \le |\eta - \xi|$, as we may by symmetry, we have

$$
\begin{aligned}
I^\pm &\lesssim \left\| \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{r-1} \lfloor\psi'\rfloor} \right\|_{L^2_{t,x}} \\
&\le \left\| \lfloor\psi\rfloor \right\|_{L^\infty_{t,x}} \left\| \langle D\rangle^{r-1} \lfloor\psi'\rfloor \right\|_{L^2_{t,x}} \\
&\lesssim \left\| \lfloor\psi\rfloor \right\|_{L^\infty_{t,x}} \left\| \psi' \right\|_{X^{s,0}_\pm} ,
\end{aligned} \tag{24}
$$

where we used the assumption $r < 1+s$. But if the support of $\widetilde{\psi}(\lambda,\eta)$ is restricted to $|\eta| \le 1$, then $\| \lfloor\psi\rfloor \|_{L^\infty_{t,x}} \le C_\varepsilon \|\psi\|_{X^{s,1/2+\varepsilon}_\pm}$ for all $s \in \mathbb{R}$, by the inequalities of Hausdorff-Young and Cauchy-Schwarz.

With the low-frequency case accounted for, we henceforth assume that in $I^\pm$,

$$
|\eta|, |\eta - \xi| \ge 1. \tag{25}
$$

It will be convenient to use the notation

$$
\begin{gathered}
F(\lambda,\eta) = \langle\eta\rangle^s \langle \lambda + |\eta| \rangle^\sigma \left| \widetilde{\psi}(\lambda,\eta) \right|, \\
G_\pm(\lambda,\eta) = \langle\eta\rangle^s \langle \lambda \pm |\eta| \rangle^\sigma \left| \widetilde{\psi}'(\lambda,\eta) \right|, \\
A = |\tau| - |\xi| , \qquad B = \lambda + |\eta|, \qquad C_\pm = \lambda - \tau \pm |\eta - \xi|, \\
R_+ = |\xi| - \big| |\eta| - |\eta - \xi| \big|, \qquad R_- = |\eta| + |\eta - \xi| - |\xi| .
\end{gathered}
$$

We shall need the easily checked estimates (see [6])

$$
\theta_+^2 \sim \frac{|\xi|\, R_+}{|\eta||\eta-\xi|}, \qquad \theta_-^2 \sim \frac{(|\eta| + |\eta-\xi|)R_-}{|\eta||\eta-\xi|} \sim \frac{R_-}{\min(|\eta|, |\eta-\xi|)}. \tag{26}
$$

The following estimates will also be needed:

$$
R_\pm \le 2\min(|\eta|, |\eta - \xi|), \tag{27}
$$

$$
R_\pm \le |A| + |B| + |C_\pm| . \tag{28}
$$

The first one is the triangle inequality, the second is proved in [6, Lemma 7].

5.1. **Estimate for I^+.** By (26), and recalling (25), we have

$$
I^+ \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1/2} R_+^{1/2} F(\lambda,\eta) G_+(\lambda-\tau, \eta-\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle A\rangle^{1/2-2\varepsilon} \langle B\rangle^\sigma \langle C_+\rangle^\sigma} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} .
$$

By (28), and since $R_+ \le |\xi|$, we get

$$
R_+^{1/2} \lesssim |A|^{1/2-2\varepsilon} |\xi|^{2\varepsilon} + |B|^{1/2} + |C_+|^{1/2} , \tag{29}
$$

hence $I^+ \lesssim I_1^+ + I_2^+ + I_3^+$, where

$$
\begin{aligned}
I_1^+ &= \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r+2\varepsilon-1/2} F(\lambda,\eta) G_+(\lambda-\tau, \eta-\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle B\rangle^\sigma \langle C_+\rangle^\sigma} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} , \\
I_2^+ &= \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1/2} F(\lambda,\eta) G_+(\lambda-\tau, \eta-\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle A\rangle^{1/2-2\varepsilon} \langle C_+\rangle^\sigma} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} , \\
I_3^+ &= \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1/2} F(\lambda,\eta) G_+(\lambda-\tau, \eta-\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle A\rangle^{1/2-2\varepsilon} \langle B\rangle^\sigma} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} .
\end{aligned}
$$

In effect, one of the "hyperbolic" weights $\langle A\rangle$, $\langle B\rangle$ or $\langle C\rangle$ in the denominator of I^+ has been traded in for the same amount of "elliptic" weights ($\langle\eta\rangle$, $\langle\eta-\xi\rangle$ or $\langle\xi\rangle$). This happens on account of the null condition, which effectively rules out the bad case where all three hyperbolic weights are ~ 1 simultaneously (unless one of the elliptic weights is ~ 1 also).

We only need to estimate I_1^+ and I_2^+, since I_2^+ and I_3^+ are symmetrical. It should be noted that for I_1^+ the only condition needed on σ is that $\sigma > 1/2$, but for I_2^+ we need to choose σ as in (20).

5.1.1. *Estimate for I_1^+.* We rewrite:

$$I_1^+ \simeq \left\| \langle D\rangle^{r+2\varepsilon-1/2} \left(\langle D\rangle^{-1/2}\lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2}\lfloor\psi'\rfloor} \right) \right\|_{L^2_{t,x}} .$$

At this point we would like to apply Corollary 1, so we need to replace $L^2_{t,x}$ by $L^q_t L^2_x$. This would be trivial if we had compact support in time, but we do not, since ψ has been replaced by $\lfloor\psi\rfloor$. Nevertheless, there is sufficient decay so that we can apply Hölder's inequality in time. To see this, fix a smooth cutoff function $\chi(t)$ such that $\chi(t) = 1$ for $|t| \le 2$. Denote its Fourier transform by $\widehat{\chi}(\tau)$, and let $\lfloor\chi\rfloor$ be the function whose Fourier transform is $|\widehat{\chi}(\tau)|$. Then $\psi = \chi\psi$, by the support assumption on ψ, hence

$$\widetilde{\lfloor\psi\rfloor} = \widetilde{\lfloor\chi\psi\rfloor} \le \widetilde{\lfloor\chi\rfloor\lfloor\psi\rfloor}.$$

Thus, for any $q \ge 2$,

$$\begin{aligned} I_1^+ &\le \left\| \langle D\rangle^{r+2\varepsilon-1/2} \left(\langle D\rangle^{-1/2}(\lfloor\chi\rfloor\lfloor\psi\rfloor) \cdot \overline{\langle D\rangle^{-1/2}\lfloor\psi'\rfloor} \right) \right\|_{L^2_{t,x}} \\ &= \left\| \lfloor\chi\rfloor \langle D\rangle^{r+2\varepsilon-1/2} \left(\langle D\rangle^{-1/2}\lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2}\lfloor\psi'\rfloor} \right) \right\|_{L^2_{t,x}} \\ &\le \|\lfloor\chi\rfloor\|_{L^p_t} \left\| \langle D\rangle^{r+2\varepsilon-1/2} \left(\langle D\rangle^{-1/2}\lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2}\lfloor\psi'\rfloor} \right) \right\|_{L^q_t L^2_x} , \end{aligned}$$

where $1/p+1/q = 1/2$. Note that $\|\lfloor\chi\rfloor\|_{L^p_t} < \infty$, by the Hausdorff-Young inequality. It remains to check, using Corollary 1, that

$$\left\| \langle D\rangle^{r+2\varepsilon-1/2} \left(\langle D\rangle^{-1/2}\lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2}\lfloor\psi'\rfloor} \right) \right\|_{L^q_t L^2_x} \lesssim \|\psi\|_{X_+^{s,\sigma}} \|\psi'\|_{X_+^{s,\sigma}} , \tag{30}$$

for some $q \ge 4$ depending on r and s. We divide into two cases: $r < 1/2$ and $r \ge 1/2$.

Assume $r < 1/2$. Then (30) follows from Corollary 1 with $s_1 = s_2 = 1/2 + s$, provided we can find $q \ge 4$ such that $s_1 + s_2 + 1/2 - r - 2\varepsilon \ge 1 - 1/q$, $s_1 + s_2 > 1/q$ and $s_1, s_2 < 1 - 1/q$. Written out, the conditions are:

$$1 + 2s \ge \frac{1}{2} - \frac{1}{q} + r + 2\varepsilon, \tag{31}$$

$$1 + 2s > \frac{1}{q} \tag{32}$$

$$s < \frac{1}{2} - \frac{1}{q}. \tag{33}$$

We distinguish two cases: $-1/4 < s < 1/4$ and $s \ge 1/4$.

Assume $-1/4 < s < 1/4$. Then we set $q = 4$, so (32) and (33) are certainly satisfied, and (31) reduces to $r + 2\varepsilon \le 3/4 + 2s$, which holds by (18).

Finally, assume $s \geq 1/4$. Then there is a lot of room in the estimate. We take $q = 4$ and discard the multiplier $\langle D\rangle^{r+2\varepsilon-1/2}$ in the left side of (30) (whose exponent is negative). Now we take $s_1 = s_2 = 3/8$ and $s_3 = 0$ in Corollary 1. This concludes the case $r < 1/2$.

Now assume $r \geq 1/2$. This can only happen if $s > -1/8$, in view of (18). Note that by symmetry, we may assume $|\eta| \geq |\eta - \xi|$ in I_1^+, hence $\langle\xi\rangle \lesssim \langle\eta\rangle$. Effectively, we can therefore move the multiplier $\langle D\rangle^{r+2\varepsilon-1/2}$ in (30) in front of $\lfloor\psi\rfloor$. Taking $q = 4$, we thus reduce (30) to

$$\left\| \langle D\rangle^{r+2\varepsilon-1} \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2} \lfloor\psi'\rfloor} \right\|_{L_t^4 L_x^2} \lesssim \|\psi\|_{X_+^{s,\sigma}} \|\psi'\|_{X_+^{s,\sigma}} . \tag{34}$$

To prove this, we apply Corollary 1, dividing into two cases: $-1/8 < s < 1/4$ and $s \geq 1/4$.

Assume $-1/8 < s < 1/4$. Set $s_2 = 1/2 + s$. Then $0 < s_2 < 3/4$, so setting $s_1 = 3/4 - s_2 = 1/4 - s$ and applying Corollary 1 gives

$$\left\| \langle D\rangle^{r+2\varepsilon-1} \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2} \lfloor\psi'\rfloor} \right\|_{L_t^4 L_x^2} \lesssim \|\psi\|_{X_+^{r+2\varepsilon-3/4-s,\sigma}} \|\psi'\|_{X_+^{s,\sigma}} ,$$

which proves (34), provided $r + 2\varepsilon - 3/4 - s \leq s$, i.e., $r + 2\varepsilon \leq 3/4 + 2s$, which holds by (18).

Now assume $s \geq 1/4$. Set $s_1 = \varepsilon$ and $s_2 = 3/4 - \varepsilon$. Then Corollary 1 gives

$$\left\| \langle D\rangle^{r+2\varepsilon-1} \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2} \lfloor\psi'\rfloor} \right\|_{L_t^4 L_x^2} \lesssim \|\psi\|_{X_+^{r+3\varepsilon-1,\sigma}} \|\psi'\|_{X_+^{1/4-\varepsilon,\sigma}} .$$

This proves (34), since $r + 3\varepsilon - 1 \leq s$, by (18).

This concludes the proof for I_1^+.

5.1.2. *Estimate for I_2^+.* Note that if $|A| \geq |\xi|$, then $R_+^{1/2} \leq |\xi|^{1/2} \leq \langle A\rangle^{1/2-2\varepsilon} |\xi|^{2\varepsilon}$, which means that I^+ reduces to I_1^+. Therefore, we may assume $|A| \leq |\xi|$, which implies

$$I_2^+ \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r+3\varepsilon-1/2} F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle A\rangle^{1/2+\varepsilon} \langle C_+\rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} .$$

We need to prove $I_2^+ \lesssim \|F\|_{L^2} \|G_+\|_{L^2}$, but by duality this is equivalent to the estimate

$$\int_{\mathbb{R}^6} \frac{\langle\xi\rangle^{r+3\varepsilon-1/2} F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi) H(\tau,\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle A\rangle^{1/2+\varepsilon} \langle C_+\rangle^{\sigma}} \, d\lambda \, d\eta \, d\tau \, d\xi \lesssim \|F\|_{L^2} \|G_+\|_{L^2} \|H\|_{L^2}$$

for all $H \in L^2(\mathbb{R}^{1+2})$, $H \geq 0$. For this, it suffices to prove

$$\left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r+3\varepsilon-1/2} G_+(\lambda-\tau,\eta-\xi) H(\tau,\xi)}{\langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2+s} \langle A\rangle^{1/2+\varepsilon} \langle C_+\rangle^{\sigma}} \, d\tau \, d\xi \right\|_{L^2_{\lambda,\eta}} \lesssim \|G_+\|_{L^2} \|H\|_{L^2} ,$$

which we rewrite as

$$\left\| \langle D\rangle^{-1/2-s} \left(\langle D\rangle^{r+3\varepsilon-1/2} u_\pm \cdot \langle D\rangle^{-1/2} \lfloor\psi'\rfloor \right) \right\|_{L^2_{t,x}} \lesssim \|u_\pm\|_{X_\pm^{0,1/2+\varepsilon}} \|\psi'\|_{X_+^{s,\sigma}} ,$$

where

$$\widetilde{u}_\pm(\tau,\xi) = \frac{H_\pm(\tau,\xi)}{\langle\tau \pm |\xi|\rangle^{1/2+\varepsilon}}$$

and

$$H_+(\tau,\xi) = \chi_{(-\infty,0)}(\tau) H(\tau,\xi), \quad H_-(\tau,\xi) = \chi_{(0,\infty)}(\tau) H(\tau,\xi).$$

By the trick used for I_1^+, we can estimate the $L^2_{t,x}$ norm on the left by the $L_t^q L_x^2$ norm for any $q \ge 2$. Indeed, in the original estimate (16) we can insert the cutoff χ in front of ψ, since $\chi\psi = \psi$, and then we can move χ in front of ψ'. Thus, it suffices to show that for some $q \ge 4$, depending on s and r,

$$\left\| \langle D \rangle^{-1/2-s} \left(\langle D \rangle^{r+3\varepsilon-1/2} u_\pm \cdot \langle D \rangle^{-1/2} \lfloor \psi' \rfloor \right) \right\|_{L_t^q L_x^2} \lesssim \left\| u_\pm \right\|_{X_\pm^{0,1/2+\varepsilon}} \left\| \psi' \right\|_{X_+^{s,\sigma}} . \tag{35}$$

We first do the low-frequency case, where $H(\tau,\xi)$ is supported in $|\xi| \le 1$. Then the left side of (35) with $q = \infty$ can be estimated by (we assume $H \ge 0$)

$$\begin{aligned} \left\| \langle D \rangle^{-1/2-s} \left(u_\pm \cdot \langle D \rangle^{-1/2} \lfloor \psi' \rfloor \right) \right\|_{L_t^\infty L_x^2} &\lesssim \left\| u_\pm \cdot \langle D \rangle^{-1/2} \lfloor \psi' \rfloor \right\|_{L_t^\infty L_x^2} \\ &\le \left\| u_\pm \right\|_{L^\infty_{t,x}} \left\| \langle D \rangle^{-1/2} \lfloor \psi' \rfloor \right\|_{L_t^\infty L_x^2} \\ &\lesssim \left\| u_\pm \right\|_{X_\pm^{0,1/2+\varepsilon}} \left\| \psi' \right\|_{X_+^{s,\sigma}} . \end{aligned}$$

With the low-frequency case out of the way, we assume from now on that $|\xi| \ge 1$ on the support of $H(\tau,\xi)$. Thus, there is effectively no difference between the multipliers $\langle D \rangle^{r+3\varepsilon-1/2}$ and $|D|^{r+3\varepsilon-1/2}$ when they act on $u_\pm$. In the following we distinguish three cases: $-1/4 < s < 0$, $0 \le s < 1/2$ and $s \ge 1/2$.

First assume $-1/4 < s < 0$. We take $q = 4$, $s_1 = s_3 = 1/2 + s$ and $s_2 = -1/4 - 2s$. Clearly, $s_1 < 3/4$, and the condition $s_2 < 3/4$ is just $s > -1/2$. The condition $s_1 + s_2 > 1/4$ reduces to $s < 0$. Thus Corollary 1 proves (35), provided $s_2 \le 1/2 - r - 3\varepsilon$, i.e., $r + 3\varepsilon \le 3/4 + 2s$, which holds by (18).

Next, assume $s \ge 1/2$. We separate the cases $r < 1/2$ and $r \ge 1/2$. If $r < 1/2$, we apply Corollary 1 with $q = \infty$ and, say, $s_1 = -3\varepsilon$, $s_2 = 1/2 + 3\varepsilon$ and $s_3 = 1/2$. Then (35) follows, provided ε is small enough. Now assume $r \ge 1/2$. Then using the triangle inequality we can estimate the left side of (35) by a sum of two terms, namely

$$\begin{aligned} J_1 &= \left\| \langle D \rangle^{-1-s+r+3\varepsilon} \left(u_\pm \cdot \langle D \rangle^{-1/2} \lfloor \psi' \rfloor \right) \right\|_{L_t^q L_x^2} , \\ J_2 &= \left\| \langle D \rangle^{-1/2-s} \left(u_\pm \cdot \langle D \rangle^{-1+r+3\varepsilon} \lfloor \psi' \rfloor \right) \right\|_{L_t^q L_x^2} . \end{aligned}$$

Now apply Corollary 1 with $q = \infty$. Choose $4\varepsilon \le 1 + s - r$, in accordance with (18). Then in J_1 we can take $s_1 = 0$, $s_2 = 1 - \varepsilon$ and $s_3 = \varepsilon$, and in J_2 we can take $s_1 = 0$, $s_2 = \varepsilon$ and $s_3 = 1 - \varepsilon$.

Finally, assume $0 \le s < 1/2$. Set $1/q = 1/4 - s/2 - \varepsilon$. Then $4 < q < \infty$, assuming that $2\varepsilon \le 1/2 - s$. Set $s_2 = s_3 = 1/2 + s$ and $s_1 = -1/4 - 3s/2 + \varepsilon$. The condition $s_2 < 1 - 1/q$ is then the same as $s < 1/2 + 2\varepsilon$, which is satisfied. The condition $s_1 < 1 - 1/q$ is nothing else than $s > -1/2$. The condition $s_1 + s_2 > 1/q$ reduces to $2\varepsilon > 0$. Corollary 1 therefore applies, and (35) follows provided $s_1 \le 1/2 - r - 3\varepsilon$, i.e., $r + 4\varepsilon \le 3/4 + 3s/2$. This covers the case $r < 3/4 + 3s/2$.

Thus, we are left with the case $3/4 + 3s/2 \le r$, which can only happen when $0 < s < 1/2$. Then we replace (29) by the following variation:

$$R_+^{1/2} \lesssim |A|^{1/2-2\varepsilon} |\xi|^{2\varepsilon} + |B|^{\sigma-1/2-\varepsilon} |\xi|^{1+\varepsilon-\sigma} + |C_+|^{\sigma-1/2-\varepsilon} |\xi|^{1+\varepsilon-\sigma} , \tag{36}$$

thus obtaining $I^+ \lesssim I_1^+ + I_2^+ + I_3^+$, where I_1^+ is the same as before, but

$$I_2^+ = \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle \xi \rangle^{r+1/2-\sigma+\varepsilon} F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle \eta \rangle^{1/2+s} \langle \eta-\xi \rangle^{1/2+s} \langle A \rangle^{1/2-2\varepsilon} \langle B \rangle^{1/2+\varepsilon} \langle C_+ \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} ,$$

and I_3^+ is symmetrical. Now we simply give up the weight $\langle A\rangle^{1/2-2\varepsilon}$, and apply Corollary 1. This is trivial if the exponent of $\langle\xi\rangle$ is negative (then we give up the $\langle\xi\rangle$ weight also), and if this exponent is positive, we can use the triangle inequality to reduce to

$$I_2^+ \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta)G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{s-r+\sigma-\varepsilon}\langle\eta-\xi\rangle^{1/2+s}\langle B\rangle^{1/2+\varepsilon}\langle C_+\rangle^{1/2+\varepsilon}}\, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}},$$

or the symmetrical expression with the exponents of $\langle\eta\rangle$ and $\langle\eta-\xi\rangle$ interchanged. Then Corollary 1 applies (after the cutoff argument) provided that $\sigma > r-s$ and $\sigma > r+1/4-2s$. Both conditions are satisfied if we choose $\sigma = 1-\varepsilon$ as in (20), recalling that $3/4+3s/2 \le r$ and $0 < s < 1/2$; in fact, the conditions then reduce to $r < 1+s-\varepsilon$ and $r < 3/4+2s-\varepsilon$, which hold by (18).

This concludes the proof of (35).

5.2. **Estimate for I^-.** Note first that if $|\eta| \ll |\eta-\xi|$, then $|\xi| \sim |\eta-\xi|$, so by (26),

$$\theta_-^2 \sim \frac{|\xi|\, R_-}{|\eta||\eta-\xi|}.$$

The same is true if $|\eta| \gg |\eta-\xi|$ or $|\xi| \sim |\eta| \sim |\eta-\xi|$. So in all these cases, we have the same estimate for θ_- as for θ_+; moreover (29) and (36) hold with R_+ replaced by R_-, in view of (27) and (28), so the analysis of I^+ in the previous subsection applies also to I^- in these cases. Thus, it suffices to consider I^- in the case where

$$|\xi| \ll |\eta| \sim |\eta-\xi|, \tag{37}$$

which we assume from now on. Recalling also (25), we then have, by (26),

$$I^- \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1} R_-^{1/2} F(\lambda,\eta)G_-(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1/4+s}\langle\eta-\xi\rangle^{1/4+s}\langle A\rangle^{1/2-2\varepsilon}\langle B\rangle^{\sigma}\langle C_-\rangle^{\sigma}}\, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}}.$$

By (27) and (28), we have

$$R_-^{1/2} \lesssim |A|^{1/2-2\varepsilon}\, |\eta|^{\varepsilon}|\eta-\xi|^{\varepsilon} + |B|^{1/2} + |C_-|^{1/2},$$

so $I^- \lesssim I_1^- + I_2^- + I_3^-$, where

$$I_1^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1} F(\lambda,\eta)G_-(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1/4+s-\varepsilon}\langle\eta-\xi\rangle^{1/4+s-\varepsilon}\langle B\rangle^{\sigma}\langle C_-\rangle^{\sigma}}\, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_2^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1} F(\lambda,\eta)G_-(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1/4+s}\langle\eta-\xi\rangle^{1/4+s}\langle A\rangle^{1/2-2\varepsilon}\langle C_-\rangle^{\sigma}}\, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_3^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle\xi\rangle^{r-1} F(\lambda,\eta)G_-(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1/4+s}\langle\eta-\xi\rangle^{1/4+s}\langle A\rangle^{1/2-2\varepsilon}\langle B\rangle^{\sigma}}\, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}}.$$

We remark that the only condition needed on σ here is $\sigma > 1/2$.

5.2.1. *Estimate for I_1^-.* Proceeding as in the estimate for I_1^+, we can reduce to proving

$$\left\| \langle D\rangle^{r-1} \left(\langle D\rangle^{\varepsilon-1/4} \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{\varepsilon-1/4} \lfloor\psi'\rfloor} \right) \right\|_{L_t^q L_x^2} \lesssim \|\psi\|_{X_+^{s,\sigma}} \|\psi'\|_{X_-^{s,\sigma}}, \tag{38}$$

for some $q \ge 4$. However, this estimate fails for $s < -1/8$. To see this, take $s_1 = s_2 = 1/4+s-\varepsilon$ and consider the conditions in Corollary 1, which as remarked are sharp up to endpoints. The condition $1 - r + s_1 + s_2 \geq 1 - 1/q$ becomes $2\varepsilon \leq 1/2 + 1/q + 2s - r$, forcing $q = 4$, since r can be arbitrarily close to $3/4 + 2s$, by (18). The condition $s_1 + s_2 \geq 1/q$ then forces $s \geq -1/8 + \varepsilon$.

Fortunately, there is a way around this difficulty. The solution is to keep the L^2 norm, instead of passing to a higher L^q norm in time. Recall (37), which says that we are in the high-high frequency case with output at low frequency. Moreover, the interaction is of the type $(+,+)$, because of the conjugation in the second factor. Indeed, observe that $\langle C_- \rangle = \langle \lambda - \tau - |\eta - \xi| \rangle = \langle \tau - \lambda + |\xi - \eta| \rangle$, so we can rewrite I_1^- in a form which is manifestly of type $(+,+)$, namely

$$I_1^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle \xi \rangle^{r-1} F(\lambda,\eta) G'(\tau - \lambda, \xi - \eta)}{\langle \eta \rangle^{1/4+s-\varepsilon} \langle \eta - \xi \rangle^{1/4+s-\varepsilon} \langle \lambda + |\eta| \rangle^{\sigma} \langle \tau - \lambda + |\xi - \eta| \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}$$

where we have set $G'(\lambda,\eta) = G_-(-\lambda,-\eta)$. We now see from Corollary 2, using (37), that

$$I_1^- \lesssim \left\| F \right\|_{L^2} \left\| G' \right\|_{L^2} = \left\| \psi \right\|_{X_+^{s,\sigma}} \left\| \psi' \right\|_{X_-^{s,\sigma}},$$

provided $2\varepsilon \leq 1 + 2s - r$, $s \leq \varepsilon + 3/8$ and $0 < s + 1/4 - \varepsilon$. These conditions are satisfied for $-1/4 < s \leq 3/8$, in view of (18). If $s > 3/8$, we can estimate $\langle \xi \rangle^{r-1} \leq \langle \xi \rangle^s$, since $r < 1 + s$, and apply Corollary 1 with $q = 4$ (in fact, this works for $s > 1/4$). This concludes the proof for I_1^-.

5.2.2. *Estimate for* I_2^-. We may assume $|A| \ll |\eta| + |\eta - \xi|$, since otherwise $I^- \lesssim I_1^-$, by (27). Recalling also (37), we then get

$$I_2^- \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{\langle \xi \rangle^{r-1} F(\lambda,\eta) G_-(\lambda - \tau, \eta - \xi)}{\langle \eta - \xi \rangle^{1/2+2s-3\varepsilon} \langle A \rangle^{1/2+\varepsilon} \langle C_- \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Estimating by duality as in the proof of I_2^+, and using the cutoff argument, we reduce to proving

$$\left\| \langle D \rangle^{r-1} u_{\pm} \cdot \langle D \rangle^{3\varepsilon - 1/2 - s} \lfloor \psi' \rfloor \right\|_{L_t^4 L_x^2} \lesssim \left\| u_{\pm} \right\|_{X_{\pm}^{0,1/2+\varepsilon}} \left\| \psi' \right\|_{X_-^{s,\sigma}}.$$

Assuming $r < 1$, this follows from Corollary 1, provided $1 - r + 1/2 + 2s - 3\varepsilon \geq 3/4$, i.e., $3\varepsilon \leq 3/4 + 2s - r$, which is compatible with (18). On the other hand, if $r \geq 1$, then using (37), we see that

$$I_2^- \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_-(\lambda - \tau, \eta - \xi)}{\langle \xi \rangle^{a/2} \langle \eta - \xi \rangle^{a/2} \langle A \rangle^{1/2+\varepsilon} \langle C_- \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

where $a = 3/2 + 2s - r - 3\varepsilon$. Hence we reduce to, by duality and cutoff,

$$\left\| \langle D \rangle^{-a/2} u_{\pm} \cdot \langle D \rangle^{-a/2+s} \lfloor \psi' \rfloor \right\|_{L_t^4 L_x^2} \lesssim \left\| u_{\pm} \right\|_{X_{\pm}^{0,1/2+\varepsilon}} \left\| \psi' \right\|_{X_-^{s,\sigma}},$$

and this follows from Corollary 1, provided $a \geq 3/4$, i.e., $3\varepsilon \leq 3/4 + 2s - r$.

5.2.3. *Estimate for* I_3^-. The argument used for I_2^- applies also here. In fact, by a change of variables, I_3^- can be transformed to I_2^-, except that the minus sign changes to a plus sign: we will have C_+ instead of C_-. However, the argument used for I_2^- is not sensitive to changes of sign.

This concludes the proof of Lemma 2.

6. Proof of Lemma 3

Again we take $\pm_1 = +$ and write $\pm_2 = \pm$. We assume (21), and choose ρ and σ according to (22) and (23). Let $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+2})$ with ψ is supported in $[-2,2] \times \mathbb{R}^2$. As before, when we say that some condition involving ε holds, we mean that it holds for all $\varepsilon > 0$ small enough.

By (8), we can then reduce (15′) to

$$I^{\pm} \lesssim \|\psi\|_{X_+^{s,\sigma}} \|\psi'\|_{X_\pm^{-s,1-\sigma-\varepsilon}}, \tag{39}$$

where now

$$I^{\pm} = \left\| \int_{\mathbb{R}^{1+2}} \frac{\theta_\pm}{\langle\xi\rangle^r \langle |\tau| - |\xi| \rangle^{1/2+\varepsilon}} \left|\widetilde{\psi}(\lambda,\eta)\right| \left|\widetilde{\psi}'(\lambda-\tau,\eta-\xi)\right| \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

and $\theta_\pm = \measuredangle\big(\eta, \pm(\eta-\xi)\big)$ as before. We proceed as in the proof of Lemma 2, and use the notation introduced there, except that now

$$G_\pm(\lambda,\eta) = \langle\eta\rangle^{-s} \langle \lambda \pm |\eta| \rangle^{1-\sigma-\varepsilon} \left|\widetilde{\psi}'(\lambda,\eta)\right|.$$

Consider the low-frequency case, $\min(|\eta|, |\eta-\xi|) \le 1$. Then

$$\langle\xi\rangle \sim \langle \max(|\eta|, |\eta-\xi|) \rangle.$$

If $|\eta| \le 1$, then (24) applies, with $r-1$ replaced by $-r$, and s replaced by $-s$, hence we need $-r \le -s$, which holds by (21). On the other hand, if $|\eta-\xi| \le 1$, then using $-r \le s$, which holds by (21), and using also the Sobolev type estimate

$$\|u\|_{L_t^\infty L_x^2} \simeq \|\widehat{u}\|_{L_t^\infty L_\xi^2} \le \|\widehat{u}\|_{L_\xi^2 L_t^\infty} \le \|\widetilde{u}\|_{L_\xi^2 L_\tau^1} \le C_\varepsilon \|u\|_{X_\pm^{0,1/2+\varepsilon}},$$

we can write

$$\begin{aligned} I^{\pm} &\lesssim \left\| \langle D\rangle^{-r} \lfloor\psi\rfloor \cdot \overline{\lfloor\psi'\rfloor} \right\|_{L^2_{t,x}} \\ &\le \left\| \langle D\rangle^{-r} \lfloor\psi\rfloor \right\|_{L_t^\infty L_x^2} \|\lfloor\psi'\rfloor\|_{L_t^2 L_x^\infty} \\ &\lesssim \|\psi\|_{X_\pm^{s,1/2+\varepsilon}} \|\lfloor\psi'\rfloor\|_{L_t^2 L_x^\infty}. \end{aligned}$$

But since we are assuming now that $\widetilde{\psi}'(\lambda-\tau,\eta-\xi)$ vanishes unless $|\eta-\xi| \le 1$, we have $\|\lfloor\psi'\rfloor\|_{L_t^2 L_x^\infty} \lesssim \|\psi'\|_{X_\pm^{-s,0}}$ for all $s \in \mathbb{R}$, by Sobolev embedding. This concludes the proof of the low-frequency case.

From now on we assume the high-frequency case, so that in $I^\pm$,

$$|\eta|, |\eta-\xi| \ge 1. \tag{40}$$

6.1. **Estimate for I^+.** From (26) and (40) we get

$$I^+ \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{R_+^{1/2} F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2} \langle\eta\rangle^{1/2+s} \langle\eta-\xi\rangle^{1/2-s} \langle A\rangle^{1/2+\varepsilon} \langle B\rangle^{\sigma} \langle C_+\rangle^{1-\sigma-\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

By (27) and (28),

$$R_+^{1/2} \lesssim |A|^{1/2} + |B|^{1/2} + |C_+|^{1-\sigma-\varepsilon} |\eta-\xi|^{\sigma-1/2+\varepsilon},$$

hence $I^+ \lesssim I_1^+ + I_2^+ + I_3^+$, where

$$I_1^+ = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1/2-s}\langle B\rangle^{\sigma}\langle C_+\rangle^{1-\sigma-\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_2^+ = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1/2-s}\langle A\rangle^{1/2+\varepsilon}\langle C_+\rangle^{1-\sigma-\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_3^+ = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1-s-\sigma-\varepsilon}\langle A\rangle^{1/2+\varepsilon}\langle B\rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

6.1.1. *Estimate for I_1^+.* Note that the sum of the exponents of the "elliptic" weights $\langle\xi\rangle$, $\langle\eta\rangle$ and $\langle\eta-\xi\rangle$ in I_1^+ is $1/2+r$. If $r > 1/2$, it therefore follows by Theorem 5 that $I_1^+ \lesssim \|F\|_{L^2} \|G_+\|_{L^2}$; here we also need $r \ge |s|$ to satisfy the hypotheses of the theorem, but this is satisfied in view of (21). Note that this argument applies for any $1/2 < \sigma < 1$.

Having disposed of the case $r > 1/2$, we may henceforth assume $r \le 1/2$. Thus, $\sigma = 1/2+\varepsilon$, according to (23), and we may restrict to $-1/2 < s < 1/2$, since $|s| \ge 1/2$ forces $r > 1/2$, by (21).

Next, note that if $|C_+| \ge |\eta-\xi|$, then the sum of the elliptic exponents increases to $1/2+r+1-\sigma-\varepsilon$, which exceeds 1, since now $\sigma = 1/2+\varepsilon$ and since we assume $r > 1/4$ (cf. (21)). Since we still have the weight $\langle B\rangle$ to a power greater than 1/2, it then follows that $I_1^+ \lesssim \|F\|_{L^2} \|G_+\|_{L^2}$, by Theorem 5.

In view of the above reductions, we can from now on assume

$$r \le \frac{1}{2}, \qquad -\frac{1}{2} < s < \frac{1}{2}, \qquad \sigma = \frac{1}{2}+\varepsilon, \qquad |C_+| \le |\eta-\xi|, \tag{41}$$

in I_1^+. Thus, $\langle\xi\rangle^{1/2-r} \lesssim \langle\eta\rangle^{1/2-r} + \langle\eta-\xi\rangle^{1/2-r}$, and we may assume

$$\|\psi'\|_{X_+^{-s-3\varepsilon,1/2+\varepsilon}} \le \|\psi'\|_{X_+^{-s,1/2-2\varepsilon}} = \|\psi'\|_{X_+^{-s,1-\sigma-\varepsilon}}.$$

By the same cutoff argument as in subsection 5.1.1, we then reduce to proving

$$\left\| \langle D\rangle^{-r} \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-1/2} \lfloor\psi'\rfloor} \right\|_{L_t^4 L_x^2} \lesssim \|\psi\|_{X_+^{s,1/2+\varepsilon}} \|\psi'\|_{X_+^{-s-3\varepsilon,1/2+\varepsilon}}, \tag{42}$$

$$\left\| \langle D\rangle^{-1/2} \lfloor\psi\rfloor \cdot \overline{\langle D\rangle^{-r} \lfloor\psi'\rfloor} \right\|_{L_t^4 L_x^2} \lesssim \|\psi\|_{X_+^{s,1/2+\varepsilon}} \|\psi'\|_{X_+^{-s-3\varepsilon,1/2+\varepsilon}}. \tag{43}$$

Now we apply Corollary 1 with $q = 4$ and $s_3 = 0$. Sufficient conditions for this to work are, for (42) and (43) respectively,

$$r+s > 0, \qquad 1/2-s-3\varepsilon > 0, \qquad r+1/2-3\varepsilon \ge 3/4,$$

and

$$1/2+s > 0, \qquad r-s-3\varepsilon > 0, \qquad r+1/2-3\varepsilon \ge 3/4.$$

All these conditions are satisfied, in view of (21) and the assumption $-1/2 < s < 1/2$. This concludes the proof for I_1^+.

6.1.2. *Estimate for I_2^+.* By the same reductions as in the previous subsection, we may assume (41), and by the duality and cutoff argument used in subsection 5.1.2, we then reduce to proving

$$\left\| \langle D \rangle^{-r-s} \left(u_\pm \cdot \langle D \rangle^{-1/2} \lfloor \psi' \rfloor \right) \right\|_{L_t^q L_x^2} \lesssim \left\| u_\pm \right\|_{X_\pm^{0,1/2+\varepsilon}} \left\| \psi' \right\|_{X_+^{-s-3\varepsilon,1/2+\varepsilon}}, \tag{44}$$

$$\left\| \langle D \rangle^{-1/2-s} \left(u_\pm \cdot \langle D \rangle^{-r} \lfloor \psi' \rfloor \right) \right\|_{L_t^q L_x^2} \lesssim \left\| u_\pm \right\|_{X_\pm^{0,1/2+\varepsilon}} \left\| \psi' \right\|_{X_+^{-s-3\varepsilon,1/2+\varepsilon}}, \tag{45}$$

for some $q \geq 4$.

For (44), we apply Corollary 1 with $s_1 = 0$ and $s_2 = 1/2 - s - 3\varepsilon$. Thus, we need

$$r + \frac{1}{2} - 3\varepsilon \geq 1 - \frac{1}{q}, \tag{46}$$

$$\frac{1}{q} < \frac{1}{2} - s - 3\varepsilon < 1 - \frac{1}{q}. \tag{47}$$

We divide into three cases: $-1/4 < s < 1/4$, $-1/2 < s \leq -1/4$ and $1/4 \leq s < 1/2$.

If $-1/4 < s < 1/4$, then (47) is satisfied with $q = 4$, and then (46) reduces to $3\varepsilon \leq r - 1/4$, which is in accordance with (21). If $-1/2 < s \leq -1/4$, then the left inequality in (47) is certainly satisfied, and to optimize we choose $q \geq 4$ as small as possible, but such that the right inequality in (47) is satisfied. Thus, we set $1/q = s + 1/2$, so $4 \leq q < \infty$. Then (47) holds, and (46) reduces to $3\varepsilon \leq r + s$, which is compatible with (21). Finally, if $1/4 \leq s < 1/2$, we set $1/q = -s + 1/2 - 4\varepsilon$. Then (47) again holds, and (46) becomes $7\varepsilon \leq r - s$, in agreement with (21).

For (45), we apply Corollary 1 with $s_1 = 0$ and $s_2 = r - s - 3\varepsilon$. We then require (46) to hold, and also

$$\frac{1}{q} < r - s - 3\varepsilon < 1 - \frac{1}{q}. \tag{48}$$

We distinguish the cases $-1/2 < s \leq -1/4$, $-1/4 < s < 0$ and $0 \leq s < 1/2$.

If $-1/2 < s \leq -1/4$, we take $1/q = s + 1/2$. Then (46) is satisfied due to (21), and (48) becomes $1/2 + 2s + 3\varepsilon < r < 1/2 + 3\varepsilon$, which holds since $1/4 < r \leq 1/2$ by (21) and (41). If $-1/4 < s < 0$, set $q = 4$. Then since $1/4 < r \leq 1/2$ by (21) and (41), we see that (46) and (48) hold. Finally, assume $0 \leq s < 1/2$. Then the right inequality in (48) is certainly satisfied, since $r \leq 1/2$ by (41), so we are left with (46) and the left inequality in (48), which we can restate as two lower bounds for r:

$$r \geq \frac{1}{2} - \frac{1}{q} + 3\varepsilon, \qquad r > s + \frac{1}{q} + 3\varepsilon.$$

To optimize, we take $1/q = 1/4 - s/2$. Then we get the condition $r > 1/4 + s/2 + 3\varepsilon$, which is compatible with (21). This concludes the proof for I_2^+.

6.1.3. *Estimate for I_3^+.* Whenever $\sigma = 1/2 + \varepsilon$, then if we replace s by $-s$, I_3^+ is essentially the same as I_2^+ (modulo epsilons in some exponents). The argument used for I_2^+ can therefore be applied also to I_3^+. In particular, we need here the condition $r > 1/4 - s/2$.

It remains to consider the cases where σ is different from $1/2 + \varepsilon$. This happens, according to (21) when $s > 0$ and $r \geq 3/4 + 3s/2$. Then for any $1/2 + \varepsilon \leq \sigma \leq 1 - \varepsilon$,

$$I_3^+ \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda, \eta) G_+(\lambda - \tau, \eta - \xi)}{\langle \xi \rangle^{r-1/2} \langle \eta \rangle^{1/2+s} \langle \eta - \xi \rangle^{-s} \langle A \rangle^{1/2+\varepsilon} \langle B \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Since $\langle \eta - \xi \rangle^s \lesssim \langle \eta \rangle^s + \langle \xi \rangle^s$, we reduce to either of the following:

$$I_{3,1}^+ = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r-1/2} \langle \eta \rangle^{1/2} \langle A \rangle^{1/2+\varepsilon} \langle B \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_{3,2}^+ = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r-s-1/2} \langle \eta \rangle^{1/2+s} \langle A \rangle^{1/2+\varepsilon} \langle B \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Both reduce to Corollary 1 with $q = 4$, via duality and the cutoff argument, since the exponents of the elliptic weights $\langle \xi \rangle$ and $\langle \eta \rangle$ are both positive (recall that $s > 0$ and $r \geq 3/4 + 3s/2$), and their sum r exceeds $3/4$.

This concludes the proof for I^+.

6.2. **Estimate for I^-.** By the argument in subsection 5.2, we may assume

$$|\xi| \ll |\eta| \sim |\eta - \xi|$$

in I^-. Combining this with (26) and (40), we get

$$I^- \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{R_-^{1/2} F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r} \langle \eta \rangle^{1/4} \langle \eta - \xi \rangle^{1/4} \langle A \rangle^{1/2+\varepsilon} \langle B \rangle^{\sigma} \langle C_+ \rangle^{1-\sigma-\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Note that s no longer appears. By (27) and (28),

$$R_-^{1/2} \lesssim |A|^{1/2} + |B|^{1/2} + |C_-|^{1-\sigma-\varepsilon} |\eta - \xi|^{\sigma-1/2+\varepsilon},$$

hence $I^- \lesssim I_1^- + I_2^- + I_3^-$, where

$$I_1^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r} \langle \eta \rangle^{1/4} \langle \eta - \xi \rangle^{1/4} \langle B \rangle^{\sigma} \langle C_- \rangle^{1-\sigma-\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_2^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r} \langle \eta \rangle^{1/4} \langle \eta - \xi \rangle^{1/4} \langle A \rangle^{1/2+\varepsilon} \langle C_- \rangle^{1-\sigma-\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_3^- = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r} \langle \eta \rangle^{1/4} \langle \eta - \xi \rangle^{3/4-\sigma-\varepsilon} \langle A \rangle^{1/2+\varepsilon} \langle B \rangle^{\sigma}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

For I_1^- and I_2^-, the sum of the exponents of the elliptic weights is $1/2 + r$, so the same argument as for I^+ shows that if $r > 1/2$, then the estimate is true for all $1/2 < \sigma < 1$. If $r \leq 1/2$ but $|C_-| \geq |\eta - \xi|$, we again get the estimate, since then $\sigma = 1/2 + \varepsilon$, and the sum of the elliptic exponents is $1 + r - 2\varepsilon > 1$, since $r > 1/4$. Hence, we may assume

$$r \leq \frac{1}{2}, \qquad \sigma = \frac{1}{2} + \varepsilon, \qquad |C_-| \leq |\eta - \xi|.$$

Using the cutoff argument, and possibly duality, we can then apply Corollary 1 with $q = 4$, using the fact that $r > 1/4$, on account of (21). This concludes the proof for I_1^- and I_2^-.

It remains to consider I_3^-. If $\sigma = 1/2 + \varepsilon$, we can apply Corollary 1 with $q = 4$ (via duality and the cutoff argument), so we only need to consider the case where $s > 0$ and $r \geq 3/4 + 3s/2$, with any $1/2 + \varepsilon \leq \sigma \leq 1 - \varepsilon$. Then

$$I_3^- \lesssim \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle \xi \rangle^{r} \langle \eta \rangle^{1/4} \langle \eta - \xi \rangle^{-1/4} \langle A \rangle^{1/2+\varepsilon} \langle B \rangle^{1/2+\varepsilon}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Using the triangle inequality, we reduce to either of the following:

$$I_{3,1}^{-} = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta)G_{-}(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r}\langle A\rangle^{1/2+\varepsilon}\langle B\rangle^{1/2+\varepsilon}}\,d\lambda\,d\eta \right\|_{L^2_{\tau,\xi}},$$

$$I_{3,2}^{-} = \left\| \int_{\mathbb{R}^{1+2}} \frac{F(\lambda,\eta)G_{-}(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/4}\langle\eta\rangle^{1/4}\langle A\rangle^{1/2+\varepsilon}\langle B\rangle^{1/2+\varepsilon}}\,d\lambda\,d\eta \right\|_{L^2_{\tau,\xi}}.$$

Both reduce to Corollary 1 with $q = 4$, via duality and the cutoff argument, since $r > 3/4$.

This concludes the proof of Lemma 3.

7. Counterexamples

Note that the main bilinear estimates (see Section 4) $(15')$ and (16) are both of the general form

$$\|\langle \beta\Pi_{\pm_1}(D)\psi, \Pi_{\pm_2}(D)\psi' \rangle\|_{H^{-c,-\gamma}} \lesssim \|\psi\|_{X^{a,\alpha}_{\pm_1}} \|\psi'\|_{X^{b,\beta}_{\pm_2}}. \tag{49}$$

We shall prove the following necessary conditions for this estimate to hold:

Theorem 6. *Let $a,b,c \in \mathbb{R}$ and $\alpha,\beta,\gamma \geq 0$. If (49) holds (with ψ assumed to be supported in $[-2,2]\times\mathbb{R}^2$), then the following conditions hold:*

$$a+b+c \geq \frac{1}{4}, \tag{50}$$

$$a+c \geq 0, \tag{51}$$

$$b+c \geq 0, \tag{52}$$

$$a+b+\gamma \geq 0. \tag{53}$$

$$\frac{a+\alpha}{2}+b+c \geq \frac{1}{2}, \tag{54}$$

$$a+\frac{b+\beta}{2}+c \geq \frac{1}{2}. \tag{55}$$

Applying this to (16), we get the following necessary conditions:

$$r \leq \frac{3}{4}+2s, \tag{56}$$

$$r \leq 1+s, \tag{57}$$

$$s \geq \frac{\rho+\varepsilon}{2}-\frac{1}{2}, \tag{58}$$

$$r \leq \frac{1}{2}+\frac{\sigma}{2}+\frac{3s}{2}, \tag{59}$$

and for $(15')$ we get the conditions

$$r \geq \frac{1}{4}, \tag{60}$$

$$r \geq |s|, \tag{61}$$

$$r \geq \frac{1-\sigma}{2}+\frac{s}{2}, \tag{62}$$

$$r \geq \frac{\sigma+\varepsilon}{2}-\frac{s}{2}, \tag{63}$$

as well as $\rho \geq 0$, which is uninteresting since we only consider $\rho,\sigma > 1/2$. We conclude:

Theorem 7.

(a) *If* $r > 3/4 + 2s$ *or* $r > 1 + s$ *or* $s \le -1/4$, *then* (16) *fails for all* $\rho, \sigma > 1/2$.
(b) *If* $r \le 1/4 - s/2$ *or* $r \le 1/4$ *or* $r < |s|$, *then* (15′) *fails for all* $\rho, \sigma > 1/2$.

Proof. This is more or less immediate. One point to note is that summation of (62) and (63) gives the necessary condition $r > 1/4$, which is stronger than (60). Note also that (58) implies the necessary condition $s > -1/4$, since $\rho > 1/2$. Similarly, since $\sigma > 1/2$, (63) implies the condition $r > 1/4 - s/2$. □

Thus, the sufficient conditions that we have obtained in Section 4 are also seen to be necessary (up to equality in most cases), with the exception of the condition $r > 1/4 + s/2$. So it remains an open question what happens in the region where $\max(1/4, s) \le r \le 1/4 + s/2$ and $0 \le s \le 1/2$.

The remainder of this section is devoted to proving Theorem 6. The following counterexamples all depend on a parameter $L \gg 1$ going to infinity. Note that

$$\begin{aligned}(64)\quad & \|\langle\, \beta\Pi_+(D)\psi, \Pi_\pm(D)\psi'\,\rangle\|_{H^{-c,-\gamma}} \\ &= \left\| \int_{\mathbb{R}^{1+2}} \frac{\Big\langle\, \Pi_\pm(\eta-\xi)\beta\Pi_+(\eta)\widetilde{\psi}(\lambda,\eta), \widetilde{\psi}'(\lambda-\tau,\eta-\xi)\,\Big\rangle}{\langle\xi\rangle^c\langle|\tau|-|\xi|\rangle^\gamma}\, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}}.\end{aligned}$$

In each counterexample we choose either the plus or the minus sign, and we choose sets $A, B, C \subset \mathbb{R}^2$, depending on L and concentrated along the ξ_1-direction, with the property

$$\eta \in A,\ \xi \in C \implies \eta - \xi \in B. \tag{65}$$

We then construct ψ and ψ' depending on L, one of which is supported in $[-2, 2] \times \mathbb{R}^2$, such that

$$\frac{\|\langle\, \beta\Pi_+(D)\psi, \Pi_\pm(D)\psi'\,\rangle\|_{H^{-c,-\gamma}}}{\|\psi\|_{X_+^{a,\alpha}}\, \|\psi'\|_{X_\pm^{b,\beta}}} \ge \frac{1}{CL^\delta}, \tag{66}$$

where C is independent of L, and $\delta = \delta(a, b, c, \alpha, \beta, \gamma)$ depends on the choice of A, B, C. In each case we get a necessary condition $\delta(a, b, c, \alpha, \beta, \gamma) \ge 0$. We shall use the notation $\mathbf{1}_{(\cdot)}$ for the function whose value is 1 if the condition $(\cdot)$ in the subscript is satisfied, and 0 otherwise.

To start with, we take the plus sign in (64), and let $\widetilde{\psi}, \widetilde{\psi}'$ be characteristic functions of slabs cut out of a thickened null hyperplane $\tau + \xi_1 = O(1)$, multiplied by an eigenvector of $\Pi_+(\xi)$ (see (5)),

$$v_+(\xi) = \big[1, \hat{\xi}_1 + i\hat{\xi}_2\big]^T, \qquad \text{where} \quad \hat{\xi} \equiv \frac{\xi}{|\xi|}. \tag{67}$$

Assuming A, B, C have been chosen, we set

$$\widetilde{\psi}(\lambda, \eta) = \mathbf{1}_{\lambda+\eta_1=O(1)}\mathbf{1}_{\eta\in A}v_+(\eta), \tag{68}$$

$$\widetilde{\psi}'(\lambda-\tau, \eta-\xi) = \mathbf{1}_{\lambda-\tau+\eta_1-\xi_1=O(1)}\mathbf{1}_{\eta-\xi\in B}v_+(\eta-\xi), \tag{69}$$

and we restrict the L^2 norm to the region

$$\tau + \xi_1 = O(1), \quad \xi \in C. \tag{70}$$

The fact that we work in a thickened null hyperplane trivializes the convolution structure, since obviously,

$$\tau + \xi_1 = O(1), \ \lambda + \eta_1 = O(1) \implies \lambda - \tau + \eta_1 - \xi_1 = O(1). \tag{71}$$

Using this and (65), we get from (64),

$$\begin{aligned} &\| \langle \beta \Pi_+(D)\psi, \Pi_+(D)\psi' \rangle \|_{H^{r-1,-1/2}} \\ &= \left\| \int \frac{\langle \beta v_+(\eta), v_+(\eta-\xi) \rangle}{\langle \xi \rangle^c \langle |\tau| - |\xi| \rangle^\gamma} \mathbf{1}_{\left\{ \substack{\eta \in A \\ \lambda+\eta_1 = O(1)} \right\}} \mathbf{1}_{\left\{ \substack{\eta-\xi \in B \\ \lambda-\tau+\eta_1-\xi_1 = O(1)} \right\}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} \\ &\geq \left\| \int \frac{\langle \beta v_+(\eta), v_+(\eta-\xi) \rangle}{\langle \xi \rangle^c \langle |\tau| - |\xi| \rangle^\gamma} \mathbf{1}_{\left\{ \substack{\eta \in A \\ \lambda+\eta_1 = O(1)} \right\}} \mathbf{1}_{\left\{ \substack{\xi \in C \\ \tau+\xi_1 = O(1)} \right\}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}} \\ &\geq \left\| \int \frac{\operatorname{Im} \langle \beta v_+(\eta), v_+(\eta-\xi) \rangle}{\langle \xi \rangle^c \langle |\tau| - |\xi| \rangle^\gamma} \mathbf{1}_{\left\{ \substack{\eta \in A \\ \lambda+\eta_1 = O(1)} \right\}} \mathbf{1}_{\left\{ \substack{\xi \in C \\ \tau+\xi_1 = O(1)} \right\}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}, \end{aligned} \tag{72}$$

where in the second step we restrict the L^2 norm to the region (70). But

$$\langle \beta v_+(\eta), v_+(\zeta) \rangle = 1 - \hat{\eta} \cdot \hat{\zeta} + i \hat{\eta} \wedge \hat{\zeta}, \qquad \text{where} \quad \hat{\eta} \wedge \hat{\zeta} = \hat{\eta}_1 \hat{\zeta}_2 - \hat{\eta}_2 \hat{\zeta}_1, \tag{73}$$

hence

$$\operatorname{Im} \langle \beta v_+(\eta), v_+(\eta-\xi) \rangle = \pm \sin \theta_+ \sim \pm \theta_+, \qquad \text{where} \quad \theta_+ = \measuredangle(\eta, \eta-\xi). \tag{74}$$

The choice of sign in front of $\sin \theta_+$ depends on the orientation: The sign is $+$ if rotating η counterclockwise through the angle θ_+ makes it line up with $\eta - \xi$. But the sets A, B, C will be chosen so that the orientation of the pair $(\eta, \eta - \xi)$ is fixed. From (72) we therefore conclude:

$$\begin{aligned} &\| \langle \beta \Pi_+(D)\psi, \Pi_+(D)\psi' \rangle \|_{H^{r-1,-1/2}} \geq I^+, \\ &\text{where} \quad I^+ = \left\| \int \frac{\theta_+}{\langle \xi \rangle^c \langle |\tau| - |\xi| \rangle^\gamma} \mathbf{1}_{\left\{ \substack{\eta \in A \\ \lambda+\eta_1 = O(1)} \right\}} \mathbf{1}_{\left\{ \substack{\xi \in C \\ \tau+\xi_1 = O(1)} \right\}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}. \end{aligned} \tag{75}$$

So far we ignored the requirement that ψ or ψ' be supported in $[-2,2] \times \mathbb{R}^2$, but this is easily fixed. Let $\chi(t)$ be a smooth, even cutoff function such that $0 \leq \chi \leq 1$, $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$. Since χ is even, its Fourier transform $\widehat{\chi}(\tau)$ is real-valued. Moreover, $\widehat{\chi}(\tau) \sim 1$ for $|\tau| = O(1)$, if we interpret $O(1)$ to mean that $|\tau| \leq \delta$ for some constant $\delta > 0$ determined by χ. For convenience we choose to localize ψ' rather than ψ. Note that

$$\widetilde{\chi\psi'}(\lambda - \tau, \eta - \xi) = f(\lambda - \tau + \eta_1 - \xi_1) \mathbf{1}_{\eta - \xi \in B} v_+(\eta - \xi), \tag{76}$$

where

$$f(\tau) = \int_{\mathbb{R}} \widehat{\chi}(\mu) \mathbf{1}_{\tau - \mu = O(1)} \, d\mu.$$

So the only difference from (69) is that $\mathbf{1}_{\lambda-\tau+\eta_1-\xi_1}$ has been replaced by $f(\lambda - \tau + \eta_1 - \xi_1)$. But from (71) and the fact that $f(\tau) \sim 1$ for $\tau = O(1)$, we then conclude that (75) still holds if we replace ψ' by $\chi\psi'$. On the other hand, by (17) we have (recall that $\beta \geq 0$ by assumption)

$$\|\psi'\|_{X_\pm^{b,\beta}} \gtrsim \|\chi\psi'\|_{X_\pm^{b,\beta}},$$

and we conclude that if (66) holds without the cutoff, then it also holds with the cutoff. In what follows we can therefore ignore the cutoff function.

We now choose the sets A, B, C. Note that in I^+,

$$\begin{aligned} &\eta \in A, \qquad \xi \in C, \qquad \eta - \xi \in B, \\ &\lambda + \eta_1 = O(1), \qquad \tau + \xi_1 = O(1), \qquad \lambda - \tau + \eta_1 - \xi_1 = O(1). \end{aligned} \tag{77}$$

7.1. **Necessity of** (50)**.** Here we consider high frequencies interacting to give output at high frequency. Set (see Figure 2)

$$A = \left\{ \xi \in \mathbb{R}^2 : |\xi_1 - L| \le L/4, \ \left| \xi_2 - L^{1/2} \right| \le L^{1/2}/4 \right\},$$
$$B = \left\{ \xi \in \mathbb{R}^2 : |\xi_1 - 2L| \le L/2, \ |\xi_2| \le L^{1/2}/2 \right\},$$
$$C = \left\{ \xi \in \mathbb{R}^2 : |\xi_1 + L| \le L/4, \ \left| \xi_2 - L^{1/2} \right| \le L^{1/2}/4 \right\}.$$

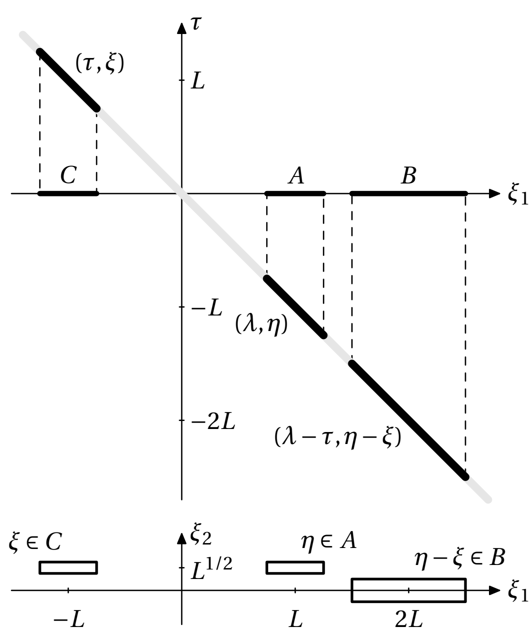

FIGURE 2. Geometry of the sets used to prove necessity of (50).

Then (65) holds. Recalling (77), we see that

$$\theta_+ = \measuredangle(\eta, \eta - \xi) \sim \frac{1}{L^{1/2}}, \qquad |\xi|, |\eta|, |\eta - \xi| \sim L,$$

and

$$\lambda + |\eta| = \lambda + \eta_1 + |\eta| - \eta_1 = \lambda + \eta_1 + \frac{\eta_2^2}{|\eta| + \eta_1} = O(1). \tag{78}$$

Similarly,

$$\lambda - \tau + |\eta - \xi| = O(1), \qquad |\tau| - |\xi| = \tau - |\xi| = O(1). \tag{79}$$

Thus, denoting by $|A|$ the area of A, we see that

$$I^+ \sim \frac{|A|\,|C|^{1/2}}{L^{1/2+c}} \qquad \|\psi\|_{X_+^{a,\alpha}} \sim L^a\,|A|^{1/2}\,, \qquad \|\psi'\|_{X_+^{b,\beta}} \sim L^b\,|B|^{1/2}\,.$$

Since $|A|\,,|B|\,,|C| \sim L^{3/2}$, we conclude that (66) holds with $\delta = a+b+c-1/4$, proving the necessity of (50).

7.2. **Necessity of** (54) **and** (55)**.** Here we use a high/low frequency interaction with output at high frequency. Define (see Figure 3)

$$\begin{aligned} A &= \left\{\xi \in \mathbb{R}^2 : |\xi_1| \le L^{1/2}/2,\ \left|\xi_2 - L^{1/2}\right| \le L^{1/2}/2\right\}, \\ B &= \left\{\xi \in \mathbb{R}^2 : |\xi_1 - L| \le L^{1/2},\ |\xi_2| \le L^{1/2}\right\}, \\ C &= \left\{\xi \in \mathbb{R}^2 : |\xi_1 + L| \le L^{1/2}/2,\ \left|\xi_2 - L^{1/2}\right| \le L^{1/2}/2\right\}. \end{aligned}$$

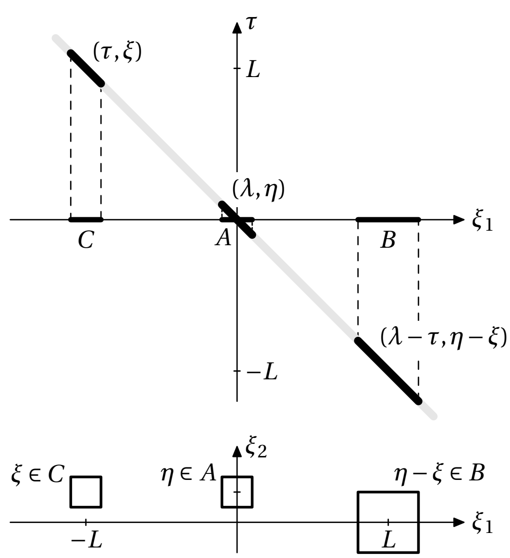

FIGURE 3. Geometry of sets used to prove necessity of (54)/(55) and (51)/(52), the squares A,B,C having sides $\sim L^{1/2}$ or ~ 1, respectively.

Then $\theta_+ = \measuredangle(\eta, \eta-\xi) \sim 1$, $|\eta| \sim L^{1/2}$ and $|\xi|\,,|\eta-\xi| \sim L$. Further, (79) still holds, whereas the calculation in (78) shows that $\lambda + |\eta| \sim L^{1/2}$. Thus,

$$I^+ \sim \frac{|A|\,|C|^{1/2}}{L^c}, \qquad \|\psi\|_{X_+^{a,\alpha}} \sim L^{a/2+\alpha/2}\,|A|^{1/2}\,, \qquad \|\psi'\|_{X_+^{b,\beta}} \sim L^b\,|B|^{1/2}\,.$$

But $|A|\,,|B|\,,|C| \sim L$, hence (66) holds with $\delta = a/2+\alpha/2+b+c-1/2$, proving the necessity of (54). By symmetry we also get (55).

7.3. **Necessity of** (51) **and** (52)**.** The configuration is the same as in the previous subsection, except that the squares A, B, C now have side length ~ 1. We set

$$A = \left\{\xi \in \mathbb{R}^2 : |\xi_1| \le 1/2,\ |\xi_2 - 1| \le 1/2\right\},$$
$$B = \left\{\xi \in \mathbb{R}^2 : |\xi_1 - L| \le 1,\ |\xi_2| \le 1\right\},$$
$$C = \left\{\xi \in \mathbb{R}^2 : |\xi_1 + L| \le 1/2,\ |\xi_2 - 1| \le 1/2\right\}.$$

Then $\theta_+ \sim 1$, $|\eta| \sim 1$, $|\xi|, |\eta - \xi| \sim L$. Since (78) and (79) still hold, we conclude:

$$I^+ \sim \frac{|A|\,|C|^{1/2}}{L^c}, \qquad \|\psi\|_{X_+^{a,\alpha}} \sim |A|^{1/2}, \qquad \|\psi'\|_{X_+^{b,\beta}} \sim L^b\,|B|^{1/2}.$$

But $|A|, |B|, |C| \sim 1$, so (66) holds with $\delta = b + c$, proving necessity of (52). By symmetry we also get (51).

7.4. **Necessity of** (53)**.** Here we consider high frequencies interacting to give output at low frequency, and we choose the minus sign in (64). Let (see Figure 4)

$$A = \left\{\xi \in \mathbb{R}^2 : |\xi_1 - L| \le 1/4,\ |\xi_2 - 1| \le 1/4\right\},$$
$$B = \left\{\xi \in \mathbb{R}^2 : |\xi_1 - L| \le 1/2,\ |\xi_2| \le 1/2\right\},$$
$$C = \left\{\xi \in \mathbb{R}^2 : |\xi_1| \le 1/4,\ |\xi_2 - 1| \le 1/4\right\}.$$

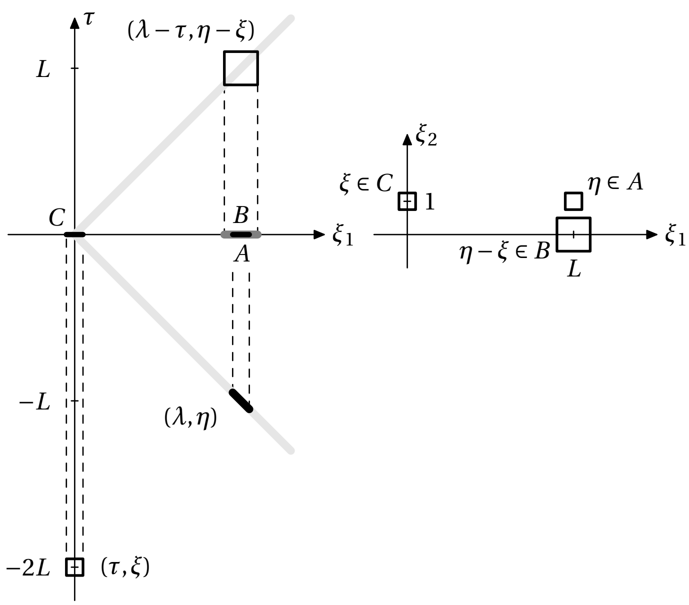

FIGURE 4. Geometry of sets used to prove necessity of (53). The squares A, B, C have side length ~ 1.

In (64) we now restrict the integration to

$$\eta \in A, \qquad \lambda + |\eta| = O(1), \qquad \xi \in C, \qquad \tau + 2L = O(1),$$

which implies

$$\eta - \xi \in B, \qquad \lambda - \tau - |\eta - \xi| = \lambda + |\eta| - \tau - 2L + L - |\eta| + L - |\eta - \xi| = O(1),$$

since $L - |\eta| = L - \eta_1 - \eta_2^2/(|\eta| + \eta_1) = O(1)$ and, similarly, $L - |\eta - \xi| = O(1)$. Now set

$$\widetilde{\psi}(\lambda, \eta) = \mathbf{1}_{\lambda+|\eta|=O(1)} \mathbf{1}_{\eta \in A} v_+(\eta),$$
$$\widetilde{\psi}'(\lambda - \tau, \eta - \xi) = \mathbf{1}_{\lambda-\tau-|\eta-\xi|=O(1)} \mathbf{1}_{\eta-\xi \in B} v_-(\eta - \xi),$$

where $v_-(\xi) = v_+(-\xi)$ and $v_+(\xi)$ is given by (67). Thus, $v_-(\xi)$ is an eigenvector of $\Pi_-(\xi) = \Pi_+(-\xi)$. Since $\theta_- = \measuredangle(\eta, \xi - \eta) \sim 1$, we then get, arguing as in (72), and using (73),

$$\| \langle \beta \Pi_+(D)\psi, \Pi_-(D)\psi' \rangle \|_{H^{-c,-\gamma}} \geq I^-,$$
$$\text{where} \quad I^- = \left\| \int \frac{1}{\langle \xi \rangle^c \langle |\tau| - |\xi| \rangle^\gamma} \mathbf{1}_{\left\{ \substack{\eta \in A \\ \lambda + |\eta| = O(1)} \right\}} \mathbf{1}_{\left\{ \substack{\xi \in C \\ \tau + 2L = O(1)} \right\}} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Since $|\xi| \sim 1$, $|\eta|, |\eta - \xi| \sim L$ and $|\tau| - |\xi| \sim |\tau| \sim L$, we see that

$$I^- \sim \frac{|A| \, |C|^{1/2}}{L^\gamma}, \qquad \|\psi\|_{X_+^{a,\alpha}} \sim L^a \, |A|^{1/2}, \qquad \|\psi'\|_{X_+^{b,\beta}} \sim L^b \, |B|^{1/2}.$$

But $|A|, |B|, |C| \sim 1$, hence (66) holds with $\delta = a + b + \gamma$, proving necessity of (53).

This concludes the proof of Theorem 6.

8. Proof of Theorem 2

We use the following sharp dyadic decomposition in frequency space. By a *dyadic number* we mean a number of the form 2^j, where $j \in \mathbb{Z}$. We let κ, λ and μ be dyadic numbers. Given $f \in \mathcal{S}(\mathbb{R}^2)$, we denote by f_λ the function whose Fourier transform is $\chi_{A_\lambda} \widehat{f}$, where A_λ is the annulus $\lambda \leq |\xi| \leq 2\lambda$. We also use a decomposition of $\mathbb{R}^2$ into almost disjoint dyadic squares $Q = [j\mu, (j+1)\mu] \times [k\mu, (k+1)\mu]$ of side length μ, where $j, k \in \mathbb{Z}$; we shall refer to these squares as *μ-squares*. We denote by f_λ^Q the function whose Fourier transform is $\chi_{A_\lambda \cap Q} \widehat{f}$. Recall that $S_\pm(t) = e^{\mp it|D|}$ is the free propagator for $-i\partial_t \pm |D|$. We let $f, g \in \mathcal{S}(\mathbb{R}^2)$ and write, for any combination of signs,

$$u(t) = S_\pm(t) f, \qquad v(t) = S_\pm(t) g.$$

We also write $u_\lambda(t) = S_\pm(t) f_\lambda$ and $u_\lambda^Q(t) = S_\pm(t) f_\lambda^Q$, and similarly for v.

An exponent pair (q, r) is said to be *wave admissible* if $2/q + 1/r \leq 1/2$ and $r \neq \infty$. For (q, r) wave admissible, the Strichartz estimate holds:

$$\|u\|_{L_t^q L_x^r} \leq C_{q,r} \|f\|_{\dot{H}^s} \qquad \left(s = 1 - \frac{2}{r} - \frac{1}{q} \right). \tag{80}$$

Note that if $r = 4$, then all $q \geq 8$ are admissible, and the following bilinear version holds:

$$\|uv\|_{L_t^q L_x^2} \leq C_{q,\delta} \|f\|_{\dot{H}^\delta} \|g\|_{\dot{H}^{1-1/q-\delta}} \qquad (4 \leq q \leq \infty, \quad 0 < \delta < 1 - 1/q). \tag{81}$$

This follows from (80) using Hölder's inequality with suitable wave admissible pairs (q_1, r_1) and (q_2, r_2). In fact, by interpolation it suffices to consider $q = 4$ and $q = \infty$.

As proved in [17], the constant in (80) can be improved if $\widehat{f}$ is supported in a small set: If $\widehat{f}$ is supported in a μ-square at distance $\sim \lambda$ from the origin, then the constant can be replaced by $C_{q,r}(\mu/\lambda)^{1/2-1/r}$. In fact, we shall need only the case $(q,r)=(8,4)$:

$$\left\|u_\lambda^Q\right\|_{L_t^8 L_x^4} \le C\left(\frac{\mu}{\lambda}\right)^{1/4}\left\|f_\lambda^Q\right\|_{\dot H^{3/8}} \sim \mu^{1/4}\lambda^{1/8}\left\|f_\lambda^Q\right\|_{L^2}, \tag{82}$$

where Q is a μ-square at distance $\sim \lambda$ from the origin. In order to prove (82) it is sufficient to assume $\lambda = 1$ after rescaling. By a standard TT^* argument, (82) is then equivalent to the estimate

$$\left\|\int S_\pm(t-s)F(s)ds\right\|_{L_t^8 L_q^4} \le C\mu^{1/2}\left\|F\right\|_{L_t^{8/7}L_x^{4/3}} \tag{83}$$

provided the support in ξ of $\widehat{F}(t,\xi)$ is contained in Q. Now we recall the estimate (A.67) from [17], i.e., in our notations

$$\|S_\pm(t)f_1^Q\|_{L^r} \le c\mu^{1-\frac{2}{r}}\,(1+|t|)^{-\frac{n-1}{2}(1-\frac{2}{r})}\|f_1^Q\|_{L^{r'}}.$$

Applying this estimate for $r=4$, $n=2$ we have

$$\left\|\int S_\pm(t-s)F(s)ds\right\|_{L_x^4} \le C\mu^{1/2}\int(1+|t-s|)^{-\frac{1}{4}}\left\|F\right\|_{L_x^{4/3}}$$

and the Hardy-Sobolev inequality as usual gives (83).

We also need the analogous small-support improvement of the Sobolev inequality $\left\|f\right\|_{L_x^4} \le C\left\|f\right\|_{\dot H^{1/2}}$. By the Hausdorff-Young inequality,

$$\left\|f_\lambda^Q\right\|_{L_x^4} \le \left\|\chi_{A_\lambda\cap Q}\widehat{f}\right\|_{L_\xi^{4/3}} \le \left\|\chi_{A_\lambda\cap Q}\right\|_{L_\xi^4}\left\|\chi_{A_\lambda\cap Q}\widehat{f}\right\|_{L_\xi^2} \le \mu^{1/2}\left\|f_\lambda^Q\right\|_{L_x^2},$$

where Q is a μ-square. Hence,

$$\left\|S_\pm(t)f_\lambda^Q\right\|_{L_t^\infty L_x^4} \le \mu^{1/2}\left\|S_\pm(t)f_\lambda^Q\right\|_{L_t^\infty L_x^2} = \mu^{1/2}\left\|f_\lambda^Q\right\|_{L_x^2},$$

and interpolation between this and (82) gives

$$\left\|u_\lambda^Q\right\|_{L_t^q L_x^4} \le C\mu^{1/2-2/q}\lambda^{1/q}\left\|f_\lambda^Q\right\|_{L^2} \qquad (8\le q\le\infty). \tag{84}$$

Using this and (81) we now prove Theorem 2.

Write $u=\sum_\kappa u_\kappa$, $v=\sum_\lambda v_\lambda$, $u_{\ll\lambda}=\sum_{\kappa\le\lambda/4}u_\kappa$ and $v_{\ll\kappa}=\sum_{\lambda\le\kappa/4}v_\lambda$, where κ and λ are dyadic numbers. Then

$$uv=\sum_\lambda u_{\ll\lambda}v_\lambda+\sum_\kappa\sum_{\kappa/2\le\lambda\le2\kappa}u_\kappa v_\lambda+\sum_\kappa u_\kappa v_{\ll\kappa}=S_1+S_2+S_3.$$

The estimate for S_2 reduces to (84), by the argument used in the proof of Theorem 4 in Appendix A of [17]. Indeed, it is sufficient to estimate the diagonal terms $u_\lambda v_\lambda$ since the other terms $u_{2\lambda}v_\lambda$ and $u_{\lambda/2}v_\lambda$ can be handled in an identical way. Denoting by Δ_μ the spectral projection on the annulus $\mu/2\le|\xi|\le2\mu$, we can write

$$\begin{aligned}\||D|^{-s_3}\sum_\lambda u_\lambda v_\lambda\|_{L_t^qL_x^2} &\le \sum_\lambda\sum_{\mu\le4\lambda}\||D|^{-s_3}\Delta_\mu(u_\lambda v_\lambda)\|_{L_t^qL_x^2}\\ &\sim \sum_\lambda\sum_{\mu\le4\lambda}\mu^{-s_3}\|\Delta_\mu(u_\lambda v_\lambda)\|_{L_t^qL_x^2}\end{aligned}$$

since the product $u_\lambda v_\lambda$ vanishes at frequencies larger than 4λ. Now assume we can prove the following estimate:

$$\|\Delta_\mu(u_\lambda v_\lambda)\|_{L^q_t L^2_x} \lesssim \left(\frac{\mu}{\lambda}\right)^{1-2/q} \lambda^{1-1/q}\|f_\lambda\|_{L^2}\|g_\lambda\|_{L^2}; \tag{85}$$

then we get

$$\||D|^{-s_3}\sum_\lambda u_\lambda v_\lambda\|_{L^q_t L^2_x} \lesssim \sum_\lambda \sum_{\mu\le 4\lambda}\left(\frac{\mu}{\lambda}\right)^{1-2/q-s_3}\lambda^{1-1/q-s_3}\|f_\lambda\|_{L^2}\|g_\lambda\|_{L^2}$$

from which the required estimate follows immediately. Thus we are reduced to prove (85) for $\mu \le 4\lambda$, $4 \le q \le \infty$. By rescaling, it is sufficient to prove (85) for $\lambda = 1$, i.e.,

$$\|\Delta_\mu(uv)\|_{L^q_t L^2_x} \lesssim \mu^{1-2/q}\|f\|_{L^2}\|g\|_{L^2} \tag{86}$$

where $\mu \le 4$ and f, g are frequency localized in the annulus $|\xi| \sim 1$. Notice that for $1/4 \le \mu \le 4$ (86) follows from Hölder's inequality and the standard Strichartz estimate, thus we can assume $\mu << 1$. We now resort to the finer frequency decomposition into the squares Q of side μ. The key remark here is that, fixing any square Q at a distance ~ 1 from the origin, the product $\Delta_\mu(u_Q v_{Q'})$ is different from zero only for Q' belonging to a finite set M_Q of μ-squares, with the property that $\#M_Q$ is bounded by an absolute constant. Thus we can write

$$\begin{aligned}\|\Delta_\mu(uv)\|_{L^q_t L^2_x} &\lesssim \sum_Q \sum_{Q'\in M_Q}\|\Delta_\mu(u^Q v^{Q'})\|_{L^q_t L^2_x}\\ &\lesssim \sum_Q \sum_{Q'\in M_Q}\|u^Q\|_{L^{2q}_t L^4_x}\|v^{Q'}\|_{L^{2q}_t L^4_x}.\end{aligned} \tag{87}$$

Applying now the improved Strichartz estimate (84) and the Cauchy-Schwarz inequality, we obtain (86).

We can now restrict our attention to S_1 and S_3. By symmetry, it suffices to consider S_1.

Since the Fourier transform of $u_{\ll\lambda}v_\lambda$ is supported in the annulus $\lambda/2 \le |\xi| \le 4\lambda$, we have, by orthogonality,

$$\left\||D|^{-s_3}S_1\right\|_{L^2_x} \sim \Big(\sum_\lambda \left\||D|^{-s_3}(u_{\ll\lambda}v_\lambda)\right\|^2_{L^2_x}\Big)^{1/2} \sim \Big(\sum_\lambda \lambda^{-2s_3}\left\|u_{\ll\lambda}v_\lambda\right\|^2_{L^2_x}\Big)^{1/2},$$

so by Minkowski's integral inequality,

$$\left\||D|^{-s_3}S_1\right\|_{L^q_t L^2_x} \lesssim \Big(\sum_\lambda \lambda^{-2s_3}\left\|u_{\ll\lambda}v_\lambda\right\|^2_{L^q_t L^2_x}\Big)^{1/2}.$$

Now we apply (81), considering separately the cases $s_1 > 0$ and $s_1 \le 0$. If $s_1 > 0$, we get, noting that $s_2 + s_3 = 1 - 1/q - s_1 > 0$,

$$\begin{aligned}\left\||D|^{-s_3}S_1\right\|_{L^q_t L^2_x} &\lesssim \Big(\sum_\lambda \lambda^{-2s_3}\left\|f_{\ll\lambda}\right\|^2_{\dot H^{s_1}}\|g_\lambda\|^2_{\dot H^{s_2+s_3}}\Big)^{1/2}\\ &\lesssim \|f\|_{\dot H^{s_1}}\Big(\sum_\lambda \|g_\lambda\|^2_{\dot H^{s_2}}\Big)^{1/2}.\end{aligned}$$

If, on the other hand, $s_1 \le 0$, then $s_2 > 1/q - s_1 \ge 0$, so

$$\begin{aligned}\left\| |D|^{-s_3} S_1 \right\|_{L_t^q L_x^2} &\lesssim \Big(\sum_\lambda \lambda^{-2s_3} \left\| f_{\ll\lambda} \right\|_{\dot H^{s_1+s_3}}^2 \left\| g_\lambda \right\|_{\dot H^{s_2}}^2 \Big)^{1/2} \\ &\sim \Big(\sum_\lambda \sum_{\kappa \le \lambda/4} \Big(\frac{\kappa}{\lambda}\Big)^{2s_3} \left\| f_\kappa \right\|_{\dot H^{s_1}}^2 \left\| g_\lambda \right\|_{\dot H^{s_2}}^2 \Big)^{1/2} \\ &\lesssim \left\| f \right\|_{\dot H^{s_1}} \Big(\sum_\lambda \left\| g_\lambda \right\|_{\dot H^{s_2}}^2 \Big)^{1/2}.\end{aligned}$$

In the last step we used the fact that $s_3 > 0$, since $s_3 = (1-1/q-s_2)-s_1 > -s_1 \ge 0$.

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