CONVERGENCE OF THE DIRAC-MAXWELL SYSTEM TO THE VLASOV-POISSON SYSTEM

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ABSTRACT. We study the behavior of solutions of the (mixed state) Dirac-Maxwell system in the simultaneous non-relativistic and semi-classical limits, i.e., as the speed of light c tends to infinity and the Planck constant $\hbar$ tends to zero. Under a constraint of the form $\frac{1}{c}\exp\left(\frac{1}{\hbar^M}\right) = o(1)$, where M is a sufficiently large integer, and with suitable conditions on the initial data, we prove that the scalar Wigner transform of the Dirac spinors converges weakly to a solution of the classical Vlasov-Poisson system. Our key technique is a blend of null form bilinear estimates with the machinery of the Wigner transform.

1. INTRODUCTION

The Dirac-Maxwell system (DM) is one of the key equations in quantum electrodynamics, describing the self-consistent interaction between spinor (representing a 4-component electron-positron state) and electromagnetic fields. According to quantum statistical mechanics we do not assume that the initial state is defined/prepared as only one "pure state" but rather as a statistical mixture of infinitely many states, which is essential when applying Wigner transform techniques, since it allows for specific assumptions on the occupation probabilities as an additional initial datum.

We write the Maxwell equations as wave equations for the potentials, choosing the Coulomb gauge condition $\operatorname{div}\mathbf{A} = 0$, which has the crucial advantage that the main bilinear terms in the Dirac equation appear as null bilinear forms. Hence we consider the following formulation of the DM system on $\mathbb{R}_t \times \mathbb{R}^3_x$:

$$
\begin{cases}
i\hbar\partial_t\psi_j^\varepsilon = -i\hbar c\alpha^k\partial_k\psi_j^\varepsilon + c^2\beta\psi_j^\varepsilon - A_k^\varepsilon\alpha^k\psi_j^\varepsilon - A_0^\varepsilon\psi_j^\varepsilon, \quad j\in\mathbb{N}, \\
\Delta A_0^\varepsilon = \rho^\varepsilon, \qquad \rho^\varepsilon = \sum_{j=1}^\infty \lambda_j^\varepsilon \left|\psi_j^\varepsilon\right|^2, \\
\square_c\mathbf{A}^\varepsilon = \mathbb{P}\left(\mathbf{J}^\varepsilon/c\right), \qquad \mathbf{J}^\varepsilon = c\sum_{j=1}^\infty \lambda_j^\varepsilon \left\langle \vec{\alpha}\psi_j^\varepsilon, \psi_j^\varepsilon\right\rangle_{\mathbb{C}^4}.
\end{cases} \tag{1}
$$

The unknowns are (i) the sequence of 4-spinors $\{\psi_j^\varepsilon(t,x)\}_{j\in\mathbb{N}} \subset \mathbb{C}^4$, regarded as column vectors, (ii) the electric potential $A_0^\varepsilon(t,x) \in \mathbb{R}$ and (iii) the magnetic potential $\mathbf{A}^\varepsilon(t,x) = (A_1^\varepsilon, A_2^\varepsilon, A_3^\varepsilon) \in \mathbb{R}^3$. Here, the particle charge and rest mass have been set equal to unity. The superscript

$$\varepsilon := \left(\frac{1}{c}, \hbar\right)$$

is used to emphasize the dependence of the solution on the parameters c, the speed of light, and $\hbar$, the Planck constant. Our aim is to perform the mathematically rigorous limit $\varepsilon \to 0$. Thus, c tends to infinity (the *non-relativistic* limit) and $\hbar$ tends to zero (the *semi-classical* limit).

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We write $\partial_t = \frac{\partial}{\partial t}$, $\partial_k = \frac{\partial}{\partial x^k}$ for $k = 1, 2, 3$, $\nabla = (\partial_1, \partial_2, \partial_3)$ for the spatial gradient, $\partial = (\partial_0, \nabla) = (\frac{1}{c}\partial_t, \nabla)$ for the spacetime gradient, and $\Delta = \sum_{j=1}^3 \partial_j^2$ for the spatial Laplacian. Further, $\square_c = \frac{1}{c^2}\partial_t^2 - \Delta$ is the wave operator, $\mathbb{P}$ is the projection onto divergence free vector fields in $\mathbb{R}_x^3$ and the α^k's and β are the Dirac matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix},$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. Repeated lower and upper roman indices are understood to be summed over 1, 2, 3. Thus, $\alpha^k \partial_k = \sum_{k=1}^3 \alpha^k \partial_k$, for example. Finally, $\langle \cdot, \cdot \rangle_{\mathbb{C}^4}$ is the standard inner product on $\mathbb{C}^4$ and $\langle \vec{\alpha}\psi, \psi \rangle$ denotes the 3-vector with components $\langle \alpha^k \psi, \psi \rangle$ for $k = 1, 2, 3$.

The data for (1) consist of (i) a non-negative sequence $\{\lambda_j^\varepsilon\}_{j\in\mathbb{N}} \in l^1$ (the probability densities) and (ii) initial values for the dynamical variables:

$$\psi_j^\varepsilon|_{t=0} = \psi_{j,I}^\varepsilon \quad \text{for } j \in \mathbb{N}; \quad \mathbf{A}^\varepsilon|_{t=0} = \mathbf{a}_I^\varepsilon, \quad \partial_t \mathbf{A}^\varepsilon|_{t=0} = \mathbf{b}_I^\varepsilon, \tag{2}$$

where we must require $\operatorname{div} \mathbf{a}_I^\varepsilon = \operatorname{div} \mathbf{b}_I^\varepsilon = 0$ due to the Coulomb condition.

Throughout we assume that for all ε,

$$\lambda_j^\varepsilon \geq 0 \quad \text{and} \quad \sum \lambda_j^\varepsilon = 1, \tag{3}$$

$$\left\{\psi_{j,I}^\varepsilon\right\}_{j\in\mathbb{N}} \quad \text{is orthonormal in} \quad L^2(\mathbb{R}_x^3)^4, \tag{4}$$

$$\left(\sum \lambda_j^\varepsilon \hbar^2 \left\| \nabla \psi_{j,I}^\varepsilon \right\|_{L_x^2}^2\right)^{1/2} + \left\| \nabla \mathbf{a}_I^\varepsilon \right\|_{L_x^2} + \frac{1}{c} \left\| \mathbf{b}_I^\varepsilon \right\|_{L_x^2} \leq B, \tag{5}$$

where $B > 1$ is a constant independent of ε. Note that (3) means that the weights λ_j^ε are indeed "occupation probabilities", whereas (5) says that the kinetic energy and the energy of the electromagnetic field are bounded initially.

Furthermore, we assume

$$\left(\sum \lambda_j^\varepsilon \left\| \psi_{j,I}^\varepsilon \right\|_{H^5}^2\right)^{1/2} + \left\| \nabla \mathbf{a}_I^\varepsilon \right\|_{H^4} + \frac{1}{c} \left\| \mathbf{b}_I^\varepsilon \right\|_{H^4} \leq B \exp\left(\frac{1}{\hbar^{m_0}}\right), \tag{6}$$

$$\left(\sum \lambda_j^\varepsilon \left\| \eta_{j,I}^\varepsilon \right\|_{H^2}^2\right)^{1/2} \leq \frac{B}{c} \exp\left(\frac{1}{\hbar^{m_0}}\right) \tag{7}$$

for all ε. Here m_0 is a fixed positive integer, $H^s = H^s(\mathbb{R}_x^3)$ is the standard L^2-based Sobolev space, and we use the following notation for the upper and lower 2-spinors of the 4-spinor fields ψ_j^ε:

$$\psi_j^\varepsilon = \begin{pmatrix} \chi_j^\varepsilon \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_j^\varepsilon \end{pmatrix}, \tag{8}$$

and similarly for the initial data $\psi_{j,I}^\varepsilon$. A rough heuristic is that $\{\chi_j^\varepsilon\}$ represents electrons and $\{\eta_j^\varepsilon\}$ positrons. Thus, (7) guarantees that the energy initially is concentrated in the upper component, or electron part, of the spinor fields. As it turns out, this property persists in the evolution.

In our work we are interested in the asymptotic analysis of DM (and the related Klein-Gordon-Maxwell system) with respect to the non-relativistic limit (also called post-Newtonian limit) and the (semi-)classical limit.

The following sketch shows the connection of the different levels of asymptotic approximation of DM :

$$\begin{array}{ccc} \boxed{\textit{Dirac-Maxwell (DM)}} & \overset{\hbar\to 0}{\longrightarrow} & \boxed{\textit{Relativistic Vlasov-Maxwell (RVM)}} \\ \Big\downarrow {\scriptstyle c\to\infty} & \searrow \begin{smallmatrix} c\to\infty \\ \hbar\to 0 \end{smallmatrix} & \Big\downarrow {\scriptstyle c\to\infty} \\ \boxed{\textit{Schrödinger-Poisson (SP)}} & \overset{\hbar\to 0}{\longrightarrow} & \boxed{\textit{Vlasov-Poisson (VP)}} \end{array}$$

The limit represented by the vertical arrow on the right was established by J. Schaeffer [15]; see also P. Degond [5].

The vertical arrow on the left, and also the corresponding limit for the related Klein-Gordon-Maxwell (KGM) system, was proved recently by the authors in joint work with P. Bechouche [2, 3, 4] and independently by N. Masmoudi and K. Nakanishi [14].

The lower horizontal arrow is due to P.-L. Lions and T. Paul [9] and, independently, P. A. Markowich and the first author [10] for a slightly regularized system; both papers deal with mixed state case, the only pure state result available so far being for the one-dimensional case, see [16].

As for the upper horizontal arrow, the linear problem, i.e., the passage from Dirac with given electromagnetic field to rel. Vlasov with given Lorentz force has been given in [7] — however, for the nonlinear problem this formal limit remains at the level of a conjecture.[1]

In this paper we prove a result corresponding to the diagonal arrow, under the constraint, however, that c tends to infinity much faster than $1/\hbar$ (cf. condition (9) below). Also, for the three dimensional case that we consider, the use of mixed states and the condition (30) on the initial data is indispensable.

Let us mention that there are also ”semi-nonrelativistic” models like the selfconsistent Pauli equation which contains corrections to the Schrödinger equation at $O(1/c)$ (cf [13], [4], where the situation concerning both rigorous modeling and asymptotic limits is still open, with work of the authors in progress.

Note that for the purpose of the combined non-relativistic/semi-classical limit from Dirac-Maxwell to Vlasov-Poisson, the local well-posedness result in [14] is not strong enough, since it only gives existence on a uniform (w.r.t. c) time interval for fixed $\hbar$. The result in [4] however, yields a rate of growth in c, which is crucial to ensure a nontrivial time interval for the limit considered here.

2. STATEMENT OF RESULTS

A major open question is whether solutions of DM exist globally in time for, say, smooth compactly supported data of any size. For small data, this question has been answered in the affirmative by Georgiev [6]. Although the case of large data remains open, it was proved by the present authors in joint work with P. Bechouche [4] that the local existence time $T = T(c)$ grows at least as fast as $\log(c)$ as c tends to infinity, if the data are uniformly bounded in H^1. In this result, $\hbar$ was set equal to unity, but this can easily be achieved by

[1] In recent work by P. A. Markowich and C. Sparber [12] a WKB analysis of DM in the semiclassical limit is carried out for small data and for small times.

rescaling, hence the results of [4] can be applied also to the limit $\varepsilon \to 0$; in so doing, we need to assume

$$c \geq \exp\left(\frac{aT_0}{\hbar^M}\right) \tag{9}$$

in order to ensure a uniform time interval $[0, T_0]$ of existence as $\varepsilon \to 0$. This condition will be tacitly assumed from now on. Here, and throughout the paper, we use a and M, and later also C, to denote sufficiently large constants which are independent of ε and T_0, but may depend on the fixed parameters B and m_0; moreover, these constants change from line to line. Thus, in the end we choose a and M in (9) so large that the right hand sides of (14), (20), (22) etc. are $o(1)$ as $\varepsilon \to 0$.

From now on, we consider the arbitrary, but finite, time interval $[0, T_0]$ to be fixed. We then have the following existence result, which follows rather easily from our earlier paper [4]:

Theorem 1. ***(Uniform local existence for DM.)***

(i) *For any fixed ε, the system (1) is locally well posed for data (2) satisfying (3)–(5). The solution $t \mapsto (\{\psi_j^\varepsilon\}, \mathbf{A}^\varepsilon, \partial_t \mathbf{A}^\varepsilon)(t)$ exists in a non-trivial time interval $0 \leq t \leq T = T(\varepsilon, B)$ and describes a continuous curve in the space*[2] *determined by the conditions (4), (5). In particular, the L^2 orthonormality (4) is conserved in time:*

$$\int \left\langle \psi_j^\varepsilon(t,x), \psi_k^\varepsilon(t,x) \right\rangle_{\mathbb{C}^4} dx = \delta_{jk} \quad \text{for all } j,k \text{ and } 0 \leq t \leq T. \tag{10}$$

(ii) *Assume (3)–(5) hold for all ε. Then there exist absolute constants a and M, and there exists a $\delta = \delta(B) > 0$ depending on the bound B in (5), such that for all $\varepsilon = (1/c, \hbar)$ satisfying $|\varepsilon| \leq \delta$ and the condition (9), we have $T(\varepsilon, B) \geq T_0$, i.e., the solution exists in the time interval $0 \leq t \leq T_0$. Moreover,*[3]

$$\left(\sum \lambda_j^\varepsilon \hbar^2 \left\| \nabla \psi_j^\varepsilon(t) \right\|_{L_x^2}^2 \right)^{1/2} + \frac{1}{c^{1/4}} \left\| \partial \mathbf{A}^\varepsilon(t) \right\|_{L_x^2} \leq C \exp\left(\frac{aT_0}{\hbar^M}\right), \tag{11}$$

for all $0 \leq t \leq T_0$, where as before $\partial = (\frac{1}{c}\partial_t, \nabla)$ is the spacetime gradient. (Recall also the convention concerning constants, set out above.)

(iii) *Assume in addition that (6), (7) are satisfied for all ε. Then the solution $t \mapsto (\{\psi_j^\varepsilon\}, \mathbf{A}^\varepsilon, \partial_t \mathbf{A}^\varepsilon)(t)$, for $0 \leq t \leq T_0$, describes a continuous curve in the space determined by the norm in the left hand side of (6), and we have bounds*

$$\left(\sum \lambda_j^\varepsilon \left\| \psi_j^\varepsilon(t) \right\|_{H^5}^2 \right)^{1/2} + \frac{1}{c^{1/4}} \left\| \partial \mathbf{A}^\varepsilon(t) \right\|_{H^4} \leq C \exp\left(\frac{aT_0}{\hbar^M}\right), \tag{12}$$

$$\left\| \partial \mathbf{A}^\varepsilon(t) \right\|_{H^2} \leq C \exp\left(\frac{aT_0}{\hbar^M}\right), \tag{13}$$

$$\left(\sum \lambda_j^\varepsilon \left\| \eta_j^\varepsilon(t) \right\|_{H^2}^2 \right)^{1/2} \leq \frac{C}{c} \exp\left(\frac{aT_0}{\hbar^M}\right) \tag{14}$$

for all ε and $0 \leq t \leq T_0$.

[2] The regularity of of the non-dynamical variable $A_0^\varepsilon = \Delta^{-1}\rho^\varepsilon$, on the other hand, is of course directly determined by that of the ψ_j^ε.

[3] Here the factor $c^{-1/4}$ is chosen somewhat arbitrarily; it could in fact be replaced by $c^{-\alpha}$ for any $\alpha > 0$, but then C in the right hand side would of course depend on α. In any event, the choice $\alpha = 1/4$ works well enough for our present purposes.

A key idea behind our main convergence result is that for the kinetic energy

$$\sum \lambda_j^\varepsilon \hbar^2 \|\nabla \psi_j^\varepsilon(t)\|_{L_x^2}^2,$$

the exponential bound in (11) can be improved to $O(1)$ by writing the DM system as a perturbed Schrödinger-Poisson (SP) system for which the total energy is “almost conserved”. This involves the decomposition

$$\psi_j^\varepsilon = \psi_{j,+}^\varepsilon + \psi_{j,-}^\varepsilon \tag{15}$$

where

$$\psi_{j,\pm}^\varepsilon = \frac{1}{2}\left\{\psi_j^\varepsilon \pm \left[\mu^c(\hbar D)\right]^{-1}\left(i\hbar\partial_t \psi_j^\varepsilon + A_0^\varepsilon \psi_j^\varepsilon\right)\right\} \tag{16}$$

and $\mu^c(D)$ is the (spatial) Fourier multiplier with symbol

$$\mu^c(\xi) = \sqrt{c^2\,|\xi|^2 + c^4}, \quad \text{hence} \quad \mu^c(D) = \sqrt{-c^2\Delta + c^4}, \tag{17}$$

ξ being the Fourier variable corresponding to x. The decomposition (15) is related to positive and negative energy, and to first order in powers of $1/c$ it coincides with the splitting (8) into upper and lower components; see [4], and Sect. 4 below. In view of (14) it is therefore not surprising that $\psi_{j,-}$ is negligibly small for our present purposes. Thus, we retain only the positive energy part, from which we subtract the rest energy, i.e., we define (the “Foldy-Wouthuysen” transformation)

$$\phi_j^\varepsilon = e^{itc^2/\hbar}\psi_{j,+}^\varepsilon. \tag{18}$$

Then we have the following result:

Theorem 2. ***(DM as a perturbed SP system.)*** *Consider the solution of (1), (2) obtained in Theorem 1, with initial assumptions (3)–(7), and assuming also (9), as always. Then the 4-spinors ϕ_j^ε, defined as above, satisfy the perturbed SP system*

$$\begin{cases} i\hbar\partial_t\phi_j^\varepsilon = -\dfrac{\hbar^2}{2}\Delta\phi_j^\varepsilon + V^\varepsilon\phi_j^\varepsilon + \mathrm{Error}_j^\varepsilon, \quad j \in \mathbb{N}, \\ \Delta V^\varepsilon = -n^\varepsilon, \qquad n^\varepsilon = \sum \lambda_j^\varepsilon \left|\phi_j^\varepsilon\right|^2, \end{cases} \tag{19}$$

on $[0, T_0] \times \mathbb{R}_x^3$, *with data*

$$\phi_j^\varepsilon|_{t=0} = \frac{1}{2}\left\{\psi_{j,I}^\varepsilon + \left[\mu^c(\hbar D)\right]^{-1}\left(-i\hbar c\alpha^k\partial_k\psi_{j,I}^\varepsilon + c^2\beta\psi_{j,I}^\varepsilon - (\mathbf{a}_I^\varepsilon)_k\alpha^k\psi_{j,I}^\varepsilon\right)\right\},$$

and for the spinor valued error terms we have the estimate

$$\left(\sum \lambda_j^\varepsilon \left\|\mathrm{Error}_j^\varepsilon(t)\right\|_{H^1}^2\right)^{\frac{1}{2}} \leq \frac{C}{c}\exp\left(\frac{aT_0}{\hbar^M}\right) \tag{20}$$

for all ε and $0 \leq t \leq T_0$. The perturbed conservation law

$$\partial_t n^\varepsilon + \operatorname{div}\mathbf{J}_S^\varepsilon = \frac{2}{\hbar}\sum \lambda_j^\varepsilon \,\mathrm{Im}\left\langle \mathrm{Error}_j^\varepsilon, \phi_j^\varepsilon\right\rangle_{\mathbb{C}^4}, \tag{21}$$

holds, where $\mathbf{J}_{\mathrm{S}}^{\varepsilon} = \hbar \sum \lambda_j^{\varepsilon} \operatorname{Im}\left\langle \nabla\phi_j^{\varepsilon}, \phi_j^{\varepsilon} \right\rangle_{\mathbb{C}^4}$ *is the Schrödinger current density. Furthermore, comparing with the solution of (1), we have*

$$\left(\sum \lambda_j^{\varepsilon} \left\| \phi_j^{\varepsilon}(t) - e^{itc^2/\hbar} \psi_j^{\varepsilon}(t) \right\|_{H^2}^2 \right)^{\frac{1}{2}} \le \frac{C}{c} \exp\left(\frac{aT_0}{\hbar^M}\right), \tag{22}$$

$$\left\| \rho^{\varepsilon}(t) - n^{\varepsilon}(t) \right\|_{L^p} \le \frac{C}{c} \exp\left(\frac{aT_0}{\hbar^M}\right) \quad \text{for} \quad 1 \le p \le \infty, \tag{23}$$

for all ε *and* $0 \le t \le T_0$*, and*

$$\mathbf{J}^{\varepsilon} - \mathbf{J}_{\mathrm{S}}^{\varepsilon} - \nabla \times \left(\sum \lambda_j^{\varepsilon} \langle \vec{S}\phi_j^{\varepsilon}, \phi_j^{\varepsilon} \rangle\right) \to 0 \quad \text{in } \mathcal{D}'\big((0,T_0)\times\mathbb{R}_x^3\big) \text{ as } \varepsilon \to 0. \tag{24}$$

Here $\langle \vec{S}\phi_j^{\varepsilon}, \phi_j^{\varepsilon} \rangle$ *denotes the three-vector with components* $\langle S^k\phi_j^{\varepsilon}, \phi_j^{\varepsilon} \rangle$*, where* $S^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$.
Finally, the total energy is uniformly bounded:

$$E_{\mathrm{tot}}^{\varepsilon}(t) = \frac{1}{2} \int_{\mathbb{R}_x^3} \sum \lambda_j^{\varepsilon} \hbar^2 \left| \nabla\phi_j^{\varepsilon}(t,x) \right|^2 + \left| \nabla V^{\varepsilon}(t,x) \right|^2 \, dx \le C. \tag{25}$$

To be precise, we have initially $E_{\mathrm{tot}}^{\varepsilon}(0) \le C$*, and furthermore*

$$E_{\mathrm{tot}}^{\varepsilon}(t) \le E_{\mathrm{tot}}^{\varepsilon}(0) + \frac{C}{c^2} \exp\left(\frac{aT_0}{\hbar^M}\right) \tag{26}$$

for all ε *and* $0 \le t \le T_0$.

The semi-classical limit $\hbar \to 0$ from the (exact) SP system to Vlasov-Poisson was done independently by Lions-Paul [9] and (for a slightly regularized system) Markowich-Mauser [10]. Since the error in Theorem 2 converges strongly to zero, we can adapt the argument used in the aforementioned papers to our present, more general case.

The key tool for the semi-classical limit is the *Wigner transform* (see [7] and [9] for exhaustive surveys), which for spinor valued $f, g \in L^2(\mathbb{R}_x^3)^4$ is given by

$$W^{\hbar}(f,g)(x,v) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}_y^3} e^{iv\cdot y} f\left(x - \frac{\hbar}{2}y\right) \left[g\left(x + \frac{\hbar}{2}y\right) \right]^* dy$$

for $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$, where g^* denotes the conjugate transpose of g. Thus $W^{\hbar}(f,g)$ takes values in the space of 4×4 complex matrices. We then define, for a solution of DM,

$$W^{\varepsilon}(t,x,v) = \sum \lambda_j^{\varepsilon} W^{\hbar}\left(\psi_j^{\varepsilon}(t,\cdot), \psi_j^{\varepsilon}(t,\cdot)\right)(x,v), \tag{27}$$

whose trace gives the *scalar* transform

$$w^{\varepsilon}(t,x,v) = \sum \frac{\lambda_j^{\varepsilon}}{(2\pi)^3} \int_{\mathbb{R}_y^3} \left\langle \psi_j^{\varepsilon}\left(t, x - \frac{\hbar}{2}y\right), \psi_j^{\varepsilon}\left(t, x + \frac{\hbar}{2}y\right) \right\rangle_{\mathbb{C}^4} e^{iv\cdot y} \, dy. \tag{28}$$

We denote the initial values of these quantities by $W_I^{\varepsilon}(x,v)$ and $w_I^{\varepsilon}(x,v)$.

For our main result we need, in addition to (3)–(7), the assumption that the Wigner transform is uniformly bounded in L^2 on the phase space. This is possible if and only if we consider mixed states with special properties of the occupation probabilities that have to depend on the semiclassical parameter in a very particular way:

$$\left\| W_I^{\varepsilon} \right\|_{L_{x,v}^2} \le B \tag{29}$$

for all ε; as we shall see, this is equivalent to

$$\sum (\lambda_j^{\varepsilon})^2 \le (2\pi\hbar)^3 B. \tag{30}$$

For a discussion of this condition and a comparison to the results in the 1-d "pure state" case, see [11].

In (29) the L^2 norm is taken with respect to the Euclidean matrix norm $|A| = \sqrt{\langle A, A \rangle}$ associated to the inner product $\langle A, B \rangle = \sum_{j,k} a_{jk}\overline{b_{jk}}$ for complex 4×4 matrices $A = (a_{jk})$ and $B = (b_{jk})$. Thus, (29) implies $\left\| w_I^\varepsilon \right\|_{L^2_{x,v}} = O(1)$.

Theorem 3. *Consider the solution of (1), (2) obtained in Theorem 1, with initial assumptions (3)–(7), and assuming also (9), as always. Consider the Wigner transform $W^\varepsilon(t, x, v)$ and its trace $w^\varepsilon = \operatorname{tr} W^\varepsilon$, given by (27) and (28). As before, we denote their initial values by $W_I^\varepsilon(x, v)$ and $w_I^\varepsilon(x, v)$. Assume (29) (or equivalently (30)) is satisfied.*

Let f_I be a weak $L^2_{x,v}$ limit point of w_I^ε (in view of (29) there exists at least one such limit point). Then

$$f_I \geq 0 \text{ (a.e.)}, \qquad f_I \in L^1_{x,v} \cap L^2_{x,v} \quad \text{and} \quad \int |v|^2 f_I(x, v)\, dx\, dv < \infty. \tag{31}$$

Moreover, there exists a sequence $\varepsilon_l \to 0$ such that as $l \to \infty$,

$$w_I^{\varepsilon_l} \longrightarrow f_I \qquad \text{in } L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)) \text{ weakly}, \tag{32}$$

$$W^{\varepsilon_l} \longrightarrow W^0 \qquad \text{in } L^\infty\big((0, T_0), L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)^{4\times 4}\big) \text{ weak } *, \tag{33}$$

$$w^{\varepsilon_l} \longrightarrow f \qquad \text{in } L^\infty\big((0, T_0), L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)\big) \text{ weak } *, \tag{34}$$

$$-A_0^{\varepsilon_l} \longrightarrow V \qquad \text{in } L^\infty\big((0, T_0), L^6(\mathbb{R}^3_x)\big) \text{ weak } *, \tag{35}$$

$$-\nabla A_0^{\varepsilon_l} \longrightarrow \nabla V \qquad \text{in } L^\infty\big((0, T_0), L^2(\mathbb{R}^3_x)\big) \text{ weak } *, \tag{36}$$

$$\rho^{\varepsilon_l} \longrightarrow n \qquad \text{in } L^\infty\big((0, T_0), L^{7/5}(\mathbb{R}^3_x)\big) \text{ weak } * \tag{37}$$

$$\mathbf{J}^{\varepsilon_l} \longrightarrow \mathbf{J} \qquad \text{in } \mathcal{D}'\big((0, T_0) \times \mathbb{R}^3_x\big), \tag{38}$$

where the matrix valued W^0 is hermitian and positive semi-definite (a.e.) and $f = \operatorname{tr} W^0 \geq 0$ (a.e.) is a weak solution of the Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = 0, & 0 < t < T_0, \quad x, v \in \mathbb{R}^3, \\ \Delta V = -n, \qquad n = \displaystyle\int f\, dv, & 0 < t < T_0, \quad x \in \mathbb{R}^3, \end{cases} \tag{39}$$

with data

$$f(0, x, v) = f_I(x, v). \tag{40}$$

The limit $\mathbf{J}$ of the current density $\mathbf{J}^{\varepsilon_l}$ is given by

$$\mathbf{J}(t, x) = \int v f(t, x, v)\, dv + \nabla_x \times \left(\int \operatorname{tr}\left(\vec{S} W^0(t, x, v) \right) dv \right), \tag{41}$$

where $\operatorname{tr}\big(\vec{S} W^0\big)$ denotes the three-vector with components $\operatorname{tr}\big(S^k W^0\big)$, with S^k defined as in Theorem 2. The conservation law

$$\partial_t n + \operatorname{div} \mathbf{J} = 0 \tag{42}$$

holds in $\mathcal{D}'\big((0, T_0) \times \mathbb{R}^3_x\big)$. Furthermore, $f \in L^\infty\big((0, T_0); L^1_{x,v} \cap L^2_{x,v}\big)$, and the energy is uniformly bounded:

$$\int |v|^2 f(t, x, v)\, dx\, dv + \int |\nabla_x V(t, x)|^2\, dx \leq C \tag{43}$$

for a.e. $0 < t < T_0$.

Remark. The initial assumptions can be restated in a more synthetic form in terms of the initial density $\kappa_I^\varepsilon(x,y) = \sum \lambda_j^\varepsilon \psi_{j,I}^\varepsilon(x)[\psi_{j,I}^\varepsilon(y)]^*$ and the associated *density operator (state operator)*

$$T_I^\varepsilon(f)(x) = \int_{\mathbb{R}^3} \kappa_I^\varepsilon(x,y) f(y)\, dy$$

acting on 4-spinor valued functions. For example, (3) and (4) say that T_I^ε is a non-negative Hilbert-Schmidt operator on $L^2(\mathbb{R}_x^3)^4$, with $\mathrm{tr}(T_I^\varepsilon) = 1$, and (29) translates to $\mathrm{tr}([T_I^\varepsilon]^2) = O(\hbar^3)$. Conversely, given a family of operators $\{T_I^\varepsilon\}_\varepsilon$ with these properties, diagonalization produces an orthonormal basis $\{\psi_{j,I}^\varepsilon\}_{j\in\mathbb{N}}$ of eigenvectors, and associated eigenvalues $\{\lambda_j^\varepsilon\}_{j\in\mathbb{N}}$ satisfying (3), (4) and (29).

3. Local existence in a uniform time interval

Here we prove Theorem 1 by applying results from [4] concerning the local well-posedness of DM and the asymptotic behaviour of solutions as $c \to \infty$. In these results, however, $\hbar$ was set equal to unity, so first we have to rescale the system, setting

$$\tilde{\psi}_j^\varepsilon(t,x) = \hbar \psi_j^\varepsilon(\hbar t, \hbar x), \quad \tilde{A}_0^\varepsilon(t,x) = A_0(\hbar t, \hbar x) \quad \text{and} \quad \tilde{\mathbf{A}}^\varepsilon(t,x) = \mathbf{A}(\hbar t, \hbar x).$$

Then $(\{\tilde{\psi}_j^\varepsilon\}_{j\in\mathbb{N}}, \tilde{A}_0^\varepsilon, \tilde{\mathbf{A}}^\varepsilon)$ solves DM with $\hbar\partial_\mu$ replaced by ∂_μ. Thus, $\hbar$ no longer appears explicitly in the equations, and the dependence on $\hbar$ is exclusively through the initial data

$$\tilde{\psi}_j^\varepsilon(0,x) = \hbar \psi_{j,I}^\varepsilon(\hbar x), \quad \tilde{\mathbf{A}}^\varepsilon(0,x) = \mathbf{a}_I^\varepsilon(\hbar x), \quad \partial_t \tilde{\mathbf{A}}^\varepsilon(0,x) = \hbar \mathbf{b}_I^\varepsilon(\hbar x).$$

From (4)–(7) we have

$$\left(\sum \lambda_j^\varepsilon \left\| \tilde{\psi}_j^\varepsilon(t=0) \right\|_{H^1}^2\right)^{\frac{1}{2}} + \left\| \partial \tilde{\mathbf{A}}^\varepsilon(t=0) \right\|_{L^2} \le \frac{10B}{\sqrt{\hbar}}, \tag{44}$$

$$\left(\sum \lambda_j^\varepsilon \left\| \tilde{\psi}_j^\varepsilon(t=0) \right\|_{H^5}^2\right)^{\frac{1}{2}} + \left\| \partial \tilde{\mathbf{A}}^\varepsilon(t=0) \right\|_{H^4} \le 10B \exp\left(\frac{1}{\hbar^{m_0}}\right), \tag{45}$$

$$\left(\sum \lambda_j^\varepsilon \left\| \tilde{\eta}_j^\varepsilon(t=0) \right\|_{H^2}^2\right)^{\frac{1}{2}} \le \frac{10B}{c} \exp\left(\frac{1}{\hbar^{m_0}}\right), \tag{46}$$

where $\tilde{\eta}_j^\varepsilon$ denotes the lower 2-spinor of $\tilde{\psi}_j^\varepsilon$ (analogously to (8)). Note that λ_j^ε is unaffected by the rescaling.

We can now apply the results from [4] to $(\{\tilde{\psi}_j^\varepsilon\}_{j\in\mathbb{N}}, \tilde{A}_0^\varepsilon, \tilde{\mathbf{A}}^\varepsilon)$. Define norms

$$\mathcal{N}_k^\varepsilon(t) = \left(\sum \lambda_j^\varepsilon \left\| \tilde{\psi}_j^\varepsilon(t) \right\|_{H^k}^2\right)^{\frac{1}{2}} + c^{-\alpha} \left\| \partial \tilde{\mathbf{A}}^\varepsilon(t) \right\|_{H^{k-1}}$$

where $0 < \alpha < 1/2$ is a fixed parameter; in fact, as in Theorem 1 we set $\alpha = 1/4$, just to make a definite choice. Again we emphasize that although $\hbar$ is still present in the superscript ε, the dependence on $\hbar$ is only through the initial data. Thus, until we use the information given by (44)–(46), the only important asymptotic parameter is c.

Local well-posedness (for fixed c) for data with $\mathcal{N}_1^\varepsilon(t=0) < \infty$ was proved in [4, Theorem 6.1], and this gives part (i) of Theorem 1. The conservation (10) of the L^2 inner product is an easy consequence of the Dirac equations for the fields ψ_j^ε and ψ_k^ε: Using the L^2 inner product, multiply the Dirac equation for ψ_j^ε on the right by ψ_k^ε, then do the same with j and k interchanged, and subtract the results. In particular, this implies

$$\sum \lambda_j^\varepsilon \left\| \tilde{\psi}_j^\varepsilon(t) \right\|_{L^2}^2 = \frac{1}{\hbar} \tag{47}$$

for all t in the interval of existence.

For part (ii) of Theorem 1, we use the following key result from [4]:

Proposition 1. *(See* [4, Theorem 14.3].*) There exist*

(i) $T^* > 0$, *depending only on* $R = \sup_{c\geq 1} \sum_j \lambda_j^\varepsilon \left\| \tilde{\psi}_j^\varepsilon(t=0) \right\|_{L^2}^2$, *and*
(ii) *absolute constants* $C, M, c_0 > 0$,

such that if $\mathcal{N}_1^\varepsilon(0) \leq B$ *uniformly, then* $\mathcal{N}_1^\varepsilon(t) \leq CB$ *for all* $0 \leq t \leq T^*$ *and all* $c \geq c_0 B^M$. *Moreover, the dependence of* T^* *on* R *can be taken to be*

$$T^*(R) = \frac{\delta}{1+R^K},$$

where $\delta > 0$ *and* $K \in \mathbb{N}$ *are absolute constants.*

This result can be iterated N times to give existence up to time NT^*, for c sufficiently large. A crucial fact is that T^* remains constant, because of the L^2 conservation law (10). Moreover, higher regularity persists:

Proposition 2. *(See* [4, Proposition 14.4].*) Fix* $k \in \mathbb{N}$, *and assume* $\mathcal{N}_k^\varepsilon(0) \leq B$ *uniformly. Given* $0 < T < \infty$, *let* N *be the smallest integer such that* $NT^* \geq T$, *with* T^* *as in Theorem 1. Then*

$$\mathcal{N}_k^\varepsilon(t) \leq P_k\big(C^N B\big) \quad \textit{for} \quad 0 \leq t \leq T, \quad \textit{provided} \quad c \geq c_0 (C^{N-1}B)^M.$$

Here C, M, c_0 *are absolute constants and* P_k *is a polynomial.*

By (44) we have $\mathcal{N}_1^\varepsilon(0) \leq 10Bh^{-1/2}$ here, and from (47) we see that we may take $T^* = \delta\hbar^K$ for some choice of absolute constants δ and K. So applying the last theorem with $T = T_0/\hbar$ (and hence with N the smallest integer such that $N \geq \frac{T_0}{\delta\hbar^{K+1}}$), and then scaling back to the original variables, we get part (ii) of Theorem 1. Moreover, from (45) we have $\mathcal{N}_5^\varepsilon(0) \leq 10B\exp\left(\frac{1}{\hbar^{m_0}}\right)$, so the same argument also gives part (iii) of the theorem, except for the bounds (13) and (14). To prove those, we apply the following:

Proposition 3. *(See* [4, Theorem 14.5 and Lemma 1.5].*) Fix* $k \in N$. *Given* $0 < T < \infty$, *let* N *be the smallest integer such that* $NT^* \geq T$, *with* T^* *as in Theorem 1. Suppose*

$$\mathcal{N}_k^\varepsilon(0) \leq B \quad \textit{and} \quad \Big(\sum \lambda_j^\varepsilon \left\| \tilde{\eta}_j^\varepsilon(t=0) \right\|_{H^{k-1}}^2\Big)^{\frac{1}{2}} \leq \frac{B}{c}$$

hold uniformly. Then

$$\Big(\sum \lambda_j^\varepsilon \left\| \tilde{\eta}_j^\varepsilon(t) \right\|_{H^{k-1}}^2\Big)^{\frac{1}{2}} \leq \frac{1}{c} P_k(C^N B)$$

for all $0 \leq t \leq T$, *provided* $c \geq c_0(C^{N-1}B)^M$. *Here* C, M, c_0 *are absolute constants and* P_k *is a polynomial.*

Moreover, if we strengthen the assumption $\mathcal{N}_k^\varepsilon(0) \leq B$ *by additionally requiring that (recall the presence of the small parameter* α *in the definition of* $\mathcal{N}_k^\varepsilon$*)* $\left\| \partial\tilde{\mathbf{A}}^\varepsilon(t=0) \right\|_{H^{k-1}} \leq B$ *uniformly, then*

$$\left\| \partial\tilde{\mathbf{A}}^\varepsilon(t) \right\|_{H^{k-1}} \leq P_k(C^N B)$$

for all $0 \leq t \leq T$, *provided* c *satisfies the same condition as above.*

Applying this with $k = 3$, $T^* = \delta\hbar^K$ (as above), $T = T_0/\hbar$ and N the smallest integer such that $N \geq \frac{T_0}{\delta\hbar^{K+1}}$, and of course making use of the bounds (45) and (46), we then get (13) and (14), after scaling back to the original variables.

4. DM AS PERTURBED SP: PROOF OF THEOREM 2

Let us first look at the decomposition (15) in a bit more detail. We want to relate it to the splitting into positive and negative energy parts for the free Dirac equation $i\partial_t u = \mathcal{Q}^c(D)u$, where the free Dirac operator $\mathcal{Q}^c(D) = -ic\alpha^k\partial_k + c^2\beta$ has Fourier symbol $c\xi_k\alpha^k + c^2\beta$. This symbol has two eigenvalues $\pm\mu^c(\xi)$, each of multiplicity 2, with μ^c given by (17). The corresponding eigenspace projections are

$$\Pi^c_\pm(D) = \frac{1}{2}\left\{I \pm [\mu^c(D)]^{-1}\mathcal{Q}^c(D)\right\}.$$

Note that the formal limits as $c \to \infty$ are, in 2×2 block form,

$$\Pi^\infty_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Pi^\infty_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{48}$$

(Cf. the decomposition (8).) In fact (see [1, Lemma 2.1]) we have:

Lemma 1. *For all $s \in \mathbb{R}$, $\Pi^c_\pm(D)$ is bounded from $H^s \to H^s$ uniformly in c. Moreover,*

$$\Pi^c_\pm(D) = \Pi^\infty_\pm \pm \frac{1}{c}\mathcal{R}^c_1(D) = \Pi^\infty_\pm \pm \frac{1}{2ci}\alpha^k\partial_k \pm \frac{1}{c^2}\mathcal{R}^c_2(D)$$

where the $\mathcal{R}^c_j(D)$ are multipliers bounded from $H^s \to H^{s-j}$ uniformly in c. (In fact, the symbols $\mathcal{R}^c_j(\xi)$ are $O(|\xi|^j)$ for $j = 1, 2$.)

This suggests that the decomposition

$$\psi^\varepsilon_j = \Pi^c_+(\hbar D)\psi^\varepsilon_j + \Pi^c_-(\hbar D)\psi^\varepsilon_j$$

of the Dirac spinors is related to positive and negative energies, or to electrons and positrons, and we now wish to compare it to the decomposition given by (15) and (16). But using the Dirac equations for the ψ^ε_j, from (1), we see that

$$\Pi^c_\pm(\hbar D)\psi^\varepsilon_j - \psi^\varepsilon_{j,\pm} = \pm\frac{1}{2}\left\{\mu^c(\hbar D)\right\}^{-1}\left(A^\varepsilon_k\alpha^k\psi^\varepsilon_j\right). \tag{49}$$

From this and the bounds in Theorem 1, we deduce the following estimates, recalling that H^s is an algebra for $s > 3/2$:

$$\left(\sum \lambda^\varepsilon_j \left\|\psi^\varepsilon_{j,\pm}(t) - \Pi^c_\pm(\hbar D)\psi^\varepsilon_j(t)\right\|^2_{H^5}\right)^{\frac{1}{2}} \le \frac{C}{c^{7/4}}\exp\left(\frac{aT_0}{\hbar^M}\right), \tag{50}$$

$$\left(\sum \lambda^\varepsilon_j \left\|\psi^\varepsilon_{j,\pm}(t)\right\|^2_{H^5}\right)^{\frac{1}{2}} \le C\exp\left(\frac{aT_0}{\hbar^M}\right), \tag{51}$$

$$\left(\sum \lambda^\varepsilon_j \left\|\phi^\varepsilon_j(t)\right\|^2_{H^5}\right)^{\frac{1}{2}} \le C\exp\left(\frac{aT_0}{\hbar^M}\right), \tag{52}$$

$$\left(\sum \lambda^\varepsilon_j \left\|\psi^\varepsilon_{j,-}(t)\right\|^2_{H^2}\right)^{\frac{1}{2}} \le \frac{C}{c}\exp\left(\frac{aT_0}{\hbar^M}\right) \tag{53}$$

for all ε and $0 \le t \le T_0$. Here, as we recall,

$$\phi^\varepsilon_j = e^{itc^2/\hbar}\psi^\varepsilon_{j,+} = e^{itc^2/\hbar}\psi^\varepsilon_j - e^{itc^2/\hbar}\psi^\varepsilon_{j,-}, \quad j \in \mathbb{N}. \tag{54}$$

Thus, (52) follows directly from (51). But (51) follows from (50) and the uniform boundedness of $\Pi^c_\pm(D)$ on H^5. In view of (50), it suffices to prove (53) with $\psi^\varepsilon_{j,-}$ replaced by $\Pi^c_-(\hbar D)\psi^\varepsilon_j$, but in that case it reduces, by Lemma 1, to the estimates (14) and (12) from Theorem 1.

Observe that the estimate (22) in Theorem 2 is a direct consequence of (53) and (54). Since, for any $p \geq 1$,

$$\left\| \rho^\varepsilon(t) - n^\varepsilon(t) \right\|_{L^p} \leq \left(\sum \lambda_j^\varepsilon \left\{ \left\| \psi_j^\varepsilon(t) \right\|_{L^{2p}}^2 + \left\| \phi_j^\varepsilon(t) \right\|_{L^{2p}}^2 \right\} \right)^{1/2} \times \left(\sum \lambda_j^\varepsilon \left\| \phi_j^\varepsilon(t) - e^{itc^2/\hbar} \psi_j^\varepsilon(t) \right\|_{L_x^{2p}}^2 \right)^{1/2},$$

we then immediately get (23).

Let us then turn to the proof of the crucial error estimate in Theorem 2. The first step is to note that (see [4, Lemma 2.1] for this)

$$i\hbar \partial_t \phi_j^\varepsilon - \left\{ \mu^c(\hbar D) - c^2 \right\} \phi_j^\varepsilon = -A_0^\varepsilon \phi_j^\varepsilon + \frac{1}{2} e^{itc^2/\hbar} R_j^\varepsilon$$

where

$$\begin{aligned} R_j^\varepsilon &= \hbar c \big\{ \mu^c(\hbar D) \big\}^{-1} \left(2i \mathbf{A}^\varepsilon \cdot \nabla \psi_j^\varepsilon + i E_k^\varepsilon \alpha^k \psi_j^\varepsilon - B_k^\varepsilon S^k \psi_j^\varepsilon \right) \\ &\quad + \big\{ \mu^c(\hbar D) \big\}^{-1} \left(\left| \mathbf{A}^\varepsilon \right|^2 \psi_j^\varepsilon + \left[\mu^c(\hbar D) - c^2, A_0^\varepsilon \right] (\psi_{j,+}^\varepsilon - \psi_{j,-}^\varepsilon) \right). \end{aligned} \tag{55}$$

Here $[\cdot, \cdot]$ denotes the commutator, $S^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$, $\mathbf{E}^\varepsilon = \nabla A_0^\varepsilon - \frac{1}{c} \partial_t \mathbf{A}^\varepsilon$ is the electric field vector and $\mathbf{B}^\varepsilon = \nabla \times \mathbf{A}^\varepsilon$ the magnetic field vector. Next, note that

$$\mu^c(\hbar D) - c^2 = -\frac{\hbar^2}{2} \Delta + \frac{1}{c^2} \mathcal{R}_4^c(\hbar D)$$

where $\mathcal{R}_4^c(D)$ is a multiplier bounded from $H^s \to H^{s-4}$ uniformly in c. (In fact, the symbol $\mathcal{R}_4^c(\xi)$ is $O(|\xi|^4)$.) Finally, writing $-A_0^\varepsilon = V^\varepsilon + r^\varepsilon$, where

$$\Delta r^\varepsilon = -\sum \lambda_j^\varepsilon \left(2 \operatorname{Re} \left\langle \psi_{j,+}^\varepsilon, \psi_{j,-}^\varepsilon \right\rangle_{\mathbb{C}^4} + \left\langle \psi_{j,-}^\varepsilon, \psi_{j,-}^\varepsilon \right\rangle_{\mathbb{C}^4} \right),$$

we conclude that (19) is satisfied, with

$$\mathrm{Error}_j^\varepsilon = \frac{1}{2} e^{itc^2/\hbar} R_j^\varepsilon + \frac{1}{c^2} \mathcal{R}_4^c(\hbar D) \phi_j^\varepsilon + r^\varepsilon \phi_j^\varepsilon,$$

and it only remains to prove the estimate (20) for each of the three terms in the right hand side.

The term $\frac{1}{c^2} \mathcal{R}_4^c(\hbar D) \phi_j^\varepsilon$ satisfies (20) with $1/c^2$ instead of $1/c$, in view of (52). The term $r^\varepsilon \phi_j^\varepsilon$ we estimate as follows. By Leibniz' rule, Hölder's inequality and Sobolev embedding, $\| fg \|_{H^1} \leq C \| \Delta f \|_{L^{6/5}} \| g \|_{H^2}$. Thus,

$$\left(\sum \lambda_j^\varepsilon \left\| r^\varepsilon \phi_j^\varepsilon(t) \right\|_{H^1}^2 \right)^{\frac{1}{2}} \leq C \left\| \Delta r^\varepsilon(t) \right\|_{L^{6/5}} \left(\sum \lambda_j^\varepsilon \left\| \phi_j^\varepsilon(t) \right\|_{H^2}^2 \right)^{\frac{1}{2}},$$

so recalling (52), and estimating

$$\begin{aligned} \left\| \Delta r^\varepsilon(t) \right\|_{L^{6/5}} &\leq C \sum \lambda_j^\varepsilon \left[\sum_{\pm} \left\| \psi_{j,\pm}^\varepsilon(t) \right\|_{L^2} \right] \left\| \psi_{j,-}^\varepsilon(t) \right\|_{H^1} \\ &\leq C \left[\sum_{\pm} \left(\sum \lambda_j^\varepsilon \left\| \psi_{j,\pm}^\varepsilon(t) \right\|_{L^2}^2 \right)^{\frac{1}{2}} \right] \left(\sum \lambda_j^\varepsilon \left\| \psi_{j,-}^\varepsilon(t) \right\|_{H^1}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

we conclude, using (51) and (53), that (20) holds for $r^\varepsilon \phi_j^\varepsilon$.

Finally, we consider terms from R_j^ε. For the bilinear terms

$$\hbar c\big\{\mu^c(\hbar D)\big\}^{-1}\left(2i\mathbf{A}^\varepsilon\cdot\nabla\psi_j^\varepsilon + iE_k^\varepsilon\alpha^k\psi_j^\varepsilon - B_k^\varepsilon S^k\psi_j^\varepsilon\right)$$

we use the rather crude estimate $\left\|\big\{\mu^c(\hbar D)\big\}^{-1}(fg)\right\|_{H^1} \le \frac{C}{c^2}\|f\|_{H^2}\|g\|_{H^1}$, which in view of (12) yields (20) for these terms. Similarly, for the cubic term

$$\big\{\mu^c(\hbar D)\big\}^{-1}\left(\left|\mathbf{A}^\varepsilon\right|^2\psi_j^\varepsilon\right)$$

we use $\left\|\big\{\mu^c(\hbar D)\big\}^{-1}(fgh)\right\|_{H^1} \le \frac{C}{c\hbar}\|fgh\|_{L^2} \le \frac{C}{c\hbar}\|f\|_{H^1}\|g\|_{H^1}\|h\|_{H^1}$. Finally, for the commutator terms

$$\big\{\mu^c(\hbar D)\big\}^{-1}\left(\left[\mu^c(\hbar D)-c^2, A_0^\varepsilon\right](\psi_{j,\pm}^\varepsilon)\right), \tag{56}$$

we can in effect treat $\big\{\mu^c(\hbar D)\big\}^{-1}$ as $1/c^2$, and $\mu^c(\hbar D)-c^2$ as $c\hbar\nabla$. In fact, since $|\mu^c(\xi+\eta)-\mu^c(\eta)| \le c\,|\xi|$, it follows readily that

$$\left\|[\mu^c(\hbar D), f]g\right\|_{H^1} \le Cc\hbar\,\|\Delta f\|_{L^2}\,\|g\|_{H^2}\,.$$

Hence, the term (56) taken in the norm $\left(\sum\lambda_j^\varepsilon\|\cdot\|_{H^1}^2\right)^{1/2}$ is dominated by

$$\frac{\hbar}{c}\left\|\Delta A_0^\varepsilon(t)\right\|_{L^2}\left(\sum\lambda_j^\varepsilon\left\|\psi_{j,\pm}^\varepsilon(t)\right\|_{H^2}^2\right)^{\frac12},$$

and since $\left\|\Delta A_0^\varepsilon(t)\right\|_{L^2} \le \sum\lambda_j^\varepsilon\left\|\psi_j^\varepsilon(t)\right\|_{L^4}^2$, we get the desired estimate from (12) and (51). This concludes the proof of the error estimate.

Next, we bound the energy. Let us first prove the "almost conservation" estimate (26), and then at the end of this section we bound the initial energy, using a Lieb-Thirring type inequality from [9]. Write $(f,g) = \int\langle f,g\rangle\,dx$ for the L^2 inner product. From (21), whose easy proof we omit, we have

$$\begin{aligned}\frac{d}{dt}\frac12\int_{\mathbb{R}_x^3}\left|\nabla V^\varepsilon(t)\right|^2 dx &= \operatorname{Re}\left(\nabla\partial_t V^\varepsilon, \nabla V^\varepsilon\right) = -\operatorname{Re}\left(\Delta\partial_t V^\varepsilon, V^\varepsilon\right)\\ &= \operatorname{Re}\left(\partial_t n^\varepsilon, V^\varepsilon\right) = \left(\mathbf{J}_\mathrm{S}^\varepsilon, \nabla V^\varepsilon\right) + \frac{2}{\hbar}\sum\lambda_j^\varepsilon\operatorname{Im}\left(\mathrm{Error}_j^\varepsilon, V\phi_j^\varepsilon\right)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\frac12\int_{\mathbb{R}_x^3}\sum\lambda_j^\varepsilon\hbar^2\left|\nabla\phi_j^\varepsilon(t)\right|^2 dx &= \sum\lambda_j^\varepsilon\hbar\operatorname{Re}\left(\hbar\nabla\partial_t\phi_j^\varepsilon, \nabla\phi_j^\varepsilon\right)\\ &= \hbar\sum\lambda_j^\varepsilon\operatorname{Im}\left([\nabla V^\varepsilon]\phi_j^\varepsilon, \nabla\phi_j^\varepsilon\right) + \hbar\sum\lambda_j^\varepsilon\operatorname{Im}\left(\nabla\,\mathrm{Error}_j^\varepsilon, \nabla\phi_j^\varepsilon\right).\end{aligned}$$

In the last equality, the first term on the right hand side is nothing else than $-(\mathbf{J}_\mathrm{S}^\varepsilon, \nabla V^\varepsilon)$, and so it follows that

$$\frac{dE_\mathrm{tot}^\varepsilon}{dt} = \sum\lambda_j\operatorname{Im}\left(\nabla\,\mathrm{Error}_j^\varepsilon, \hbar\nabla\phi_j^\varepsilon\right) + \frac{2}{\hbar}\sum\lambda_j^\varepsilon\operatorname{Im}\left(\mathrm{Error}_j^\varepsilon, V\phi_j^\varepsilon\right) = I^\varepsilon + II^\varepsilon.$$

We now estimate the terms I^ε, II^ε on the right hand side in absolute value.

By the Schwarz inequality,

$$\left|I^\varepsilon\right| \le \left(\sum\lambda_j^\varepsilon\left\|\nabla\,\mathrm{Error}_j^\varepsilon\right\|_{L^2}^2\right)^{\frac12}\left(\sum\lambda_j^\varepsilon\hbar^2\left\|\nabla\phi_j^\varepsilon\right\|_{L^2}^2\right)^{\frac12}.$$

By Hölder's inequality and Sobolev embedding,

$$\begin{aligned}\left|II^\varepsilon\right| &\leq \frac{2}{\hbar}\sum \lambda_j^\varepsilon \left\|\mathrm{Error}_j^\varepsilon\right\|_{L^3} \|V\|_{L^6} \left\|\phi_j^\varepsilon\right\|_{L^2} \\ &\leq \frac{2}{\hbar}\|\nabla V\|_{L^2}\left(\sum \lambda_j^\varepsilon \left\|\mathrm{Error}_j^\varepsilon\right\|_{H^1}^2\right)^{\frac{1}{2}}\left(\sum \lambda_j^\varepsilon \left\|\phi_j^\varepsilon\right\|_{L^2}^2\right)^{\frac{1}{2}}.\end{aligned}$$

But from (49) we have $\left\|\phi_j^\varepsilon\right\|_{L^2} \leq \left\|\psi_j^\varepsilon\right\|_{L^2} + \left\|\psi_{j,-}^\varepsilon\right\|_{L^2}$, so by (53),

$$\left(\sum \lambda_j^\varepsilon \left\|\phi_j^\varepsilon(t)\right\|_{L^2}^2\right)^{\frac{1}{2}} \leq 1 + \frac{C}{c}\exp\left(\frac{aT_0}{\hbar^M}\right). \tag{57}$$

Using (20) we then conclude:

$$\left|\frac{dE_{\mathrm{tot}}^\varepsilon}{dt}\right| \leq \left(E_{\mathrm{tot}}^\varepsilon\right)^{\frac{1}{2}}\frac{C}{c}\exp\left(\frac{aT_0}{\hbar^M}\right)$$

uniformly in $0 \leq t \leq T_0$, and this clearly implies (26).

It remains to prove $E_{\mathrm{tot}}^\varepsilon(0) \leq C$. By (5), this reduces to bounding $\|\nabla V^\varepsilon(0)\|_{L_x^2}$. But

$$\left\|\nabla V^\varepsilon\right\|_{L_x^2} \leq C\left\|\Delta V^\varepsilon\right\|_{L_x^{6/5}} = C\left\|n^\varepsilon\right\|_{L_x^{6/5}} \leq C\left\|n^\varepsilon\right\|_{L_x^1}^{5/12}\left\|n^\varepsilon\right\|_{L_x^{7/5}}^{7/12},$$

and $\|n^\varepsilon\|_{L_x^1} \leq \sum \lambda_j^\varepsilon \left\|\phi_j^\varepsilon\right\|_{L_x^2}^2$ is uniformly bounded by (57), so it suffices to bound $\|n^\varepsilon(0)\|_{L^{7/5}}$. To this end, we apply a Lieb-Thirring type inequality from [9] (there the inequality was proved for complex valued functions, but the same argument works for spinors):

Theorem. *(See* [9, Appendix]*.) Assume given an orthonormal spinor valued sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^2(\mathbb{R}_x^3)^4$ and a sequence $\{\mu_j\}_{j\in\mathbb{N}}$ of non-negative numbers, such that $\sum \mu_j^2 < \infty$ and $\int \sum \mu_j \left|\nabla u_j(x)\right|^2 dx < \infty$. Define*

$$a(x) = \sum \mu_j \left|u_j(x)\right|^2 \quad \textit{and} \quad b(x) = \sum \mu_j \operatorname{Im}\left\langle \nabla u_j(x), u_j(x)\right\rangle_{\mathbb{C}^4}.$$

Then

$$\|a\|_{L^{7/5}} \leq C\left\|\{\mu_j\}\right\|_{l^2}^{4/7}\left(\int \sum \mu_j \left|\nabla u_j(x)\right|^2 dx\right)^{3/7}, \tag{58}$$

$$\|b\|_{L^{7/6}} \leq C\left\|\{\mu_j\}\right\|_{l^2}^{2/7}\left(\int \sum \mu_j \left|\nabla u_j(x)\right|^2 dx\right)^{5/7}. \tag{59}$$

Applying this to the solution of (1) gives

$$\left\|\rho^\varepsilon(t)\right\|_{L^{7/5}} \leq C\left(\frac{1}{\hbar^3}\sum (\lambda_j^\varepsilon)^2\right)^{2/7}\left(\sum \lambda_j^\varepsilon \hbar^2 \left\|\nabla \psi_j^\varepsilon(t)\right\|_{L^2}^2\right)^{3/7}, \tag{60}$$

which implies $\|\rho^\varepsilon(0)\|_{L^{7/5}} \leq C$, on account of (5) and (30).But from this and (23) we get also $\|n^\varepsilon(0)\|_{L^{7/5}} \leq C$.

Finally, we prove (24). With notation as in (15), define $\phi_{j,\pm}^\varepsilon = e^{\pm itc^2/\hbar}\psi_{j,+}^\varepsilon$. Then (see [4, Lemma 2.1])

$$i\hbar\partial_t \phi_{j,\pm}^\varepsilon = \pm\left\{\mu^c(\hbar D) - c^2\right\}\phi_{j,\pm}^\varepsilon - A_0^\varepsilon \phi_{j,\pm}^\varepsilon \pm \frac{1}{2}e^{itc^2/\hbar}R_j^\varepsilon$$

where R_j^ε is given by (55). Thus, from (51), (53) and the proof of (20), we conclude that

$$
\begin{aligned}
\Big(\sum \lambda_j^\varepsilon \left\|\partial_t \phi_{j,+}(t)\right\|_{L^2}^2\Big)^{1/2} &\le C \exp\left(\frac{aT_0}{\hbar^M}\right), \\
\Big(\sum \lambda_j^\varepsilon \left\|\partial_t \phi_{j,-}(t)\right\|_{L^2}^2\Big)^{1/2} &\le \frac{C}{c} \exp\left(\frac{aT_0}{\hbar^M}\right).
\end{aligned} \tag{61}
$$

Now write, for the vector components of $\mathbf{J}^\varepsilon$,

$$
\begin{aligned}
c\sum \lambda_j^\varepsilon \left\langle \alpha^k \psi_j^\varepsilon, \psi_j^\varepsilon \right\rangle &= c\sum \lambda_j^\varepsilon \left\langle \alpha^k \left(\psi_{j,+}^\varepsilon + \psi_{j,-}^\varepsilon\right), \left(\psi_{j,+}^\varepsilon + \psi_{j,-}^\varepsilon\right) \right\rangle \\
&= c\sum \lambda_j^\varepsilon \left\langle \alpha^k \psi_{j,+}, \psi_{j,+}^\varepsilon \right\rangle + c\sum \lambda_j^\varepsilon \left\langle \alpha^k \psi_{j,-}, \psi_{j,-}^\varepsilon \right\rangle \\
&\quad + c\sum \lambda_j^\varepsilon \left\langle \alpha^k \psi_{j,+}, \psi_{j,-}^\varepsilon \right\rangle + c\sum \lambda_j^\varepsilon \left\langle \alpha^k \psi_{j,-}, \psi_{j,+}^\varepsilon \right\rangle \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

First, we prove that $I_3, I_4 \to 0$ in $\mathcal{D}'\big((0,T_0)\times\mathbb{R}_x^3\big)$ as $\varepsilon \to 0$. But I_3 can be written

$$
ce^{-2tc^2/\hbar} \sum \lambda_j^\varepsilon \left\langle \alpha^k \phi_{j,+}, \phi_{j,-}^\varepsilon \right\rangle = -\frac{\hbar}{2c}\frac{\partial}{\partial t}\left(e^{-2tc^2/\hbar}\right) \sum \lambda_j^\varepsilon \left\langle \alpha^k \phi_{j,+}, \phi_{j,-}^\varepsilon \right\rangle,
$$

so we simply multiply this by a test function, integrate by parts and make use of (61), and similarly for I_4. By (53), the term I_2 converges strongly to zero in, e.g., L_x^1 uniformly in $0 \le t \le T_0$, so we are left with I_1. In view of (50), we can replace $\psi_{j,+}^\varepsilon$ by $\Pi_+^c(\hbar D)\psi_j^\varepsilon$ in I_1; this only introduces an error which converges strongly to zero in L_x^1 uniformly in $0 \le t \le T_0$. Thus, we redefine

$$
I_1 = c\sum \lambda_j^\varepsilon \left\langle \alpha^k \Pi_+^c(\hbar D)\psi_j^\varepsilon, \Pi_+^c(\hbar D)\psi_j^\varepsilon \right\rangle.
$$

Using the expansion given in Lemma 1, and basic properties of the matrices $\Pi_\pm^\infty$ as defined in (48), namely, $\Pi_\pm^\infty = \left(\Pi_\pm^\infty\right)^*$, $\left(\Pi_\pm^\infty\right)^2 = \Pi_\pm^\infty$, $\Pi_+^\infty \Pi_-^\infty = 0$ and $\alpha^k \Pi_+^\infty = \Pi_-^\infty \alpha^k$, one finds that

$$
I_1 = \operatorname{Im} \sum \lambda_j^\varepsilon \left\langle \alpha^k \alpha^l \partial_l (\Pi_+^\infty \psi_j^\varepsilon), \Pi_+^\infty \psi_j^\varepsilon \right\rangle + O\left(\frac{1}{c}\exp\left(\frac{aT_0}{\hbar^M}\right)\right)
$$

in L_x^1 uniformly in $0 \le t \le T_0$. In view of (14), we can remove the projector Π_+^∞, and using the identity (here the summation convention applies)

$$
\alpha^k \alpha^l = \delta^{kl} I_{4\times 4} + i\epsilon^{klm} S^m, \quad \text{where} \quad S^m = \begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix},
$$

we conclude that

$$
\begin{aligned}
I_1 &= \sum \lambda_j^\varepsilon \operatorname{Im}\left\langle \partial_k \psi_j^\varepsilon, \psi_j^\varepsilon \right\rangle + \sum \lambda_j^\varepsilon \operatorname{Re} \epsilon^{klm} \left\langle S^m \partial_l \psi_j^\varepsilon, \psi_j^\varepsilon \right\rangle + O\left(\frac{1}{c}\exp\left(\frac{aT_0}{\hbar^M}\right)\right) \\
&= \sum \lambda_j^\varepsilon \operatorname{Im}\left\langle \partial_k \psi_j^\varepsilon, \psi_j^\varepsilon \right\rangle + \frac{1}{2}\sum \lambda_j^\varepsilon \epsilon^{klm} \partial_l \left\langle S^m \psi_j^\varepsilon, \psi_j^\varepsilon \right\rangle + O\left(\frac{1}{c}\exp\left(\frac{aT_0}{\hbar^M}\right)\right)
\end{aligned}
$$

in L_x^1 uniformly in $0 \le t \le T_0$. Moreover, in the first two terms on the right hand side, we may replace ψ_j^ε by ϕ_j^ε, on account of (22), and this concludes the proof of (24).

5. FROM DM TO VP: PROOF OF THEOREM 3

Let $W^\varepsilon(t,x,v)$ and $w^\varepsilon = \operatorname{tr} W^\varepsilon$ be given as in (27) and (28). Since the λ_j^ε are real, W^ε is hermitian, and w^ε is real valued. In fact, the λ_j^ε are non-negative, but this does not imply that W^ε is positive semi-definite. The latter property can be recovered, however, by mollification of the Wigner transform using Gaussians. In fact, the *Husimi transform*, defined by

$$\tilde{W}^\varepsilon = W^\varepsilon *_{x,v} G^\hbar, \quad \text{where} \quad G^\hbar(x,v) = g^\hbar(x) g^\hbar(v) \quad \text{and} \quad g^\hbar(x) = \frac{e^{-|x|^2/\hbar}}{(\pi\hbar)^{3/2}},$$

is hermitian and positive semi-definite. For this and other basic facts concerning the Wigner transform, we refer to the surveys [9] and [7].

Writing

$$\begin{aligned} \kappa^\varepsilon(t,x,y) &= \sum \lambda_j^\varepsilon \psi_j^\varepsilon(t,x) \left[\psi_j^\varepsilon(t,y)\right]^*, \\ \tilde{\kappa}^\varepsilon(t,x,y) &= \kappa^\varepsilon\left(t, x - \frac{y}{2}, x + \frac{y}{2}\right), \end{aligned} \tag{62}$$

we have from Plancherel's theorem:

$$\left\| W^\varepsilon(t) \right\|_{L^2_{x,v}} = \frac{1}{(2\pi\hbar)^{3/2}} \left\| \tilde{\kappa}^\varepsilon(t) \right\|_{L^2_{x,y}} = \frac{1}{(2\pi\hbar)^{3/2}} \left\| \kappa^\varepsilon(t) \right\|_{L^2_{x,y}}. \tag{63}$$

It follows that the L^2 norm of the Wigner transform is conserved in time:

$$\left\| W^\varepsilon(t) \right\|^2_{L^2_{x,v}} = \frac{1}{(2\pi\hbar)^3} \sum_j \sum_k \lambda_j^\varepsilon \lambda_k^\varepsilon \left(\psi_j^\varepsilon(t), \psi_k^\varepsilon(t)\right) = \frac{1}{(2\pi\hbar)^3} \sum_j (\lambda_j^\varepsilon)^2 \tag{64}$$

where we used the Dirac spinor L^2 conservation (10) to get the last equality. This confirms the equivalence of (29) and (30). Next we note that

$$\left\| \rho^\varepsilon(t) \right\|_{L^p} \le C \quad \text{for} \quad 1 \le p \le 7/5. \tag{65}$$

Indeed, for $p = 1$ this follows from (3) and (10), and for $p = 7/5$ we use (60), (30) and (25). Then it follows not only that ∇A_0^ε is uniformly bounded in L^2, but in fact that for some $\delta > 0$ we have

$$\left\| \nabla A_0^\varepsilon(t) \right\|_{H^\delta} \le C. \tag{66}$$

Thus, the quantities in the left hand sides of (32)–(37) are all uniformly bounded in the appropriate spaces appearing in the right hand side, so we can extract subsequences converging as stated. Thus, from now on we may assume that (32)–(37) hold, and it only remains to verify the key properties of the limit. Our arguments here follow closely those in [9] and [10], so we give few details.

First, since the Husimi transform is hermitian and positive semi-definite, it follows immediately that the same holds for the limit W^0. Thus, $f = \operatorname{tr} W^0$ is non-negative, and by construction f is in $L^2_{x,v}$ uniformly in time. We still have to prove that

$$(1 + |v|^2) f \in L^1_{x,v} \tag{67}$$

uniformly in time. To this end, one can use the following identities involving the Husimi transform and its trace $\tilde{w}^\varepsilon = \operatorname{tr} \tilde{W}^\varepsilon$:

$$\int \tilde{w}^\varepsilon(t,x,v)\,dv = \rho^\varepsilon(t,\cdot) * g^\hbar(x), \tag{68}$$

$$\int \tilde{w}^\varepsilon(t,x,v)\,dx\,dv = \int \rho^\varepsilon(t,x)\,dx = \sum \lambda_j^\varepsilon = 1, \tag{69}$$

$$\int |v|^2\,\tilde{w}^\varepsilon(t,x,v)\,dx\,dv = \int_{\mathbb{R}^3_x} \sum \lambda_j^\varepsilon \hbar^2 \left|\nabla \psi_j^\varepsilon(t,x)\right|^2 dx + \frac{3\hbar}{2}. \tag{70}$$

From Theorem 2 we know that the right hand side of (70) is uniformly bounded, so it follows that $\int(1+|v|^2)\tilde{w}^\varepsilon\,dx\,dv$ is uniformly bounded. Hence, recalling the weak * compactness of bounded sets in the space $\mathcal{M}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ of finite measures, one concludes that the limit $(1+|v|^2)f$ must belong to $\mathcal{M}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$; but since this limit is in fact a non-negative function, this is the same as saying that it belongs to $L^1_{x,v}$. By the same type of argument we conclude from (65) that $n \in L^1_x \cap L^{7/5}_x$ uniformly in time.

It only remains to prove that (f,n) satisfies the Vlasov-Poisson system, and that (38) holds, with the limit given by (41); the conservation law (42) then follows immediately from the conservation law $\partial_t \rho^\varepsilon + \operatorname{div} \mathbf{J}^\varepsilon = 0$ for (1).

The fact that $\Delta V = -n$ is of course an immediate consequence of $\Delta A_0^\varepsilon = \rho^\varepsilon$. It is less obvious that $\int f\,dv = n$. Suppressing the time variable for notational convenience, what we have to prove is that $\int f(x,v)\theta(x)\,dv\,dx = \int n(x)\theta(x)\,dx$ for all test functions $\theta \in C_c^\infty$. In view of (68), this reduces to proving that

$$\int F_l(v)\,dv \to \int F(v)\,dv \tag{71}$$

as $l \to \infty$, where $F_l(v) = \int \tilde{w}^{\varepsilon_l}(x,v)\theta(x)\,dx$ and $F(v) = \int f(x,v)\theta(x)\,dx$ are non-negative L^1 functions. Of course, since $\tilde{w}^{\varepsilon_l} \to f$ weakly, we do know that

$$\int F_l(v)\chi(v)\,dv \to \int F(v)\chi(v)\,dv \tag{72}$$

for all test functions $\chi \in C_c^\infty$. It is easy to show that (72) implies (71) if and only if F_l is compact at infinity, in the sense that

$$\limsup_{l\to\infty} \int_{|v|>R} F_l(v)\,dv$$

converges to zero as $R \to \infty$. But in our case, $(1+|v|^2)F_l(v)$ is uniformly bounded in L^1, which clearly implies compactness at infinity. This concludes the proof of $\int f\,dv = n$.

It only remains to prove that $\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = 0$ is satisfied in the weak sense, with data $f|_{t=0} = f_I$, i.e.,

$$\iint_0^{T_0} f\,\{-\partial_t\theta - v\nabla_x\theta + \nabla_x V \cdot \nabla_v\theta\}\,dt\,dx\,dv + \int f_I(x,v)\theta(0,x,v)\,dx\,dv = 0$$

for all test functions $\theta \in C_c^\infty\big([0,T_0) \times \mathbb{R}^3_x \times \mathbb{R}^3_v\big)$. For this, it is convenient to use the Wigner transform with ψ_j^ε replaced by ϕ_j^ε, appearing in (19). Thus, we redefine

$$\begin{aligned} w^\varepsilon(t,x,v) &= \sum \frac{\lambda_j^\varepsilon}{(2\pi)^3} \int_{\mathbb{R}^3_y} \Big\langle \phi_j^\varepsilon\Big(t, x - \frac{\hbar}{2}y\Big), \phi_j^\varepsilon\Big(t, x + \frac{\hbar}{2}y\Big) \Big\rangle_{\mathbb{C}^4} e^{iv\cdot y}\, dy \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3_y} (\operatorname{tr} \tilde{\kappa}^\varepsilon)(x, \hbar y)\, e^{iv\cdot y}\, dy \end{aligned}$$

where $\tilde{\kappa}^\varepsilon(x,y)$ is defined as in (62) but with ψ_j^ε replaced by ϕ_j^ε.

The difference between this Wigner transform and our original definition (28) converges strongly to zero as $\varepsilon \to 0$, in $L^2_{x,v}$ uniformly in time, as is easily seen from the estimate (22), using Plancherel's theorem as in (63). In particular, we have

$$\frac{1}{(2\pi\hbar)^{3/2}} \left\| \operatorname{tr} \tilde{\kappa}^\varepsilon \right\|_{L^2_{x,v}} \le C \tag{73}$$

uniformly in time. Moreover, (32) and (34) still hold, with our new definition of w^ε; (35) and (36) hold with $-A_0^\varepsilon$ replaced by V^ε, and (37) holds with ρ^ε replaced by n^ε, on account of (23), which also implies that (65) and (66) hold with ρ^ε replaced by n^ε and A_0^ε replaced by V^ε. Thus,

$$\left\| \nabla V^\varepsilon(t) \right\|_{H^\delta} \le C. \tag{74}$$

for some $\delta > 0$.

A straightforward but tedious calculation, taking into account that ϕ_j^ε solves (19), reveals that w^ε satisfies

$$\partial_t w^\varepsilon + v \cdot \nabla_x w^\varepsilon - \nabla_x V^\varepsilon \cdot \nabla_v w^\varepsilon = (2\pi)^3 \left(A^\varepsilon + B^\varepsilon + C^\varepsilon\right), \tag{75}$$

where

$$\begin{aligned} A^\varepsilon &= \int_{\mathbb{R}^3_y} (\operatorname{tr} \tilde{\kappa}^\varepsilon)(x, \hbar y)(-i) F^\varepsilon(t,x,y) e^{iv\cdot y}\, dy, \\ F^\varepsilon &= \frac{V^\varepsilon(t, x - \hbar y/2) - V^\varepsilon(t, x + \hbar y/2)}{\hbar} - y \cdot \nabla V^\varepsilon(t,x), \\ B^\varepsilon &= \frac{1}{i\hbar} \int_{\mathbb{R}^3_y} \sum \lambda_j^\varepsilon \Big\langle \mathrm{Error}_j^\varepsilon\Big(t, x - \frac{\hbar}{2}y\Big), \phi_j^\varepsilon\Big(t, x + \frac{\hbar}{2}y\Big) \Big\rangle_{\mathbb{C}^4} e^{iv\cdot y}\, dy, \\ C^\varepsilon &= \overline{B^\varepsilon}. \end{aligned}$$

Let us first prove that the terms in the right hand side of (73) tend to zero in the weak sense. For B^ε and C^ε, this is immediate from (20), and in fact the convergence is strong in $L^2_{x,v}$ uniformly in time, as one can see from the appropriate analogue of (63). To prove that $A^\varepsilon \to 0$ in the sense of distributions it suffices, in view of (73), to show that F^ε converges strongly to zero in $L^2_{\mathrm{loc}}(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ uniformly in $0 \le t \le T_0$. To this end, one writes

$$F^\varepsilon(t,x,y) = \int_{-1/2}^{1/2} \big\{ \nabla_x V^\varepsilon(t, x + s\hbar y) - \nabla_x V^\varepsilon(t,x) \big\}\, ds \cdot y.$$

and applies, e.g., the elementary inequality (valid for any $\delta > 0$ and $R \ge 1$)

$$\|g(x+y) - g(x)\|_{L^2_x} \le \|g\|_{H^\delta} \Big(\sup_{|\xi|\le R} \left| e^{i\xi\cdot y} - 1 \right| + R^{-\delta} \Big),$$

combined with (74).

It now only remains to prove that the left hand side of (75) converges to $\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f$ in the sense of distributions. The issue is to prove that the nonlinear term $\nabla_x V^{\varepsilon_l} \cdot \nabla_v w^{\varepsilon_l} = \mathrm{div}_v(w^{\varepsilon_l} \nabla_x V^{\varepsilon_l})$ converges to $\nabla_x V \cdot \nabla_v f = \mathrm{div}_v(f \nabla_x V)$ in $\mathcal{D}'$, or equivalently that $w^{\varepsilon_l} \nabla_x V^{\varepsilon_l} \to f \nabla_x V$ in $\mathcal{D}'$. This is not obvious, since the factors w^{ε_l} and $\nabla_x V^{\varepsilon_l}$ are only known to converge weakly in L^2. The crucial fact, however, is that some subsequence of $\nabla_x V^{\varepsilon_l}$ converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3_x)$, uniformly in $0 \le t \le T_0$. To see this, one first notes that $\nabla_x V^\varepsilon$ is compact in $L^2_{\mathrm{loc}}(\mathbb{R}^3_x)$, in view of (74). Thus, it suffices to prove equicontinuity of $\nabla_x V^\varepsilon(t)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3_x)$. By L^p interpolation, Sobolev embedding and the estimate (74), one has

$$\begin{aligned} \left\| \nabla V^\varepsilon(t) - \nabla V^\varepsilon(s) \right\|_{L^2(B_R)} &\le C \left\| \nabla V^\varepsilon(t) - \nabla V^\varepsilon(s) \right\|^\theta_{L^{7/6}(B_R)} \\ &\le C \left(\int_s^t \left\| \partial_t \nabla V^\varepsilon(t') \right\|_{L^{7/6}(B_R)} \, dt' \right)^\theta \end{aligned}$$

for some $0 < \theta < 1$ and all $0 \le s \le t \le T_0$, where B_R denotes the ball $|x| \le R$. It is therefore enough to prove that $\|\partial_t \nabla V^\varepsilon(t)\|_{L^{7/6}(B_R)}$ is uniformly bounded. But from (19) and (21) we have

$$\partial_t \nabla V^\varepsilon = -\Delta^{-1} \nabla \partial_t n^\varepsilon = \Delta^{-1} \nabla \, \mathrm{div} \, \mathbf{J}^\varepsilon_{\mathrm{S}} - \Delta^{-1} \nabla r^\varepsilon,$$

where $\mathbf{J}^\varepsilon_{\mathrm{S}} = \hbar \sum \lambda^\varepsilon_j \, \mathrm{Im} \left\langle \nabla \phi^\varepsilon_j, \phi^\varepsilon_j \right\rangle_{\mathbb{C}^4}$ is uniformly bounded in $L^{7/6}(\mathbb{R}^3_x)$, in view of (59), and where $r^\varepsilon = \frac{2}{\hbar} \sum \lambda^\varepsilon_j \, \mathrm{Im} \left\langle \mathrm{Error}^\varepsilon_j, \phi^\varepsilon_j \right\rangle_{\mathbb{C}^4}$. Thus,

$$\begin{aligned} \left\| \partial_t \nabla V^\varepsilon(t) \right\|_{L^{7/6}(B_R)} &\le \left\| \mathbf{J}^\varepsilon_{\mathrm{S}}(t) \right\|_{L^{7/6}(\mathbb{R}^3_x)} + C_R \left\| \Delta^{-1} \nabla r^\varepsilon(t) \right\|_{L^2(\mathbb{R}^3_x)} \\ &\le C + C_R \left\| r^\varepsilon(t) \right\|_{L^{6/5}(\mathbb{R}^3_x)}, \end{aligned}$$

and by (20) and (52) the very last term is seen to be uniformly bounded.

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