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NONRELATIVISTIC LIMIT OF KLEIN-GORDON-MAXWELL TO SCHRÖDINGER-POISSON

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Abstract. We prove that in the nonrelativistic limit $c \to \infty$, where c is the speed of light, solutions of the Klein-Gordon-Maxwell system on $\mathbb{R}^{1+3}$ converge in the energy space $C([0,T];H^1)$ to solutions of a Schrödinger-Poisson system, under appropriate conditions on the initial data. This requires the splitting of the scalar Klein-Gordon field into a sum of two fields, corresponding, in the physical interpretation, to electrons and positrons. The proof relies on bilinear spacetime estimates related to the Klainerman-Machedon estimates, but taking into account the variation of the parameter c. A crucial fact is that the system has a null form structure in Coulomb gauge, as proved by Klainerman-Machedon.

1. Introduction. In this paper we study the behavior of solutions to the Klein-Gordon-Maxwell (*KGM*) system on $\mathbb{R}^{1+3}$ in the limit $c \to \infty$, where c denotes the speed of light. Coupled to the Coulomb gauge condition, the system reads:

$$
(1)\qquad \begin{cases} (\partial_\mu - \frac{i}{c}A_\mu)(\partial^\mu - \frac{i}{c}A^\mu)\phi = m^2c^2\phi, \\ \partial^\nu F_{\mu\nu} = J_\mu/c, \\ F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \\ J_\mu := \operatorname{Im}\left[\overline{\phi}\left(\partial_\mu - \frac{i}{c}A_\mu\right)\phi\right], \\ \partial^i A_i = 0. \end{cases}
$$

Here the unknowns are the scalar Klein-Gordon field $\phi(t,x) \in \mathbb{C}$ and a real electromagnetic potential $\{A_\mu(t,x)\}_{\mu=0,1,2,3}$, and $m > 0$ is the mass of the particle represented by ϕ. From now on we will for simplicity set $m = 1$. On the Minkowski spacetime $\mathbb{R}^{1+3}$ we use relativistic coordinates $x^0 = ct \in \mathbb{R}, x = (x^1, x^2, x^3) \in \mathbb{R}^3$, and indices are raised and lowered relative to the metric with signature $-1, 1, 1, 1$. We denote by ∂_μ the partial derivative $\frac{\partial}{\partial x^\mu}$. Note that $\partial_0 = \frac{1}{c}\partial_t$, where $\partial_t = \frac{\partial}{\partial t}$. We also write $\nabla = (\partial_1, \partial_2, \partial_3)$ for the spatial gradient. The usual summation convention is in effect: Repeated greek indices $\mu, \nu, \ldots$ are summed over $0, 1, 2, 3$, roman indices $i, j, \ldots$ over $1, 2, 3$. Thus, the spatial Laplacian can be written as $\Delta := \partial_i\partial^i$ and the wave operator as $\Box_c := \partial_\mu\partial^\mu = -\frac{1}{c^2}\partial_t^2 + \Delta$.

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Let us briefly recall how this system is derived. The first equation in (1) is the Klein-Gordon (*KG*) equation for a relativistic particle in the electromagnetic field $\{A_\mu\}$. Note that it is obtained from the free *KG* equation $\Box_c \phi = m^2 c^2 \phi$ through the substitution (see, e.g., [17, p. 163]), from classical physics,

$$i\partial_\mu \longrightarrow i\partial_\mu + A_\mu/c. \tag{2}$$

The next equation in (1) is Maxwell's equation. To put it in classical notation, let us split the potential $\{A_\mu\}$ into its temporal part A_0^c and its spatial part $\mathbf{A} := (A_1, A_2, A_3)$, and define the electric and magnetic field vectors by, respectively, $\mathbf{E} := \nabla A_0^c - \partial_t \mathbf{A}/c$ and $\mathbf{B} := \nabla \times \mathbf{A}$. Then $F_{i0} = E_i$ and $F_{ij} = \epsilon_{ijk} B^k$, so $\partial^\nu F_{\mu\nu} = J_\mu/c$ becomes

$$\operatorname{div} \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \partial_t \mathbf{E}/c = \mathbf{J}/c,$$

where $\rho := J^0/c = -J_0/c$ and $\mathbf{J} := (J^1, J^2, J^3)$. The homogeneous equations

$$\operatorname{div} \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B}/c = 0,$$

which follow from the definitions of $\mathbf{E}$ and $\mathbf{B}$, complete the Maxwell system in standard form. Moving on, the fourth equation in (1) defines the four-current density associated to the *KG* equation; multiplying the *KG* equation by $\overline{\phi}$ and taking imaginary parts yields the conservation law $\partial^\mu J_\mu = 0$, i.e., $\partial_t \rho + \operatorname{div} \mathbf{J} = 0$. The final equation in (1) is the Coulomb gauge condition $\operatorname{div} \mathbf{A} = 0$. The reason for this particular choice of gauge will be explained later. It is equivalent to $\mathcal{P}\mathbf{A} = \mathbf{A}$, where $\mathcal{P}$ is the projection onto divergence free vector fields in $\mathbb{R}_x^3$. The second and fifth equations in (1) are then seen to be equivalent to

$$\Delta A_0^c = -J_0/c = \rho, \qquad \Box_c \mathbf{A} = -\mathcal{P}\mathbf{J}/c,$$

provided the initial data of $\mathbf{A}$ are divergence free.

Remark 1. *KGM* can also be derived from Hamilton's principle with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}\left[\left(\partial_\mu - \tfrac{i}{c} A_\mu\right)\phi \overline{\left(\partial^\mu - \tfrac{i}{c} A^\mu\right)\phi} + c^2 \phi\overline{\phi}\right].$$

The energy-momentum tensor

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\nu A_\lambda)} \partial_\mu A_\lambda + \frac{\partial \mathcal{L}}{\partial(\partial^\nu \phi)} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\overline{\partial^\nu \phi})} \overline{\partial_\mu \phi} - \mathcal{L}\delta_{\mu\nu}$$

satisfies $\partial^\nu T_{\mu\nu} = 0$. (See, e.g., [4, Chapter 12].) This tensor turns out not to be symmetric, but can be symmetrized by the same trick used for the Maxwell

Lagrangian (see [4, pp. 583–584]). Thus, setting $T'_{\mu\nu} = T_{\mu\nu} - \partial^\lambda \left(F_{\lambda\nu} A_\mu\right)$ we still have the conservation law $\partial^\nu T'_{\mu\nu} = 0$, which in particular implies

$$\mathcal{E}(t) := \int_{\mathbb{R}^3} T'_{00}(t,x)\,dx = \text{const.} \tag{3}$$

A calculation reveals that

$$T'_{00} = \frac{1}{2}\left(\sum_{\mu=0}^{3} \left|\left(\partial_\mu - \tfrac{i}{c}A_\mu\right)\phi\right|^2 + c^2\,|\phi|^2 + \mathbf{E}^2 + \mathbf{B}^2\right), \tag{4}$$

which is nonnegative. (The energy conservation (3) can also be deduced by direct calculation, of course.)

In view of the discussion preceding Remark 1, we can reformulate the system (1) as follows:

$$\begin{aligned} \left(\partial_t^2 + E(c)^2\right)\phi^c &= -2ic\mathbf{A}^c \cdot \nabla\phi^c \\ &\quad + 2iA_0^c\partial_t\phi^c + i(\partial_t A_0^c)\phi^c + (A_0^c)^2\phi^c - |\mathbf{A}^c|^2\,\phi^c, \end{aligned} \tag{5a}$$

$$\Delta A_0^c = \frac{1}{c^2}\,\mathrm{Im}\left(\phi^c\overline{\partial_t\phi^c}\right) + \frac{1}{c^2}\,|\phi^c|^2\,A_0^c, \tag{5b}$$

$$\Box_c \mathbf{A}^c = \frac{1}{c}\mathcal{P}\,\mathrm{Im}\left(\phi^c\overline{\nabla\phi^c}\right) + \frac{1}{c^2}\mathcal{P}\big(|\phi^c|^2\,\mathbf{A}^c\big), \tag{5c}$$

where we have put in superscripts to emphasize the dependence of (ϕ, A_μ) on c, and where

$$E(c) := \sqrt{c^4 - c^2\Delta}. \tag{6}$$

We specify finite energy initial data

$$\begin{aligned} (\mathbf{A}^c, \partial_t\mathbf{A}^c)\,|_{t=0} &= \left(\mathbf{a}_0^c, \mathbf{a}_1^c\right) \in \dot{H}^1 \times L^2, \\ (\phi^c, \partial_t\phi^c)\,|_{t=0} &= \left(\phi_0^c, \phi_1^c\right) \in H^1 \times L^2, \end{aligned} \tag{7}$$

satisfying

$$\operatorname{div}\mathbf{a}_0^c = \operatorname{div}\mathbf{a}_1^c = 0. \tag{8}$$

(Equivalently, instead of data for $\mathbf{A}$ we could specify data for $\mathbf{E}$ and $\mathbf{B}$ in L^2.) Here $H^s = H^s(\mathbb{R}^3)$ is the Sobolev space with norm $\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2}\widehat{f}\|_{L^2_\xi}$, where $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$, and $\dot{H}^s$ denotes the corresponding homogeneous space, with norm $\|f\|_{\dot{H}^s} = \||\xi|^s\widehat{f}\|_{L^2_\xi}$.

Klainerman and Machedon [8] proved global well-posedness of the Cauchy problem (5), (7), (8). (In [8] the massless case is considered, but it is a simple matter to modify their proof to the massive case; we outline the necessary changes in an appendix.) The question we address here is what happens to the solutions as $c \to \infty$. Let us now state our main result.

Some notation: The O, o notation always refers to the limit $c \to \infty$. If X is a Banach space of functions on $\mathbb{R}^3_x$, we denote by $L^p_t X$ the space with norm $\|u\|_{L^p_t X} = (\int_{-\infty}^{\infty} \|u(t,\cdot)\|_X^p \, dt)^{1/p}$, with the usual modification if $p = \infty$. The localization of this norm to a time slab $S_T := [0,T] \times \mathbb{R}^3$ is denoted $\|u\|_{L^p_t X(S_T)}$.

THEOREM 1. (Nonrelativistic limit of KGM.) *Let* $(\phi^c, A_0^c, \mathbf{A}^c)$ *be the global solution of (5), (7), (8), obtained by Klainerman-Machedon [8], and assume the data satisfy:*

(i) $\alpha := \lim_{c\to\infty} \phi_0^c$ *and* $\beta := \lim_{c\to\infty} E(c)^{-1}\phi_1^c$ *exist in* H^1,

(ii) $\|\mathbf{a}_0^c\|_{\dot H^1} + \frac{1}{c}\|\mathbf{a}_1^c\|_{L^2} = O(1)$,

where $E(c)$ *is defined by (6). Write*

$$\phi^c = \phi^c_+ + \phi^c_- = e^{-itc^2}\psi^c_+ + e^{+itc^2}\psi^c_- \tag{9}$$

where

$$\phi^c_\pm := \frac{1}{2}\left(\phi^c \pm iE(c)^{-1}\partial_t \phi^c\right), \tag{10}$$

$$\psi^c_\pm := e^{\pm itc^2}\phi^c_\pm. \tag{11}$$

Let (v_+, v_-, u) *be the solution of the Schrödinger-Poisson system*

$$\begin{cases} (i\partial_t \pm \frac{\Delta}{2})v_\pm = -uv_\pm, \\ \Delta u = |v_+|^2 - |v_-|^2, \\ v_\pm|_{t=0} = \frac{1}{2}(\alpha \pm i\beta). \end{cases} \tag{12}$$

Then for every $0 < T < \infty$,

$$(\psi^c_+, \psi^c_-, A_0^c) \longrightarrow (v_+, v_-, u) \quad \text{in} \quad L^\infty_t(H^1 \times H^1 \times \dot H^1)(S_T)$$

as $c \to \infty$.

The corresponding problem for the linear Klein-Gordon equation in an external (i.e., fixed) electromagnetic field has a long history, at least on a formal level; rigorous results can be found in [22, 3]. (These papers only treat the static case, i.e., time-independent potential. The nonrelativistic limit for the related Dirac equation with time-dependent external potential was treated in [1].) There are also results for the nonlinear Klein-Gordon equation, see [14, 9, 13, 15, 11].

As this work was being completed, we learned of independent and concurrent work of Masmoudi and Nakanishi [12], who have established a result similar to our Theorem 1, as well as an analogous result for the somewhat similar Dirac-Maxwell (*DM*) system. The analytic tools applied in [12] are much the same as those used in this paper (i.e., variations on the Klainerman-Machedon method [8], taking into account the varying parameter c), the main difference being that our estimates are strong enough to obtain uniform (w.r.t. c) estimates on any finite time interval assuming only the uniform boundedness of the data (cf. Theorem 2 below), whereas in [12] the convergence assumption (i.e., condition (i) in Theorem 1 above) is essential. (The reason being that in [12], uniform estimates are only obtained on an arbitrarily short time interval, and to push the result to later times they must use estimates on the limiting Schrödinger-Poisson system.) For *KGM* this distinction is admittedly not very important, but for *DM* the situation is different, since global well-posedness remains an open question. Thus, in a follow-up to this work we establish a logarithmic lower bound on the local existence time of *DM* as $c \to \infty$, for data uniformly bounded in H^1.

Let us now provide some motivation for our result. The splitting of the ϕ field defined by (10) is arrived at by diagonalizing the two-component form of the free *KG* equation, which reads

$$\partial_t \begin{pmatrix} \phi_{(0)} \\ \phi_{(1)} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -E(c)^2 & 0 \end{pmatrix} \begin{pmatrix} \phi_{(0)} \\ \phi_{(1)} \end{pmatrix},$$

where $\phi_{(0)} = \phi$ and $\phi_{(1)} = \partial_t \phi$. As one can easily check, the transformation (10) diagonalizes this system, producing

$$i\partial_t \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} E(c) & 0 \\ 0 & -E(c) \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$

if ϕ solves the free *KG* equation. Then with notation as in (11) (i.e., subtracting the rest energy),

$$i\partial_t \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} E(c) - c^2 & 0 \\ 0 & -[E(c) - c^2] \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

and one observes that the Fourier symbol of $E(c) - c^2$ is

$$\sqrt{c^2 \left|\xi\right|^2 + c^4} - c^2 = \frac{\left|\xi\right|^2}{1 + \sqrt{1 + \left|\xi\right|^2 / c^2}} \longrightarrow \frac{\left|\xi\right|^2}{2} \quad \text{as} \quad c \to \infty, \tag{13}$$

where ξ is the Fourier variable corresponding to x. Thus, for the simple model case where ϕ is a solution of the *free KG* equation it is trivial to see that $\psi_\pm$

converge to solutions of Schrödinger equations, assuming the limits in (i) of Theorem 1 exist.

Having thus motivated (10), we proceed to modify it through the so-called minimal substitution (2). From now on we therefore set (this transformation appears to be well known, and can be found in physics textbooks in connection with the nonrelativistic limit of *KG* in an external field)

$$\phi_\pm^c = \frac{1}{2}\left\{\phi^c \pm E(c)^{-1}(i\partial_t\phi^c + A_0^c\phi^c)\right\}. \tag{14}$$

Since it turns out (see Theorem 2 below) that

$$\left\|E(c)^{-1}(A_0^c\phi^c)\right\|_{L_t^\infty H^1(S_T)} = O(c^{-1})$$

under the hypotheses of Theorem 1, it is clear that as far as Theorem 1 is concerned, it is immaterial whether we use (10) or (14). The latter, however, is more natural to work with, since the evolution equations satisfied by $\psi_\pm$ turn out to be much nicer. In fact, we have the following.

LEMMA 1. *In terms of the splitting (9) defined via (14) and (11), the KG equation (5) is equivalent to the system of two equations*

$$L_\pm(c)\psi_\pm^c = -A_0^c\psi_\pm^c \pm e^{\pm itc^2}R^c, \tag{15}$$

where

$$L_\pm(c) := i\partial_t \mp (E(c) - c^2), \tag{16}$$

$$R^c := \frac{1}{2}E(c)^{-1}\left\{2ic\mathbf{A}^c\cdot\nabla\phi^c + [E(c)-c^2, A_0^c](\phi_+^c - \phi_-^c) + |\mathbf{A}^c|^2\phi^c\right\} \tag{17}$$

and $[\cdot,\cdot]$ *denotes the commutator.*

Proof. This is a straightforward calculation. Apply $i\partial_t$ to both sides of (14) and substitute for $\partial_t^2\phi^c$ using the *KG* equation (5a), and then for $i\partial_t\phi^c$ using [recall (14)]

$$i\partial_t\phi^c = E(c)(\phi_+^c - \phi_-^c) - A_0^c\phi^c.$$

Using also $\phi^c = \phi_+^c + \phi_-^c$, the result is

$$i\partial_t\phi_\pm^c = \pm E(c)\phi_\pm^c - A_0^c\phi_\pm^c \pm R^c.$$

Now multiply by $e^{\pm itc^2}$ and recall (11). □

Since $E(c) - c^2$ behaves like $-\frac{\Delta}{2}$ as $c \to \infty$ (cf. (13)), and since it turns out that R^c vanishes in the limit (see Theorem 2 below), it is hardly surprising that (15) tends to the Schrödinger equation in (12). Furthermore,

$$\Delta A_0^c = -J_0/c = \operatorname{Re}\left[(\overline{\phi_+^c} + \overline{\phi_-^c})\tfrac{E(c)}{c^2}(\phi_+^c - \phi_-^c)\right] = |\psi_+^c|^2 - |\psi_-^c|^2 + R'^c, \tag{18}$$

where

$$\begin{aligned} R'^c := \; & \operatorname{Re}\left[\overline{\psi_+^c}\left(\tfrac{E(c)}{c^2} - 1\right)\psi_+^c\right] - \operatorname{Re}\left[\overline{\psi_-^c}\left(\tfrac{E(c)}{c^2} - 1\right)\psi_-^c\right] \\ & - \operatorname{Re}\left[e^{+2itc^2}\overline{\psi_+^c}\left(\tfrac{E(c)}{c^2} - 1\right)\psi_-^c\right] + \operatorname{Re}\left[e^{-2itc^2}\overline{\psi_-^c}\left(\tfrac{E(c)}{c^2} - 1\right)\psi_+^c\right] \end{aligned} \tag{19}$$

vanishes in the limit (see Theorem 2), motivating the fact that (5) tends to the Poisson equation in (12).

The main difficulty in proving Theorem 1 is to obtain $O(1)$ bounds in H^1 as $c \to \infty$. The bounds obtained from the conservation of the KGM energy are not good enough. For example, energy conservation gives $\|\phi^c\|_{L_t^\infty H^1} = O(c)$ (see Section 3), but this can be improved to $O(1)$ (on finite time intervals) using spacetime estimates of Strichartz type. Energy conservation does, however, give the important global-in-time bound $\|\phi_\pm^c\|_{L_t^\infty L_x^2} = O(1)$, which is not surprising in view of the fact that for the limiting system (12), the L^2 norms of $v^\pm$ are exactly conserved in time.

The main estimates are contained in the following theorem.

THEOREM 2. *Let $(\phi^c, A_0^c, \mathbf{A}^c)$ be the global solution of (5), (7), (8), obtained by Klainerman-Machedon [8], and assume the data satisfy:*

$$\|\mathbf{a}_0^c\|_{\dot H^1} + \frac{1}{c}\|\mathbf{a}_1^c\|_{L^2} = O(1), \tag{20}$$

$$\|\phi_0^c\|_{H^1} + \left\|E(c)^{-1}\phi_1^c\right\|_{H^1} = O(1). \tag{21}$$

Then we have the global-in-time bound

$$\|\phi_\pm^c\|_{L_t^\infty L_x^2} = \|\psi_\pm^c\|_{L_t^\infty L_x^2} = O(1). \tag{22}$$

Moreover, for every $0 < T < \infty$,

(i) $\|\phi_\pm^c\|_{L_t^\infty H^1(S_T)} = \|\psi_\pm^c\|_{L_t^\infty H^1(S_T)} = O(1)$,

(ii) $\|\mathbf{A}^c\|_{L_t^\infty \dot H^1(S_T)} + \frac{1}{c}\|\partial_t \mathbf{A}^c\|_{L_t^\infty L_x^2(S_T)} = O(1)$,

(iii) $\|L_\pm(c)\psi_\pm^c\|_{L_t^1 H^1(S_T)} + c\,\|\Box_c \mathbf{A}^c\|_{L_t^1 L_x^2(S_T)} = O(1)$,

(iv) $\|\nabla A_0^c\|_{L_t^\infty L_x^r(S_T)} = O(1)$ *for* $3/2 < r \le 3$,

(v) $\|R^c\|_{L_t^1 H^1(S_T)} = O(c^{-1/2})$,

(vi) $\|R'^c\|_{L_t^1 L_x^r(S_T)} = O(c^{-1/2})$ *for* $1 \le r \le (3/2)^+$,

(vii) $\|E(c)^{-1}(A_0^c \phi^c)\|_{L_t^\infty H^1(S_T)} = O(c^{-1})$,

where $\phi_\pm^c$, $\psi_\pm^c$, R^c *and* R'^c *are given by (14), (11), (17) and (19).*

Let us briefly comment on the technical tools used to prove this result. Our main source of inspiration is the paper of Klainerman-Machedon [8] on the global H^1 well-posedness of *KGM*, so it is worthwhile to recall some key points from their paper. First, in view of the conservation of energy, it suffices to prove *local* well-posedness in H^1. Second, if one assumes slightly more regularity of the data, i.e., $H^{1+\varepsilon}$ instead of H^1, then one can prove local well-posedness using linear Strichartz estimates for the homogeneous wave equation; see [16]. To get the sharp H^1 result proved in [8], however, is much more involved, and requires certain bilinear generalizations of Strichartz' L^4 estimate (cf. [8, Section 2]) to handle the first terms in the right hand sides of (5a,c). A key point is that, due to the Coulomb gauge condition, these terms have a null form structure, without which the estimates would in fact fail. Here we prove modifications of these estimates (see Section 4) where the wave operator $\square_c$ may be replaced by the operator $L_\pm(c)$ defined in Lemma 1, which essentially behaves like the Schrödinger operator at frequency $\lesssim c$, and like the wave operator at frequency $\gg c$. Linear Strichartz estimates for this operator have been proved in [13], but we include a proof here for the convenience of the reader.

The rest of this paper is organized as follows: In the next section we recall some facts concerning the limit system (12), and we collect some inequalities that will be used repeatedly. In Section 3 we use energy conservation to prove (22), and Section 4 deals with linear and bilinear spacetime estimates for the operators $\square_c$ and $L_\pm(c)$. In Section 5 we prove parts (i)–(vii) of Theorem 2, and finally in Section 6 we prove the main result, Theorem 1.

Throughout the paper, the following notational conventions will be in effect:

- To avoid cumbersome notation, we generally skip the superscript c on the fields (ϕ, A_μ) henceforth.
- $\lesssim$ means $\le$ up to multiplication by a positive constant independent of c. $X \sim Y$ stands for $X \lesssim Y \lesssim X$.
- The O, o notation always refers to the limit $c \to \infty$.
- K, δ and N denote positive constants, independent of c, which may change from line to line. $\sigma(T)$ denotes the function $K(T^\delta + T^N)$ and $P(x)$ is the polynomial $x + x^N$.
- For exponents we use the standard shorthand p^+ (resp. p^-) for $p + \varepsilon$ (resp. $p - \varepsilon$), where $\varepsilon > 0$ is sufficiently small, independently of c. See, e.g., Lemma 5 in the next section.
- χ is a smooth cut-off on $\mathbb{R}^3$ such that $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$. Moreover, we assume that χ is radial, and we write $\chi(\xi)$ and $\chi(r = |\xi|)$

interchangeably. We use $\chi(\xi/c)$ to split functions $f(x)$ into low ($\lesssim c$) and high ($\gg c$) frequencies:

$$f = f * \theta_c + f * (1 - \theta_c) = f_l + f_h, \tag{23}$$

where θ_c is the inverse Fourier transform of $\chi(\xi/c)$. Then $\|\theta_c\|_{L^1}$ does not depend on c, so $\|f_l\|_{L^p}, \|f_h\|_{L^p} \lesssim \|f\|_{L^p}$ for $1 \le p \le \infty$ by Young's inequality.

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2. Preliminaries. Global well-posedness in L^2 for the Schrödinger-Poisson system (12) follows from the work of Castella [2]. In fact, since the L^2 norms of $v^\pm$ are conserved, it is enough to prove local well-posedness for L^2 data. It is then easy to obtain L^2 bounds for $\nabla v^\pm$ on finite time intervals. For the convenience of the reader, and since a similar but more involved argument will be used in the proof of Theorem 2, we include here a short proof of the following:

LEMMA 2. *(Cf. [2].) The system (12) is globally well-posed in L^2, and for H^1 initial data we have*

$$\|v^\pm\|_{L_t^\infty H^1(S_T)} < \infty \tag{24}$$

for all $T < \infty$.

So assume (u, v_+, v^-) solves (12), and let us derive some estimates for v_+ (the argument for v^- is of course the same). Writing $\langle f, g \rangle = \int_{\mathbb{R}^3} f\overline{g}\, dx$, we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\langle \nabla v_+, \nabla v_+ \rangle &= \operatorname{Re}\langle \nabla \partial_t v_+, \nabla v_+ \rangle \\ &= \operatorname{Im}\Big\langle -\tfrac{\Delta}{2}(\nabla v_+) + u\nabla v_+ + (\nabla u)v_+, \nabla v_+ \Big\rangle = \operatorname{Im}\langle (\nabla u)v_+, \nabla v_+ \rangle,\end{aligned}$$

since Δ and u are self-adjoint. But

$$\begin{aligned}\operatorname{Im}\langle (\nabla u)v_+, \nabla v_+ \rangle &\le \|\nabla u\|_{L_x^3} \|v_+\|_{L_x^6} \|\nabla v_+\|_{L_x^2} \lesssim \|\Delta u\|_{L_x^{3/2}} \|\nabla v_+\|_{L_x^2}^2 \\ &\lesssim \Big(\textstyle\sum_\pm \|v^\pm\|_{L_x^2} \|v^\pm\|_{L_x^6}\Big) \|\nabla v_+\|_{L_x^2}^2,\end{aligned}$$

where we used Lemma 5(ii) below and the Sobolev embedding (29). Therefore, by Gronwall's lemma applied to $f(t) = \|\nabla v_+(t)\|_{L^2}^2$,

$$\|\nabla v_+(t)\|_{L^2} \le \|\nabla v_+(0)\|_{L^2} \exp\Big(\textstyle\sum_\pm \|v^\pm(0)\|_{L^2} \displaystyle\int_0^t \|v^\pm(s)\|_{L^6}\, ds\Big), \tag{25}$$

where we used the conservation of $\|v^{\pm}(t)\|_{L^2}$. Therefore, (24) will certainly follow if we can control the norms $\|v^{\pm}\|_{L_t^2 L_x^6(S_T)}$. To this end, define

$$Z_T^{\pm} = \|v^{\pm}\|_{L_t^\infty L_x^2(S_T)} + \|v^{\pm}\|_{L_t^2 L_x^6(S_T)} = \|v^{\pm}(0)\|_{L^2} + \|v^{\pm}\|_{L_t^2 L_x^6(S_T)}$$

and set $Z_T = Z_T^+ + Z_T^-$.

We claim that (recall the notational conventions set out at the end of the Introduction)

$$Z_T \lesssim Z_0 + \sigma(T) Z_0^2 Z_T. \tag{26}$$

This would imply that $Z_T \lesssim Z_0$ up to a time $T > 0$ only depending on $Z_0 = \sum_{\pm} \|v^{\pm}(0)\|_{L^2}$. Then local well-posedness of (12) in L^2 follows by standard arguments, hence global well-posedness by L^2-conservation.

So it remains to prove (26). To this end, we use a Strichartz type inequality for the Schrödinger initial value problem on $\mathbb{R}^{1+3}$,

$$i\partial_t v \pm \frac{\Delta}{2} v = F, \quad v|_{t=0} = f. \tag{27}$$

In fact, by Corollary 1.4 in [6], if $2 \le q, r \le \infty$ and $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$, then the estimate

$$\|v\|_{L_t^2 L_x^6(S_T)} + \|v\|_{L_t^\infty L_x^2(S_T)} \lesssim \|f\|_{L^2} + \|F\|_{L_t^{q'} L_x^{r'}(S_T)} \tag{28}$$

holds for solutions of (27), where $1 = \frac{1}{q} + \frac{1}{q'}$ and $1 = \frac{1}{r} + \frac{1}{r'}$. We apply this inequality with q, r given by $\frac{2}{q} = \varepsilon$ and $\frac{1}{r} = \frac{1}{2} - \frac{\varepsilon}{3}$, where $\varepsilon > 0$ is sufficiently small. Thus, $(q', r') = (1^+, 2^-)$. Applying (28) to the Poisson equation in (12) then gives

$$Z_T^{\pm} \lesssim Z_0^{\pm} + \|u\|_{L_t^{q'} L_x^{3/\varepsilon}(S_T)} \|v^{\pm}(0)\|_{L_x^2}.$$

But by Sobolev embedding, Hölder's inequality and L^p interpolation,

$$\|u\|_{L_x^{3/\varepsilon}} \lesssim \|\Delta u\|_{L_x^{(3/2)-}} \lesssim \sum\nolimits_{\pm} \|v^{\pm}\|_{L_x^2} \|v^{\pm}\|_{L_x^{6-}} \lesssim \sum\nolimits_{\pm} \|v^{\pm}\|_{L_x^2}^{1^+} \|v^{\pm}\|_{L_x^6}^{1^-},$$

and applying Hölder's inequality in t then yields

$$\|u\|_{L_t^{q'} L_x^{3/\varepsilon}(S_T)} \lesssim \sigma(T) \sum\nolimits_{\pm} \|v^{\pm}(0)\|_{L_x^2}^{1^+} \|v^{\pm}\|_{L_t^2 L_x^6(S_T)}^{1^-} \lesssim \sigma(T) Z_0 Z_T.$$

This proves (26) and concludes the proof of Lemma 2. □

We now list some simple estimates that will be used extensively in later sections. First, for the operator $E(c)$ defined by (6), we have:

LEMMA 3. *The following operator norm estimates hold, for all $s \in \mathbb{R}$.*

(i) $\|E(c)^{-1}\|_{H^s \to H^s} = O(1/c^2)$.

(ii) $\|E(c)^{-1}\|_{H^s \to H^{s+1}} = O(1/c)$.

(iii) $\|E(c) - c^2\|_{H^{s+1} \to H^s} = O(c)$.

(iv) $\|E(c) - c^2\|_{H^{s+2} \to H^s} = O(1)$.

Proof. These statements translate to estimates on the Fourier symbols of the operators. Thus, the symbol $(c^4 + c^2 |\xi|^2)^{-1/2}$ of $E(c)^{-1}$ is bounded by c^{-2} as well as $(c|\xi|)^{-1}$, which proves (i) and (ii), respectively. The symbol of $E(c) - c^2$, given by (13), is bounded by $c|\xi|$, and also by $|\xi|^2/2$, proving parts (iii) and (iv), respectively. □

For the splitting (23) into low and high frequencies, we have:

LEMMA 4. *The following estimates hold on $\mathbb{R}^3$.*

(i) $\left\|\frac{E(c)}{c^2} f_l\right\|_{L^p} \lesssim \|f_l\|_{L^p}$ *for* $1 \le p \le \infty$.

(ii) $\|f_l\|_{H^{1+\varepsilon}} \lesssim c^\varepsilon \|f_l\|_{H^1}$ *for* $\varepsilon > 0$.

(iii) $\left\|\frac{E(c)}{c^2} f_h\right\|_{L^2} \lesssim \frac{1}{c} \|f_h\|_{\dot{H}^1}$.

(iv) $\|f_h\|_{L^2} \lesssim \frac{1}{c} \|f_h\|_{H^1}$.

Proof. Since $\frac{E(c)}{c^2} f_l = \omega_c * f_l$, where $\widehat{\omega_c}(\xi) = \left(1 + |\frac{\xi}{c}|^2\right)^{1/2} \chi\left(\frac{\xi}{2c}\right)$, and since the L^1 norm of ω_c is independent of c, we get (i) by Young's inequality. The remaining inequalities are easy to prove using Plancherel's theorem; we omit the details. □

In order to estimate A_0^c, we will need:

LEMMA 5. *The following estimates hold on $\mathbb{R}^3$.*

(i) $\|f\|_{L^\infty} \lesssim \|\Delta f\|_{L^{(3/2)^+}} + \|\Delta f\|_{L^{(3/2)^-}}$.

(ii) $\|\nabla f\|_{L^3} \lesssim \|\Delta f\|_{L^{3/2}}$.

Proof. The second inequality is immediate from Sobolev embedding and the fact that $\|\partial_i \partial_j f\|_{L^p} \lesssim \|\Delta f\|_{L^p}$ for $1 < p < \infty$ (see [21, Proposition III.3]). To prove (i), observe first that for $\delta > 0$ arbitrarily small,

$$\|f\|_{L^\infty} \lesssim \left\|(I - \Delta)^\delta f\right\|_{L^{3/\delta}} \lesssim \left\|(-\Delta)^\delta f\right\|_{L^{3/\delta}} + \|f\|_{L^{3/\delta}}.$$

The first inequality follows by Sobolev embedding, the second from [21, Lemma V.2(ii)]. By the Hardy-Littlewood-Sobolev inequality (see [19, Theorem 0.3.2]) the right hand side is $\lesssim \|\Delta f\|_{L^{3/(2-\delta)}} + \|\Delta f\|_{L^{3/(2+\delta)}}$. This concludes the proof. □

Finally, we note that the Sobolev embedding

$$\|f\|_{L^6_x} \lesssim \|f\|_{\dot H^1}, \tag{29}$$

implies

$$\|fgh\|_{L^2_x} \lesssim \|f\|_{\dot H^1} \|g\|_{\dot H^1} \|h\|_{\dot H^1} \tag{30}$$

and

$$\|fg\|_{L^2_x} \lesssim \|f\|_{\dot H^1} \|g\|_{L^2_x}^{1/2} \|g\|_{\dot H^1}^{1/2}. \tag{31}$$

To prove the latter, write $\|fg\|_{L^2} \le \|f\|_{L^6} \|g\|_{L^3}$ and $\|g\|_{L^3} \le \|g\|_{L^2}^{1/2} \|g\|_{L^6}^{1/2}$.

3. Energy conservation and uniform L^2 bounds. Throughout this section it is assumed that the hypotheses of Theorem 2 are satisfied. Our aim here is to prove the global-in-time $L_t^\infty L_x^2$ bound (22) for $\phi_\pm$. But by (14), Lemma 3(i) and (31),

$$\|\phi_\pm\|_{L^2} \lesssim \|\phi\|_{L^2} + \tfrac{1}{c^2} \|\partial_t \phi\|_{L^2} + \tfrac{1}{c^2} \|\nabla A_0\|_{L^2} \|\phi\|_{L^2}^{1/2} \|\nabla \phi\|_{L^2}^{1/2},$$

at each time t, so it suffices to prove

$$\|\phi\|_{L_t^\infty L_x^2} + \tfrac{1}{c} \|\nabla\phi\|_{L_t^\infty L_x^2} + \tfrac{1}{c^2} \|\partial_t\phi\|_{L_t^\infty L_x^2} + \tfrac{1}{c} \|\nabla A_0\|_{L_t^\infty L_x^2} = O(1). \tag{32}$$

This will be deduced from the conservation of the KGM energy $\mathcal{E}(t)$ given by (3) and (4). Thus, if we can show

$$\mathcal{E}(0) = O(c^2) \tag{33}$$

and

$$c^2 \|\phi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2 + \tfrac{1}{c^2} \|\partial_t\phi\|_{L^2}^2 + \|\nabla A_0\|_{L^2}^2 \lesssim \mathcal{E}\left(1 + \mathcal{E}/c^3 + \mathcal{E}^2/c^6\right) \tag{34}$$

at each time t, then (32) follows immediately.

Proof of (33). It is enough to prove that at $t = 0$,

$$c^2 \|\phi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2 + \tfrac{1}{c^2} \|\partial_t\phi\|_{L^2}^2 = O(c^2) \tag{35}$$

$$\|\nabla A_0\|_{L^2}^2 + \|\nabla \mathbf{A}\|_{L^2}^2 + \tfrac{1}{c^2} \|\partial_t \mathbf{A}\|_{L^2}^2 = O(c^2), \tag{36}$$

$$\tfrac{1}{c^2} \|A_0\phi\|_{L^2}^2 + \tfrac{1}{c^2} \|\mathbf{A}\phi\|_{L^2}^2 = O(1). \tag{37}$$

The first two terms on l.h.s.(35) are $O(c^2)$ at $t = 0$ by (21), and for the third term we write $\partial_t\phi = E(c)E(c)^{-1}\partial_t\phi$, which gives

$$\frac{1}{c}\left\|\partial_t\phi\right\|_{L^2} \lesssim c\left\|E(c)^{-1}\partial_t\phi\right\|_{L^2} + \left\|E(c)^{-1}\partial_t\phi\right\|_{H^1} = O(c) \tag{38}$$

by (21). This proves (35). The last two terms on l.h.s.(36) are $O(1)$ at $t = 0$ by (20), and for the first term we use the elliptic estimate (see [8, Eqs. (3.4a,b)]) $\left\|\nabla A_0(t)\right\|_{L^2} \lesssim \frac{1}{c}\left\|\partial_t\phi\right\|_{L^2}$. Therefore, by (38), $\left\|\nabla A_0(t = 0)\right\|_{L^2} = O(c)$, and this concludes the proof of (36). Finally, to prove (37) at $t = 0$, use (31) and the bounds in (21) and (36). □

Proof of (34). First, by [8, Eq. (1.3c)],

$$\left\|\nabla A_0\right\|_{L^2}^2 + \left\|\nabla \mathbf{A}\right\|_{L^2}^2 \lesssim \mathcal{E} \tag{39}$$

for all t, so we get the desired bound for the last term on l.h.s.(34). The first term is obviously bounded by $\mathcal{E}$, so it remains to consider the two middle terms. But using (31),

$$\left\|\nabla\phi\right\|_{L^2} \leq \sum_{j=1}^{3}\left\|\left(\partial_j - \frac{i}{c}A_j\right)\phi\right\|_{L^2} + \frac{1}{c}\left\|\nabla\mathbf{A}\right\|_{L^2}\left\|\phi\right\|_{L^2}^{1/2}\left\|\nabla\phi\right\|_{L^2}^{1/2}.$$

Now use the fact that if $\alpha \leq \beta + \gamma\sqrt{\alpha}$, where $\alpha, \beta, \gamma \geq 0$, then $\alpha \leq 2\beta + 4\gamma^2$. Combining this with (39) gives the bound $\left\|\nabla\phi\right\|_{L^2}^2 \lesssim \mathcal{E} + (\mathcal{E}/c^2)^3$. Similarly,

$$\frac{1}{c}\left\|\partial_t\phi\right\|_{L^2} \leq \left\|\left(\partial_0 - \frac{i}{c}A_0\right)\phi\right\|_{L^2} + \frac{1}{c}\left\|\nabla A_0\right\|_{L^2}\left\|\phi\right\|_{L^2}^{1/2}\left\|\nabla\phi\right\|_{L^2}^{1/2}.$$

Squaring this, and using (39) as well as the bounds already obtained for ϕ and $\nabla\phi$, we get the correct bound for $\frac{1}{c^2}\left\|\partial_t\phi\right\|_{L^2}^2$. □

4. Linear and bilinear spacetime estimates. Here we prove some linear and bilinear Strichartz type estimates on $\mathbb{R}^{1+3}$ for the operators $L_\pm(c)$, defined by (16).

4.1. Linear estimates. The key observation is that the propagators associated to $L_\pm(c)$,

$$U_c^\pm(t) = e^{\mp it(E(c)-c^2)}, \tag{40}$$

behave like the Schrödinger propagators

$$V^\pm(t) = e^{\pm it\Delta/2} \tag{41}$$

at low frequencies ($\lesssim c$) and like the wave equation propagators $e^{\mp itc\sqrt{-\Delta}}$ at high frequencies ($\gg c$). Indeed, $U_c^\pm(t)$ is a multiplier with Fourier symbol $e^{\mp ith_c(\xi)}$, where

$$h_c(\xi) = \frac{|\xi|^2}{1 + \sqrt{1 + |\xi|^2/c^2}} \sim \begin{cases} |\xi|^2/2 & \text{for } |\xi| \lesssim c, \\ c|\xi| & \text{for } |\xi| \gg c. \end{cases} \tag{42}$$

It is therefore not surprising that we have Strichartz estimates for $U_c^\pm$ in $L_t^q L_x^r$ for every sharp wave admissible pair (q, r) of Lebesgue exponents, and if we restrict to low frequency ($\lesssim c$), Schrödinger admissible exponents are also allowed.

Let us be more explicit. Following the terminology introduced in [6], we say that a pair (q, r) of Lebesgue exponents is *sharp wave admissible* (for $\mathbb{R}^{1+3}$) if

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2} \quad \text{and} \quad (q, r) \neq (2, \infty), \tag{43}$$

and we say (q, r) is *Schrödinger admissible* (for $\mathbb{R}^{1+3}$) if $q, r \geq 2$ and

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}. \tag{44}$$

PROPOSITION 1. *For every sharp wave admissible pair (q, r), the estimate*

$$\left\| U_c^\pm(t) f \right\|_{L_t^q L_x^r(S_T)} \lesssim \|f\|_{\dot{H}^{\frac{1}{q}}} + c^{-\frac{1}{q}} \|f\|_{\dot{H}^{\frac{2}{q}}} \tag{45}$$

holds.

The choice of norm on the right-hand side is motivated by dimensional analysis. Thus, the first term $\|f\|_{\dot{H}^{\frac{1}{q}}}$, which dominates at low frequency, is what one would get by scaling if $U_c^\pm$ were replaced by the Schrödinger propagator $V^\pm$. If instead we consider high frequencies and replace $U_c^\pm$ by the wave propagator $e^{itc\sqrt{-\Delta}}$, we get the second term $c^{-\frac{1}{q}} \|f\|_{\dot{H}^{\frac{2}{q}}}$, again by scaling.

Then using Duhamel's principle to write the solution of

$$L_\pm(c) u = F, \quad u|_{t=0} = f \tag{46}$$

as

$$u(t) = U_c^\pm(t) f + \int_0^t U_c^\pm(t - s) F(s)\, ds, \tag{47}$$

and noting that the norm on r.h.s.(45) is dominated by $\|f\|_{H^{\frac{2}{q}}}$ as $c \to \infty$, we immediately obtain the following:

COROLLARY. *For every sharp wave admissible pair (q, r), the estimate*

$$\|u\|_{L_t^q L_x^r(S_T)} + \|u\|_{L_t^\infty H^{\frac{2}{q}}(S_T)} \lesssim \|f\|_{H^{\frac{2}{q}}} + \int_0^T \|F(t)\|_{H^{\frac{2}{q}}}\, dt \tag{48}$$

holds for solutions of (46).

Next we consider estimates for Schrödinger admissible exponents.

PROPOSITION 2. *Let (q, r) and $(\widetilde{q}, \widetilde{r})$ be any two Schrödinger admissible pairs. Then for the low frequency part u_l (see (23) for definition) of the solution of (46) we have the estimate*

$$\|u_l\|_{L_t^q L_x^r(S_T)} + \|u_l\|_{L_t^\infty L_x^2(S_T)} \lesssim \|f_l\|_{L^2} + \|F_l\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(S_T)},$$

where $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1$ and $\frac{1}{\widetilde{r}} + \frac{1}{\widetilde{r}'} = 1$.

Let us turn to the proofs.

Proof of Proposition 1. Proceeding as in the standard proof of the Strichartz estimates for the homogeneous wave equation (see, e.g., [6] or [20, Section III.5]) we reduce to proving the decay estimate

$$|K_{\mu,c}(t,x)| \lesssim \begin{cases} \frac{\mu}{|t|} & \text{for } \mu \lesssim c, \\ \frac{\mu^2}{c|t|} & \text{for } \mu \gg c, \end{cases} \tag{49}$$

for the convolution kernel

$$K_{\mu,c}(t,x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{ith_c(\xi)} \beta\big(\tfrac{\xi}{\mu}\big)\, d\xi,$$

where h_c is given by (42), β is a Littlewood-Paley cut-off function supported in the annulus $|\xi| \sim 1$ and μ is a dyadic number of the form 2^j, $j \in \mathbb{Z}$. But in view of the scaling identity

$$K_{\mu,c}(t,x) = c^3 K_{\mu/c,1}(c^2 t, cx),$$

it suffices to prove (49) for $c = 1$. To simplify the notation we write $K_\mu = K_{\mu,1}$ and $h = h_1$. We shall need the following fact, whose elementary proof we omit:

LEMMA 6. *Define $\alpha(r) = \frac{r^2}{1+\sqrt{1+r^2}}$ for $r > 0$. Then $\alpha'(r) = \frac{r}{\sqrt{1+r^2}}$ and $\alpha''(r) = \frac{1}{(1+r^2)^{3/2}}$.*

To prove (49) for $c = 1$, we split into four cases:

(i) $\mu \lesssim 1$ and $|x| \gtrsim \mu\,|t|$,

(ii) $\mu \lesssim 1$ and $|x| \ll \mu\,|t|$,

(iii) $\mu \gg 1$ and $|x| \gtrsim |t|$,

(iv) $\mu \gg 1$ and $|x| \ll |t|$.

Introducing polar coordinates $\xi = r\omega$, $r > 0$, $\omega \in S^2$, we have

$$K_\mu(t,x) = \int_0^\infty \int_{S^2} e^{irx\cdot\omega} e^{it\alpha(r)} \beta\big(\tfrac{r}{\mu}\big) r^2 \, d\sigma(\omega)\, dr \tag{50}$$

$$= \int_0^\infty \widehat{\sigma}(rx) e^{it\alpha(r)} \beta\big(\tfrac{r}{\mu}\big) r^2 \, dr, \tag{51}$$

where σ is surface measure on S^2. Since $|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-1}$ (see, e.g., [20, Eq. (5.13)]) we get from (51)

$$|K_\mu(t,x)| \lesssim |x|^{-1} \int_0^\infty \beta\big(\tfrac{r}{\mu}\big) r \, dr \sim \frac{\mu^2}{|x|},$$

which proves (49) ($c = 1$) for the cases (i) and (iii). Next, rewrite (50) as $K_\mu(t,x) = \int_{S^2} I(\omega)\, d\sigma(\omega)$, where

$$I(\omega) = \int_0^\infty \frac{d}{dr}\left[e^{i(t\alpha(r)+rx\cdot\omega)}\right] \frac{\beta\big(\tfrac{r}{\mu}\big) r^2}{i\big(t\alpha'(r) + x\cdot\omega\big)}\, dr.$$

Integrating by parts and writing

$$-\frac{d}{dr}\left[\frac{\beta\big(\tfrac{r}{\mu}\big) r^2}{i\big(t\alpha'(r)+x\cdot\omega\big)}\right] = \frac{\beta\big(\tfrac{r}{\mu}\big) r^2 t\alpha''(r)}{i\big(t\alpha'(r)+x\cdot\omega\big)^2} - \frac{\frac{d}{dr}\big[\beta\big(\tfrac{r}{\mu}\big) r^2\big]}{i\big(t\alpha'(r)+x\cdot\omega\big)}$$

gives $I = I_1 + I_2$.

Consider case (ii). Then $r \sim \mu \lesssim 1$, so $|\alpha'(r)| \sim \mu$ and $|\alpha''(r)| \sim 1$ by Lemma 6. Then since $|x| \ll \mu\,|t|$, we get $|t\alpha'(r) + x\cdot\omega| \gtrsim \mu\,|t|$, and this gives $|I_j(\omega)| \lesssim \mu/\,|t|$ for $j = 1, 2$, proving (49) for this case.

Finally, consider case (iv). Then Lemma 6 gives $|\alpha'(r)| \sim 1$ and $|\alpha''(r)| \sim \mu^{-3}$, since $r \sim \mu \gg 1$. In view of the assumption $|x| \ll |t|$, we then get $|t\alpha'(r) + x\cdot\omega| \gtrsim |t|$, whence $|I_j(\omega)| \lesssim \mu^2/\,|t|$ for $j = 1, 2$. This proves (49) for case (iv), and concludes the proof of Proposition 1. □

Proof of Proposition 2. Take the convolution with θ_c in (46) and use the identity $\theta_c = \theta_c * \theta_{2c}$ to see that $L_\pm(c)_{\mathrm{low}} u_l = F_l$ with data $u_l|_{t=0} = f_l$, where $L_\pm(c)_{\mathrm{low}}$ is the operator with propagator $U^\pm_{c,\mathrm{low}}(t) = \theta_{2c} * e^{\mp it(E(c)-c^2)}$. It therefore suffices to prove

$$\|u\|_{L_t^q L_x^r(S_T)} + \|u\|_{L_t^\infty L_x^2(S_T)} \lesssim \|f\|_{L^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(S_T)}$$

for solutions of $L_\pm(c)_{\text{low}} u = F$ with data $u|_{t=0} = f$. But by [6, Theorem 1.2] (see also the proof of Corollary 1.4 there) it suffices to prove the decay estimate

$$|K_c(t,x)| \lesssim |t|^{-3/2} \tag{52}$$

for the convolution kernel $K_c(t,x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{ith_c(\xi)} \chi(\xi/c)\, d\xi$, where h_c is given by (42). In view of the scaling identity $K_c(t,x) = c^3 K_1(c^2 t, cx)$, it is enough to prove (52) for $c = 1$, in which case it follows from a standard result about decay of the Fourier transform of surface carried measures; see [19, Theorem 1.2.1]. Indeed, $K_1(t,x)$ is the (inverse) spacetime Fourier transform of the measure (recall that $h = h_1$)

$$\delta(\tau - h(\xi))\chi(\xi),$$

which is compactly supported on the hypersurface $\{(\tau,\xi) \in \mathbb{R}^{1+3} \colon \tau = h(\xi)\}$, whose curvature is nonvanishing. □

4.2. Bilinear null form estimates. In [7], Klainerman and Machedon proved that the estimate

$$\|u\nabla v\|_{L^2(\mathbb{R}^{1+3})} \lesssim \|f\|_{H^1} \|g\|_{H^1} \tag{53}$$

fails for solutions of $\Box_c u = \Box_c v = 0$ on $\mathbb{R}^{1+3}$ with initial data $(f,0)$ and $(g,0)$. In particular, this shows that the endpoint $(q,r) = (2,\infty)$ for the linear Strichartz estimates is forbidden, for if the estimate $\|u\|_{L_t^2 L_x^\infty} \lesssim \|f\|_{H^1}$ were true, it would clearly imply (53). If the bilinear form $u\nabla v$ in (53) is replaced by one of the null forms $Q_{ij}(|\nabla|^{-1} u, v)$ or $|\nabla|^{-1} Q_{ij}(u,v)$, the estimate is true, however, as proved in [7]. Here $|\nabla|^\alpha = (-\Delta)^{\alpha/2}$ and

$$Q_{ij}(u,v) = \partial_i u\, \partial_j v - \partial_j u\, \partial_i v \quad (1 \le i,j \le 3).$$

This fact was used in [8] to control the bilinear terms with derivatives in the KGM system, which turn out to have this structure when Coulomb gauge is used.

In fact (see the proof of the corollary to Proposition 2.1 in [8])

$$\|\mathcal{P}(u\nabla v)\|_{L_x^2} \lesssim \sum_{1\le i,j\le 3} \left\| |\nabla|^{-1} Q_{ij}(u,v) \right\|_{L_x^2}, \tag{54}$$

where $\mathcal{P}$ as before is the projection onto the divergence free vector fields in $\mathbb{R}_x^3$. Moreover, if $u = (u^1, u^2, u^3)$ is vector valued and divergence free, so that $\mathcal{P}u = u$, then (see the proof of Proposition 2.2 in [8], or [18, Section 1.5])

$$u \cdot \nabla v = \frac{1}{2} \sum_{1 \leq i,j \leq 3} Q_{ij}\big(|\nabla|^{-1}\, [R_j u^i - R_i u^j], v\big), \tag{55}$$

where $R_i = |\nabla|^{-1}\, \partial_i$ are the Riesz operators.

Here we prove versions of the Klainerman-Machedon null form estimates where $\square_c$ may be replaced by $L_\pm(c)$.

PROPOSITION 3. *Suppose $L_\pm(c)u = F$ and $L_\pm(c)v = G$ (independent signs) with initial data $u|_{t=0} = f$ and $v|_{t=0} = g$. Then*

$$\left\| |\nabla|^{-1}\, Q(u, v) \right\|_{L^2(S_T)} \lesssim \left(\|f\|_{H^1} + \|F\|_{L^1_t H^1(S_T)} \right) \left(\|g\|_{H^1} + \|G\|_{L^1_t H^1(S_T)} \right)$$

for $Q = Q_{ij}$.

In view of (54), this implies the following:

COROLLARY. *Under the hypotheses of Proposition 3, we have*

$$\|\mathcal{P}(u \nabla v)\|_{L^2(S_T)} \lesssim \left(\|f\|_{H^1} + \|F\|_{L^1_t H^1(S_T)} \right) \left(\|g\|_{H^1} + \|G\|_{L^1_t H^1(S_T)} \right).$$

Next, we consider the null form $Q_{ij}\big(|\nabla|^{-1}\, u, v\big)$.

PROPOSITION 4. *Suppose $\square_c u = F$ and $L_\pm(c)v = G$ with initial data $u|_{t=0} = f_0$, $\partial_t u|_{t=0} = f_1$ and $v|_{t=0} = g$. Then*

$$\begin{aligned} &\left\| Q\big(|\nabla|^{-1}\, u, v\big) \right\|_{L^2(S_T)} \\ &\quad \lesssim c^{-1/2} \left(\|f_0\|_{\dot H^1} + \tfrac{1}{c} \|f_1\|_{L^2} + c \|F\|_{L^1_t L^2_x(S_T)} \right) \left(\|g\|_{\dot H^1} + \|G\|_{L^1_t \dot H^1(S_T)} \right) \end{aligned}$$

for $Q = Q_{ij}$.

Then using (55) and noting that the Riesz operators commute with $\square_c$ and are bounded on every H^s space, we obtain:

COROLLARY. *Assume the hypotheses of Proposition 4 are satisfied. If in addition we assume that $u(t)$ is vector valued and divergence free, then*

$$\begin{aligned} &\|u \cdot \nabla v\|_{L^2(S_T)} \\ &\quad \lesssim c^{-1/2} \left(\|f_0\|_{\dot H^1} + \tfrac{1}{c} \|f_1\|_{L^2} + c \|F\|_{L^1_t L^2_x(S_T)} \right) \left(\|g\|_{\dot H^1} + \|G\|_{L^1_t \dot H^1(S_T)} \right). \end{aligned}$$

In the rest of this section, the Fourier transform of a function $u(t,x)$ (resp. $f(x)$) is denoted $\mathcal{F}u(\tau,\xi)$ (resp. $\widehat{f}(\xi)$). Then

$$\mathcal{F}[Q_{ij}(u,v)](\tau,\xi) = \int q_{ij}(\eta,\xi-\eta)\mathcal{F}u(\lambda,\eta)\mathcal{F}v(\tau-\lambda,\xi-\eta)\,d\lambda\,d\eta, \tag{56}$$

where $q_{ij}(\xi,\eta) = \xi_i\eta_j - \xi_j\eta_i$ for $\xi,\eta \in \mathbb{R}^3$. We will need the two inequalities

$$|q_{ij}(\xi,\eta)| \le |\xi \times \eta| \le |\xi+\eta|\,|\xi|^{1/2}\,|\eta|^{1/2}. \tag{57}$$

The first inequality is obvious, and to prove the second, observe that $\xi \times \eta = (\xi+\eta)\times\eta = \xi\times(\xi+\eta)$, whence $|\xi\times\eta| \le |\xi+\eta|\min(|\xi|,|\eta|)$. From (56), (57) and Plancherel's theorem, we then get

$$\left\| |\nabla|^{-1} Q_{ij}(u,v)\right\|_{L^2(\mathbb{R}^{1+3})} \le \left\| |\nabla|^{1/2} u \cdot |\nabla|^{1/2} v\right\|_{L^2(\mathbb{R}^{1+3})} \tag{58}$$

provided $\mathcal{F}u, \mathcal{F}v \ge 0$.

Proof of Proposition 3. In view of the formula (47) for the solution of (46), it suffices to prove this for $F = G = 0$. Then $\mathcal{F}u(\tau,\xi) = \delta(\tau \pm h_c(\xi))\widehat{f}(\xi)$ and $\mathcal{F}v(\tau,\xi) = \delta(\tau \pm h_c(\xi))\widehat{g}(\xi)$, where h_c is given by (42). Withough loss of generality, we assume $\widehat{f},\widehat{g} \ge 0$. Thus, (58) applies, and

$$\left\| |\nabla|^{-1} Q_{ij}(u,v)\right\|_{L^2(\mathbb{R}^{1+3})} \le \left\| |\nabla|^{1/2} u\right\|_{L^4(\mathbb{R}^{1+3})} \left\| |\nabla|^{1/2} v\right\|_{L^4(\mathbb{R}^{1+3})}.$$

Now apply Proposition 1, or its corollary, with $q = r = 4$, to conclude the proof. □

Proof of Proposition 4. Reasoning as above, we may assume $G = 0$, so that $v(t) = U_c^{\pm}(t)g$. Similarly, since the solution of

$$\square_c u = F, \quad u|_{t=0} = f_0, \quad \partial_t u|_{t=0} = f_1 \tag{59}$$

is given by (recall that $\square_c = -\frac{1}{c^2}\partial_t^2 + \Delta$)

$$\begin{aligned} u(t) &= \cos(ct\,|\nabla|)f_0 + (c\,|\nabla|)^{-1}\sin(ct\,|\nabla|)f_1 \\ &\quad - c\int_0^t |\nabla|^{-1}\sin\big(c(t-s)\,|\nabla|\big)F(s)\,ds, \end{aligned} \tag{60}$$

we reduce to the case where $f_1 = 0$, $F = 0$ and $u(t) = e^{\pm ict|\nabla|}f_0$. Without loss of generality, we choose the plus sign in the exponential. Thus, writing $f = f_0$, we only have to prove

$$\left\| Q_{ij}\big(|\nabla|^{-1} u, v\big) \right\|_{L^2(\mathbb{R}^{1+3})} \lesssim c^{-1/2} \|f\|_{\dot H^1} \|g\|_{\dot H^1},$$

where $u(t) = e^{ict|\nabla|}f$ and $v(t) = e^{\pm it(E(c)-c^2)}g$. Changing variables $t \to ct$, this becomes

$$\left\| Q_{ij}\big(|\nabla|^{-1} u', v'\big) \right\|_{L^2(\mathbb{R}^{1+3})} \lesssim \|f\|_{\dot H^1} \|g\|_{\dot H^1}, \tag{61}$$

where $u'(t) = e^{it|\nabla|}f$ and $v'(t) = e^{\pm it(E(c)-c^2)/c}g$. Thus,

$$\mathcal{F}u'(\tau,\xi) = \delta(\tau - |\xi|)\widehat{f}(\xi) \quad \text{and} \quad \mathcal{F}v'(\tau,\xi) = \delta\left(\tau \pm c\, h\big(\tfrac{\xi}{c}\big)\right)\widehat{g}(\xi),$$

where $h(\xi) = \frac{|\xi|^2}{1+\sqrt{1+|\xi|^2}}$. We may assume $\widehat{f}, \widehat{g} \geq 0$. Then by (56),

$$\begin{aligned}&\left|\mathcal{F}[Q_{ij}(\,|\nabla|^{-1} u', v')](\tau,\xi)\right| \\ &\quad \leq \int \frac{|q_{ij}(\eta,\xi-\eta)|}{|\eta|^2\,|\xi-\eta|} (\,|\nabla| f)\widetilde{\ }(\eta)(\,|\nabla|\, g)\widetilde{\ }(\xi-\eta)\delta\left(\tau - |\eta| \pm c\, h\big(\tfrac{\xi-\eta}{c}\big)\right)\, d\eta.\end{aligned}$$

Now apply the Cauchy-Schwarz inequality with respect to the measure $\delta(\ldots)\, d\eta$, square both sides and integrate in $d\tau\, d\xi$ to obtain

$$\left\| Q_{ij}\big(|\nabla|^{-1} u', v'\big) \right\|^2_{L^2(\mathbb{R}^{1+3})} \leq \left\|I^{\pm}\right\|_{L^\infty} \|f\|^2_{\dot H^1} \|g\|^2_{\dot H^1},$$

where

$$I^{\pm}(\tau,\xi) = \int \frac{|q_{ij}(\eta,\xi-\eta)|^2}{|\eta|^4\,|\xi-\eta|^2}\delta\left(\tau - |\eta| \pm c\, h\big(\tfrac{\xi-\eta}{c}\big)\right)\, d\eta.$$

This reduces (61) to proving that $I^{\pm}$ is bounded, independently of c. But by (57),

$$I^{\pm} \leq \int \frac{\sin^2\theta}{|\eta|^2}\delta\left(\tau - |\eta| \pm c\, h\big(\tfrac{\xi-\eta}{c}\big)\right)\, d\eta,$$

where θ denotes the angle between η and $\xi - \eta$. Now apply the following general result, with $k(r) = c\alpha(r/c)$ and α as in Lemma 6.

LEMMA 7. *Suppose $k(r)$ is positive and differentiable for $r > 0$, and that $|k'(r)| \leq 1$. Define*

$$I^{\pm}(\tau,\xi) = \int \frac{\sin^2\theta}{|\eta|^2}\delta\big(\tau - |\eta| \pm k(\,|\xi-\eta|)\big)\,d\eta,$$

where θ is the angle between η and $\xi - \eta$. Then $\sup_{\tau,\xi} I^{\pm}(\tau,\xi) \leq 8\pi$.

To see that this applies with $k(r) = c\alpha(r/c)$, we need only observe that $k'(r) = \alpha'(r/c)$, and $0 < \alpha' < 1$ by Lemma 6. We remark that this lemma also applies with $k(r) = r$, which corresponds to the Klainerman-Machedon estimates (then u and v both solve the homogeneous wave equation). This concludes the proof of Proposition 4. □

Proof of Lemma 7. In polar coordinates, $I^{\pm}(\tau,\xi) = \int_{S^2}\rho(\tau,\xi,\omega)\,d\omega$, where

$$\rho(\tau,\xi,\omega) = \int_0^\infty (\,\sin\theta)^2\delta\big(\tau - r \pm k(\,|\xi - r\omega|)\big)\,dr,$$

so it suffices to show that $\rho \leq 2$ for all τ,ξ, and for almost every $\omega \in S^2$.

We shall use the following fact: Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(r) < 0$, and has a zero at r_0. Then (see [5, Theorem 6.1.5])

$$\delta\big(f(r)\big)\,dr = \frac{\delta(r - r_0)\,dr}{|f'(r_0)|}. \tag{62}$$

Take $f(r) = \tau - r \pm k(\,|\xi - r\omega|)$, for fixed τ,ξ and ω. Then, since $|k'| \leq 1$,

$$f'(r) = -1 \pm k'(\,|\xi - r\omega|)\frac{(\xi - r\omega)\cdot\omega}{|\xi - r\omega|} \leq -1 + |\cos\theta|\,, \tag{63}$$

so $f'(r) < 0$ if we exclude the two points on S^2 where ω is collinear with ξ. Since (63) shows that $|f'| \geq 1 - |\cos\theta| \geq \frac{1}{2}\sin^2\theta$, we conclude from (62) that $\rho(\tau,\xi,\omega) \leq 2$. □

5. Local-in-time a priori bounds. Here we prove parts (i)–(vii) of Theorem 2. Throughout this section we assume that the hypotheses of the theorem are satisfied.

Definition 1. For $0 \leq T < \infty$ we define spacetime norms

$$X_T \;=\; \|\mathbf{A}\|_{L_t^\infty \dot{H}^1(S_T)} + \tfrac{1}{c}\,\|\partial_t\mathbf{A}\|_{L_t^\infty L_x^2(S_T)} + c\,\|\Box_c\mathbf{A}\|_{L_t^1 L_x^2(S_T)}\,,$$

$$
\begin{aligned}
Y_T^\pm &= \|\psi_\pm\|_{L_t^\infty H^1(S_T)} + \|L_\pm(c)\psi_\pm\|_{L_t^1 H^1(S_T)}\,,\\
Z_T^\pm &= \|\psi_{\pm,l}\|_{L_t^\infty L_x^2(S_T)} + \|\psi_{\pm,l}\|_{L_t^2 L_x^6(S_T)}\,,
\end{aligned}
$$

where $\psi_{\pm,l}$ is defined by (23), and we set $Y_T = Y_T^+ + Y_T^-$ and $Z_T = Z_T^+ + Z_T^-$.

The global solutions of (5), (7) obtained in [8] have the regularity

$$\partial_\mu A_\nu \in C(\mathbb{R}; L^2), \quad \phi \in C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2). \tag{64}$$

Moreover, for every $0 < T < \infty$ (see the Main Theorem and Propositions 3.2 and 2.3 in [8])

$$\int_0^T \left(\|\Box \mathbf{A}(t)\|_{L^2} + \|\Box\phi(t)\|_{L^2} \right) dt < \infty, \tag{65}$$

$$\int_0^T \left(\|\nabla A_0(t)\|_{L^3} + \|A_0(t)\|_{L^\infty} \right) dt < \infty. \tag{66}$$

This implies

$$X_T, Y_T, Z_T < \infty, \tag{67}$$

as we prove below. Thus, X_T, Y_T and Z_T depend continuously on T. They also depend on c, not only through the explicit appearance of c in the definitions, but also through the implicit dependence of $\mathbf{A}$ and $\psi_\pm$ on c.

We claim that the assumptions on the data imply

$$X_0, Y_0, Z_0 = O(1) \tag{68}$$

as $c \to \infty$. Obviously, (20) implies $X_0 = O(1)$, and to bound Y_0 and Z_0 it suffices to check that $\|\psi_\pm\|_{H^1} = O(1)$ at $t = 0$. But using (14) and Lemma 3(3),

$$\|\psi_\pm\|_{H^1} = \|\phi_\pm\|_{H^1} \le \|\phi\|_{H^1} + \left\| E(c)^{-1}\partial_t\phi \right\|_{H^1} + \tfrac{1}{c}\|A_0\phi\|_{L^2}\,,$$

and by (21) and (37), the right hand side is $O(1)$ at $t = 0$.

Our main task will be to show that (68) persists, i.e., for every $T < \infty$,

$$X_T, Y_T, Z_T = O(1) \tag{69}$$

as $c \to \infty$. In fact, we will prove (69) for a time $T = T_0 > 0$ which only depends on the size of the global-in-time bound (22), and by iterating this argument we get (69) for every finite time T. Once (69) has been proved, the local-in-time bounds in Theorem 2 follow easily, as we demonstrate in section 5.2.

5.1. Main estimates and bootstrap argument. Here we prove (69) for a time $T = T_0 > 0$ which only depends on the size of (22). Using a bootstrap argument, we reduce this to proving (recall the notational conventions set out in the Introduction)

$$X_T \lesssim X_0 + \sigma(T) Y_T^2 + \tfrac{\sigma(T)}{c} P(X_T + Y_T), \tag{70}$$

$$Y_T \lesssim Y_0 + \sigma(T) Z_T^2 Y_T + \tfrac{\sigma(T)}{c^{1/2}} P(X_T + Y_T), \tag{71}$$

$$Z_T \lesssim Z_0 + \sigma(T) \Big(\sum\nolimits_{\pm} \|\psi_\pm\|_{L_t^\infty L_x^2}^2 \Big) Z_T + \tfrac{\sigma(T)}{c^{1/2}} P(X_T + Y_T), \tag{72}$$

for, say, $0 \le T \le 1$ and $c \ge 1$.

Indeed, assuming these inequalities hold, first observe that (72) implies

$$Z_T \lesssim Z_0 + \tfrac{\sigma(T)}{c^{1/2}} P(X_T + Y_T) \quad \text{for} \quad 0 \le T \le T_0, \tag{73}$$

for some $T_0 > 0$ which only depends on (22). Plugging this into (71) gives

$$Y_T \lesssim Y_0 + \sigma(T) Z_0^2 Y_T + \tfrac{\sigma(T)}{c^{1/2}} P(X_T + Y_T) \quad \text{for} \quad 0 \le T \le T_0.$$

Thus, making T_0 smaller if necessary, but still depending only on (22), we get

$$Y_T \lesssim Y_0 + \tfrac{\sigma(T)}{c^{1/2}} P(X_T + Y_T) \quad \text{for} \quad 0 \le T \le T_0. \tag{74}$$

Inserting this into the second term on the right hand side of (70) gives

$$X_T \lesssim X_0 + Y_0^2 + \tfrac{\sigma(T)}{c} P(X_T + Y_T) \quad \text{for} \quad 0 \le T \le T_0. \tag{75}$$

Adding up (75) and (74) gives

$$f(T) \le P(f(0)) + \tfrac{\sigma(T)}{c^{1/2}} Q(f(T)) f(T) \quad \text{for} \quad 0 \le T \le T_0, \tag{76}$$

where $f(T) = X_T + Y_T$ depends continuously on T and Q is a polynomial. We claim that (76) implies

$$f(T) < 2P(f(0)) \quad \text{for} \quad 0 \le T \le T_0 \tag{77}$$

if c is sufficiently large [depending on $f(0)$]. In view of (73) and (68), this implies (69) for $T \le T_0$.

Let us prove the claim. If it fails, then by continuity we can find c arbitrarily large and $0 \le T \le T_0$ such that $f(T) = 2P(f(0))$. But by (76) this implies

$$1 \le \tfrac{\sigma(T)}{c^{1/2}} Q[2P(f(0))],$$

which fails for sufficiently large c.

Thus, we have reduced (69) to proving (70)–(72). To this end, we will use energy estimates and the spacetime estimates proved in Section 4. Let us turn to the details. We start by proving some estimates for the elliptic variable A_0.

LEMMA 8. *Let* $1 \le q < 2$. *Then*

(i) $\|\Delta A_0\|_{L_t^q L_x^r(S_T)} \lesssim \sigma(T)\, Z_T^2 + \frac{\sigma(T)}{c^{1/2}} Y_T^2$ *if* $1 \le r \le \frac{3}{2}^+$.

(ii) $\|\Delta A_0\|_{L_t^q L_x^r(S_T)} \lesssim \sigma(T) \left(\sum_\pm \|\psi_\pm\|_{L_t^\infty L_x^2(S_T)}\right) Z_T + \frac{\sigma(T)}{c^{1/2}} Y_T^2$ *if* $1 \le r \le \frac{3}{2}$.

(iii) $\|\Delta A_0\|_{L_t^\infty L_x^r(S_T)} \lesssim Y_T^2$ *if* $1 \le r \le \frac{3}{2}$.

Proof. In view of (5) this reduces to proving the same estimates for $\|I\|_{L_t^q([0,T])}$ where $I(t) := \|\overline{\psi_\pm} \frac{E(c)}{c^2} \psi_\pm\|_{L_x^r}$ and the $\pm$ signs are independent. Expanding $\psi_\pm = \psi_{\pm,l} + \psi_{\pm,h}$ as in (23) gives

$$I \le I_{l,l} + I_{l,h} + I_{h,l} + I_{h,h} \quad \text{where} \quad I_{\cdot,\cdot} := \left\| \overline{\psi_{\pm,(\cdot)}} \frac{E(c)}{c^2} \psi_{\pm,(\cdot)} \right\|_{L_x^r}.$$

Case 1. $r = \frac{3}{2}^+$. By Hölder's inequality, Lemma 4(i) and L^p interpolation,

$$I_{l,l} \le \|\psi_{\pm,l}\|_{L_x^{2+}} \|\psi_{\pm,l}\|_{L_x^6} \lesssim \|\psi_{\pm,l}\|_{L_x^2}^{1-\delta} \|\psi_{\pm,l}\|_{L_x^6}^{1+\delta}$$

for some $\delta > 0$. Since $q < 2$, and $\delta \to 0$ as $r \to 3/2$, we will have $\frac{1}{q} - \frac{1+\delta}{2} > 0$ if r is close enough to $3/2$. Applying Hölder's inequality in t then yields

$$\|I_{l,l}\|_{L_t^q([0,T])} \lesssim \|\psi_{\pm,l}\|_{L_t^\infty L_x^2(S_T)}^{1-\delta} T^{\frac{1}{q} - \frac{1+\delta}{2}} \|\psi_{\pm,l}\|_{L_t^2 L_x^6(S_T)}^{1+\delta} \lesssim \sigma(T) Z_T^2 \tag{78}$$

as desired. Next, by Hölder's inequality and Lemma 4,

$$I_{l,h} \lesssim \|\psi_{\pm,l}\|_{L_x^{6+}} \left\| \frac{E(c)}{c^2} \psi_{\pm,h} \right\|_{L_x^2} \lesssim \|\psi_{\pm,l}\|_{L_x^{6+}} \frac{1}{c} \|\psi_\pm\|_{H^1}, \tag{79}$$

$$I_{h,l} \lesssim \|\psi_{\pm,h}\|_{L_x^2} \left\| \frac{E(c)}{c^2} \psi_{\pm,l} \right\|_{L_x^{6+}} \lesssim \frac{1}{c} \|\psi_\pm\|_{H^1} \|\psi_{\pm,l}\|_{L_x^{6+}}, \tag{80}$$

$$I_{h,h} \lesssim \|\psi_{\pm,h}\|_{L_x^{6+}} \left\| \frac{E(c)}{c^2} \psi_{\pm,h} \right\|_{L_x^2} \lesssim \|\psi_{\pm,h}\|_{L_x^{6+}} \frac{1}{c} \|\psi_\pm\|_{H^1}. \tag{81}$$

Since $\|\psi_{\pm,l}\|_{L_x^{6+}} \lesssim c^\varepsilon \|\psi_\pm\|_{H^1}$ by Sobolev embedding and Lemma 4(ii), it follows that $I_{l,h}, I_{h,l} \lesssim c^{\varepsilon-1} Y_T^2$, whence

$$\|I_{l,h}\|_{L_t^q([0,T])}, \|I_{l,h}\|_{L_t^q([0,T])} \lesssim \frac{T^{1/q}}{c^{1/2}} Y_T^2$$

as desired. Finally, to control $I_{h,h}$, we have to use Strichartz estimates. Applying Hölder's inequality in t to (81) gives

$$\|I_{h,h}\|_{L_t^q([0,T])} \lesssim T^{\frac{1}{q} - \frac{1}{p}} \|\psi_\pm\|_{L_t^p L_x^{6+}(S_T)} \frac{1}{c} \|\psi_\pm\|_{L_t^\infty H^1(S_T)}.$$

Choosing $p = 3^-$ so that $(p, 6^+)$ is sharp wave admissible, we have

$$\|\psi_\pm\|_{L_t^p L_x^{6^+}(S_T)} \lesssim Y_T$$

by the corollary to Proposition 1. This proves part (i) of Lemma 8.

Case 2. $r \leq \frac{3}{2}$. This is similar, but simpler. Instead of (78), we have

$$I_{l,l} \leq \|\psi_{\pm,l}\|_{L_x^2} \|\psi_{\pm,l}\|_{L_x^p} \leq \|\psi_{\pm,l}\|_{L_x^2}^{1+\delta} \|\psi_{\pm,l}\|_{L_x^6}^{1-\delta}$$

for some $2 \leq p \leq 6$ and $0 \leq \delta \leq 1$. Thus,

$$\begin{aligned} \|I_{l,l}\|_{L_t^\infty([0,T])} &\lesssim \|\psi_\pm\|_{L_t^\infty H^1(S_T)}^2 \leq Y_T^2, \\ \|I_{l,l}\|_{L_t^q([0,T])} &\lesssim \|\psi_{\pm,l}\|_{L_t^\infty L_x^2(S_T)}^{1+\delta} T^{\frac{1}{q}-\frac{1-\delta}{2}} \|\psi_{\pm,l}\|_{L_t^2 L_x^6(S_T)}^{1-\delta}, \end{aligned}$$

as desired for parts (iii) and (ii), respectively, of Lemma 8. Next, since the estimates (79)–(81) now hold with 6^+ replaced by some $2 \leq p \leq 6$, we have

$$I_{l,h}, I_{h,l}, I_{h,h} \lesssim \frac{1}{c} \|\psi_\pm\|_{L_t^\infty H^1(S_T)}^2 \leq \frac{1}{c} Y_T^2, \tag{82}$$

by Sobolev embedding. This concludes the proof of Lemma 8. □

Next, we prove the estimates for the X, Y, Z norms.

Proof of (70). By the energy inequality for (59), which reads

$$\|u\|_{L_t^\infty \dot{H}^1(S_T)} + \frac{1}{c} \|\partial_t u\|_{L_t^\infty L_x^2(S_T)} \leq \|f_0\|_{\dot{H}^1} + \frac{1}{c} \|f_1\|_{L^2} + c \|F\|_{L_t^1 L_x^2(S_T)},$$

we have $X_T \leq X_0 + c \|\Box_c \mathbf{A}\|_{L_t^1 L_x^2(S_T)}$, so in view of (5) it suffices to prove

$$\|\mathcal{P}(\phi \overline{\nabla \phi})\|_{L_t^1 L_x^2(S_T)} \lesssim \sqrt{T} Y_T^2, \tag{83}$$

$$\||\phi|^2 \mathbf{A}\|_{L_t^1 L_x^2(S_T)} \lesssim T Y_T^2 X_T. \tag{84}$$

First note that l.h.s.(83) is bounded by a sum of terms $\sqrt{T}\|\mathcal{P}(\psi_\pm \overline{\nabla \psi_\pm})\|_{L^2(S_T)}$ with independent signs. Noting the identity $\overline{L_\pm(c)u} = -L_\mp(c)\overline{u}$, we apply the corollary to Proposition 3 to conclude that (83) holds. L.h.s.(84) is bounded by a sum of terms $T\|\psi_\pm \overline{\psi_\pm} \mathbf{A}\|_{L_t^\infty L_x^2(S_T)}$ with independent signs. Applying the inequality (30) gives (84). □

Proof of (71). For the solution of (46) we have, in view of the formula (47),

$$\|u\|_{L_t^\infty H^1(S_T)} \leq \|f\|_{H^1} + \|F\|_{L_t^1 H^1(S_T)}.$$

Thus, $Y_T^\pm \lesssim Y_0^\pm + \|L_\pm(c)\psi_\pm\|_{L_t^1 H^1(S_T)}$, so in view of Lemma 1 and Lemma 3(ii), it suffices to prove

$$\|A_0\psi_\pm\|_{L_t^1 H^1(S_T)} \lesssim \sigma(T) Z_T^2 Y_T + \tfrac{\sigma(T)}{c^{1/2}} Y_T^3, \tag{85}$$

$$\|\mathbf{A}\cdot\nabla\phi\|_{L_t^1 L_x^2(S_T)} \lesssim \sqrt{\tfrac{T}{c}} X_T Y_T, \tag{86}$$

$$\left\|\mathbf{A}^2\phi\right\|_{L_t^1 L_x^2(S_T)} \lesssim T X_T^2 Y_T, \tag{87}$$

$$\left\|\left[A_0, \tfrac{E(c)-c^2}{c}\right]\psi_\pm\right\|_{L_t^1 L_x^2(S_T)} \lesssim \sigma(T) Z_T^2 Y_T + \tfrac{\sigma(T)}{c^{1/2}} Y_T^3. \tag{88}$$

To prove (86) and (87), expand using (9), and apply, respectively, the corollary to Proposition 4 and inequality (30). Next, observe that by the product rule for derivatives, Hölder's inequality and the Sobolev embedding (29), l.h.s.(85) is dominated by

$$\int_0^T \left(\|\nabla A_0\|_{L^3} + \|A_0\|_{L^\infty}\right) dt \;\|\psi_\pm\|_{L_t^\infty H^1(S_T)}, \tag{89}$$

and in view of Lemma 3(iii), l.h.s.(88) is also $\lesssim$ (89). Thus, it is enough to show

$$\int_0^T \left(\|\nabla A_0\|_{L^3} + \|A_0\|_{L^\infty}\right) dt \lesssim \sigma(T) Z_T^2 + \tfrac{\sigma(T)}{c^{1/2}} Y_T^2, \tag{90}$$

but this follows from Lemmas 5 and 8. □

Proof of (72). Our argument here is reminiscent of that used in Section 2 to prove the well-posedness of the Schrödinger-Poisson system. Thus, we apply the Strichartz estimate in Proposition 2 to the equation (15) (Lemma 1) with $(q, r) = (2, 6)$ and $(\widetilde{q}', \widetilde{r}') = (1, 2)$ or $(1^+, 2^-)$. To be precise we have

$$Z_T^\pm \lesssim Z_0^\pm + \|A_0\psi_+\|_{L_t^a L_x^b(S_T)} + \|R\|_{L_t^1 L_x^2(S_T)},$$

where $\frac{1}{a} + \frac{\varepsilon}{2} = 1$ and $\frac{1}{b} + \frac{1}{2} - \frac{\varepsilon}{3} = 1$ for some sufficiently small $\varepsilon > 0$. Thus, it suffices to prove

$$\|A_0\psi_\pm\|_{L_t^a L_x^b(S_T)} \lesssim \sigma(T)\left(\textstyle\sum_\pm \|\psi_\pm\|_{L_t^\infty L_x^2(S_T)}^2\right) Z_T + \tfrac{\sigma(T)}{c^{1/2}} Y_T^3, \tag{91}$$

$$\|R\|_{L_t^1 L_x^2(S_T)} \lesssim \tfrac{\sigma(T)}{c^{3/2}} P(X_T + Y_T). \tag{92}$$

First, write

$$\|A_0\psi_\pm\|_{L_t^a L_x^b(S_T)} \le \|A_0\|_{L_t^a L_x^{3/\varepsilon}(S_T)} \textstyle\sum_\pm \|\psi_\pm\|_{L_t^\infty L_x^2(S_T)}.$$

Since $\|A_0\|_{L^a_t L^{3/\varepsilon}_x(S_T)} \lesssim \|\Delta A_0\|_{L^a_t L^{(3/2)-}_x(S_T)}$ by Sobolev embedding, (91) then follows from Lemma 8(ii). Next, observe that (92) follows from Lemma 3(i) and the estimates (86), (87) and

$$\left\| \left[A_0, \tfrac{E(c)-c^2}{c} \right] \psi_\pm \right\|_{L^1_t L^2_x(S_T)} \lesssim \sigma(T) Y_T^3.$$

This last inequality follows from (88) and the fact that

$$Z_T \lesssim (1 + T^{1/2}) \sum\nolimits_\pm \|\psi_\pm\|_{L^\infty_t H^1(S_T)} \leq (1 + T^{1/2}) Y_T, \tag{93}$$

where we used (29) to get the first inequality. □

Proof of (67). Here we prove our earlier claim that the regularity properties of $(A_0, \mathbf{A}, \phi)$ imply (67). First, $X_T < \infty$ follows directly from (64) and (65). Next, using the definition (14), Lemma 3(3) and (31), we conclude from (64) that

$$\phi_\pm \in C(\mathbb{R}; H^1). \tag{94}$$

In view of (93), this implies $Z_T < \infty$. Moreover, it reduces $Y_T < \infty$ to showing that

$$\|L_\pm(c)\psi_\pm\|_{L^1_t H^1(S_T)} < \infty.$$

But the latter reduces to proving that the left hand sides of (85)–(88) are finite. First recall that l.h.s.(85) and l.h.s.(88) are bounded by (89), which is finite by (66) and (94). Next, using [8, Proposition 2.2] instead of the corollary to Proposition 4, one finds that l.h.s.(86) is controlled by (65) and the norms of the initial data (7). Finally, l.h.s.(87) $< \infty$ by (64), if we use (30). □

5.2. Conclusion of the proof of Theorem 2. We conclude by showing that (69) implies the local-in-time bounds in Theorem 2. By the definitions of X_T, Y_T and Z_T, it is obvious that they control the norms in parts (i)–(iii) in Theorem 2. The bound (iv) reduces to Lemma 8 via Sobolev embedding. To prove part (vii), use Lemma 3(ii) and (31) to get

$$\left\| E(c)^{-1}(A_0\phi) \right\|_{H^1} \lesssim \tfrac{1}{c} \|\nabla A_0\|_{L^2} \|\phi\|_{H^1}^2$$

for each t. Then use the bounds in parts (i) and (iv). Next, we prove the bound for R in part (v) of Theorem 2. By Lemma 3(ii), this reduces to $\frac{1}{c} \|E(c)R\|_{L^1_t L^2_x(S_T)} = O(c^{-1/2})$. Recalling the definition (17) of R and the estimates (86) and (87), we see that it suffices to prove

$$\left\| [A_0, E(c) - c^2]\psi_\pm \right\|_{L_t^1 L_x^2(S_T)} = O(c^{1/2}).$$

To do this, expand the term inside the brackets in (18) using the frequency decomposition (23), as in the proof of Lemma 8, and write $A_0 = A_0' + A_0''$, where A_0' corresponds to terms of the type $\overline{\psi_{\pm,l}} \frac{E(c)}{c^2} \psi_{\pm,l}$, i.e., both factors are at low frequency, and A_0'' corresponds to terms where at least one factor has high frequency. Let us consider first

$$\left\| [A_0'', E(c) - c^2]\psi_\pm \right\|_{L_t^1 L_x^2(S_T)} .$$

Here we simply expand the commutator and use Lemma 3(iii) to dominate by

$$c \left(\|\nabla A_0''\|_{L_t^1 L_x^3(S_T)} + \|A_0''\|_{L_t^1 L_x^\infty(S_T)} \right) \|\psi_\pm\|_{L_t^\infty H^1(S_T)} .$$

In view of Lemma 5, it therefore suffices to check

$$\|\Delta A_0''\|_{L_t^1 L_x^{(3/2)\pm}(S_T)} = O(c^{-1/2}),$$

but this is clear from the proof of Lemma 8, since for A_0'' there is no term $I_{l,l}$. It remains to prove

$$\left\| [A_0', E(c) - c^2]\psi_\pm \right\|_{L_t^1 L_x^2(S_T)} = O(c^{1/2}).$$

In fact, applying the following lemma with $f = \overline{\psi_{\pm,l}}$, $g = \frac{E(c)}{c^2}\psi_{\pm,l}$ and $h = \psi_\pm$, and using Lemma 4, parts (i) and (ii), gives $\left\| [A_0', E(c) - c^2]\psi_\pm \right\|_{L_t^1 L_x^2(S_T)} \lesssim c^\varepsilon Y_T^3$ for $\varepsilon > 0$ arbitrarily small.

LEMMA 9. *Define*

$$T(f, g, h) = (E(c) - c^2)\Big[(-\Delta)^{-1}(fg) \cdot h\Big] - (-\Delta)^{-1}(fg) \cdot (E(c) - c^2)h.$$

Then the estimate $\|T(f, g, h)\|_{L^2} \lesssim \|f\|_{H^{1+}} \|g\|_{H^{1+}} \|h\|_{H^1}$ *holds on* $\mathbb{R}^3$.

Proof. The Fourier symbol of T is

$$\sigma(\xi, \eta, \zeta) = \frac{1}{|\xi + \eta|^2} \left[h_c(\xi + \eta + \zeta) - h_c(\zeta) \right],$$

where h_c is given by (42). We claim that

$$|\sigma(\xi,\eta,\zeta)| \lesssim 1 + \frac{|\zeta|}{|\xi+\eta|}. \tag{95}$$

Since we may assume that $\widehat{f}, \widehat{g}, \widehat{h} \geq 0$, this would imply

$$\|T(f,g,h)\|_{L^2} \lesssim \|fgh\|_{L^2} + \left\| |\nabla|^{-1}(fg) \cdot |\nabla| h \right\|_{L^2}.$$

The first term on the right-hand side is covered by (30), the second term is $\leq$

$$\left\| |\nabla|^{-1}(fg) \right\|_{L^\infty} \|h\|_{H^1} \lesssim \|fg\|_{H^{(1/2)+}} \|h\|_{H^1}.$$

Since $\|fg\|_{H^{(1/2)+}} \lesssim \|f\|_{H^{1+}} \|g\|_{H^{1+}}$, we get the desired estimate.

Thus, it only remains to prove (95), which clearly reduces to

$$|h_c(\xi+\eta) - h_c(\eta)| \lesssim |\xi| \left(|\xi| + |\eta| \right).$$

By the mean value theorem, this reduces to checking

$$|\nabla h_c(\xi)| \lesssim |\xi|. \tag{96}$$

But writing $h = h_1$, we have $h_c(\xi) = c^2 h(\xi/c)$, so

$$\nabla h_c(\xi) = c(\nabla h)(\xi/c),$$

and we know from Lemma 6 that $|\nabla h(\xi)| \lesssim |\xi|$ for all ξ. □

This concludes the proof of part (v) of Theorem 2, and it only remains to prove the estimate for R' in part (vi) of the theorem, where R' is given by (19). Note that

$$\left\| R'(t) \right\|_{L^r_x} \lesssim \sum \left\| \overline{\psi_\pm}(t) \tfrac{E(c)-c^2}{c^2} \psi_\pm(t) \right\|_{L^r_x},$$

and the sum is over all combinations of signs. Splitting the right hand side as in the proof of Lemma 8, and using the estimates obtained there, we get

$$\|R'\|_{L^1_t L^r_x(S_T)} \lesssim \sum \left\| \overline{\psi_{\pm,l}} \tfrac{E(c)-c^2}{c^2} \psi_{\pm,l} \right\|_{L^1_t L^r_x(S_T)} + \tfrac{\sigma(T)}{c^{1/2}} Y_T^2.$$

Applying Hölder's inequality, Sobolev embedding and Lemma 3(iii), we obtain

$$\left\| \overline{\psi_{\pm,l}}(t) \tfrac{E(c)-c^2}{c^2} \psi_{\pm,l}(t) \right\|_{L_x^r} \lesssim c^{\varepsilon-1} \left\| \psi_{\pm,l}(t) \right\|_{H^1}^2$$

for $\varepsilon > 0$, and this concludes the proof of Theorem 2.

6. Proof of H^1 convergence. Here we prove Theorem 1. Thus, we assume that the hypotheses of the theorem are satisfied, with one modification: As noted in the discussion following the statement of the theorem, we may use, in view of the bound in Theorem 2(vii), the definition (14) instead of (10).

We first prove the convergence $\psi_\pm \to v_\pm$. Clearly, it is enough to show that given $0 < T < \infty$, there exist constants $B, \varepsilon > 0$, independent of c, such that for every time interval $I = [t_0, t_1] \subset [0, T]$,

$$f(I) \lesssim \sum\nolimits_\pm \left\| \psi_\pm(t_0) - v_\pm(t_0) \right\|_{H^1} + B \left| I \right|^\varepsilon f(I) + o(1), \tag{97}$$

where $f(I) := \sum_\pm \| \psi_\pm - v^\pm \|_{L_t^\infty H^1(I \times \mathbb{R}^3)}$. In fact, B and ε depend only on the bounds in Theorem 2 and Lemma 2.

Without loss of generality, we assume $I = [0, T]$, and we choose the plus sign on the left hand side of (97). By the formula (47) for the initial value problem (46), and the corresponding formula for the Schrödinger equation,

$$\begin{aligned} \psi_+(t) &= U_c^+(t)\psi_+(0) + \int_0^t U_c^+(t-s)(L_+(c)\psi_+)(s)\, ds, \\ v_+(t) &= V^+(t)v_+(0) + \int_0^t V^+(t-s)(uv_+)(s)\, ds, \end{aligned}$$

where U_c^+ and V^+ are given by (40) and (41). Thus,

$$\begin{aligned} \psi_+(t) - v_+(t) &= U_c^+(t)[\psi_+(0) - v_+(0)] + [U_c^+(t) - V^+(t)]v_+(0) \\ &\quad + \int_0^t U_c^+(t-s)[L_+(c)\psi_+ - uv_+](s)\, ds \\ &\quad + \int_0^t [U_c^+(t-s) - V^+(t-s)](uv_+)(s)\, ds \\ &= I_1 + \cdots + I_4. \end{aligned}$$

Now, $(U_c^+(t)f)\widehat{\ }(\xi) = e^{-ith_c(\xi)}\widehat{f}(\xi)$, with h_c given by (42), and $(V^+(t)f)\widehat{\ }(\xi) = e^{it|\xi|^2/2}\widehat{f}(\xi)$. Using Plancherel's theorem, it is therefore clear that

$$\| I_1 \|_{L_t^\infty H^1(S_T)} \leq \| \psi_+(0) - v_+(0) \|_{H^1} .$$

Moreover,

$$\begin{aligned}\|I_2\|_{L_t^\infty H^1(S_T)} &\le \left\| \sup_{0\le t\le T} \left| e^{-ith_c(\xi)} - e^{it|\xi|^2/2} \right| (1+|\xi|^2)^{1/2} |\widehat{v_+}(0,\xi)| \right\|_{L^2_\xi} \\ &\lesssim \left\| \min\left\{1, T\left|h_c(\xi) - \tfrac{|\xi|^2}{2}\right|\right\} (1+|\xi|^2)^{1/2} |\widehat{v_+}(0,\xi)| \right\|_{L^2_\xi},\end{aligned}$$

and the latter $\to 0$ as $c\to\infty$, by the dominated convergence theorem. Similarly, with $F = uv_+$,

$$\|I_4\|_{L_t^\infty H^1(S_T)} \lesssim \int_0^T \left\| \min\left\{1, T\left|h_c(\xi) - \tfrac{|\xi|^2}{2}\right|\right\} (1+|\xi|^2)^{1/2} |\widehat{F}(s,\xi)| \right\|_{L^2_\xi} ds,$$

and this $\to 0$ as $c\to\infty$ by dominated convergence, because $\|F\|_{L_t^1 H^1(S_T)} < \infty$. To prove the latter, note that (cf. (89))

$$\|uv_+\|_{H^1} \lesssim \left(\|\nabla u\|_{L_x^3} + \|u\|_{L_x^\infty} \right) \|v_+\|_{H^1}$$

and

$$\|\nabla u\|_{L_x^3} + \|u\|_{L_x^\infty} \lesssim \|\Delta u\|_{L_x^{(3/2)-}} + \|\Delta u\|_{L_x^{(3/2)+}} \lesssim \sum\nolimits_\pm \|v^\pm\|_{H^1}^2, \tag{98}$$

where we used Lemma 5 to get the first inequality, then Hölder's inequality and Sobolev embedding to get the second one.

It only remains to estimate I_3. Write

$$L_+(c)\psi_+ - uv_+ = (A_0 - u)\psi_+ + u(\psi_+ - v_+) + e^{itc^2} R,$$

where R is given by (17). Correspondingly, we split $I_3 = I_3' + I_3'' + I_3'''$. First observe that

$$\|I_3'''\|_{L_t^\infty H^1(S_T)} \le \|R\|_{L_t^1 H^1(S_T)} = O(c^{-1/2})$$

by part (v) of Theorem 2. Next, write

$$\begin{aligned}\|I_3''\|_{L_t^\infty H^1(S_T)} &\le \int_0^T \|u(\psi_+ - v_+)\|_{H^1}\, dt \\ &\lesssim \left(\|\nabla u\|_{L_t^1 L_x^3(S_T)} + \|u\|_{L_t^1 L_x^\infty(S_T)} \right) \|\psi_+ - v_+\|_{L_t^\infty H^1(S_T)}\end{aligned}$$

and recall (98). Similarly,

$$\|I_3'\|_{L_t^\infty H^1(S_T)} \lesssim \left(\|\nabla(A_0 - u)\|_{L_t^1 L_x^3(S_T)} + \|A_0 - u\|_{L_t^1 L_x^\infty(S_T)} \right) \|\psi_+\|_{L_t^\infty H^1(S_T)},$$

so in view of Lemma 5, to finish the proof of (97), it suffices to show

$$\|\Delta(A_0 - u)\|_{L_t^1 L_x^r(S_T)} \leq B\sigma(T) \sum\nolimits_{\pm} \|\psi_{\pm} - v_{\pm}\|_{L_t^\infty H^1(S_T)} + o(1) \tag{99}$$

for $1 \leq r \leq \frac{3}{2}^+$. But

$$\Delta(A_0 - u) = -|\psi_+|^2 + |v_+|^2 + |\psi_-|^2 - |v_-|^2 + R',$$

where $\|R'\|_{L_t^1 L_x^r(S_T)} = O(c^{-1/2})$, by Theorem 2. Hölder's inequality and Sobolev embedding yield

$$\left\| |\psi_{\pm}|^2 - |v_{\pm}|^2 \right\|_{L_x^r} \lesssim \|\psi_{\pm} - v_{\pm}\|_{H^1} \left(\|\psi_{\pm}\|_{H^1} + \|v_{\pm}\|_{H^1} \right).$$

Integrating in time and using the bounds in Theorem 2 and Lemma 2 then gives the first term on the right-hand side of (99), and this concludes the proof of Theorem 1.

Appendix. As mentioned in the Introduction, the global existence result of Klainerman and Machedon [8] was for the massless KGM system. Here we indicate how their argument can be modified to handle the massive case.

First, the arguments relying on the conservation of energy require no change. Thus, [8, Proposition 1.1] holds as stated, and in fact the proof is easier in the massive case, since now the *KGM* energy includes the L^2 norm of ϕ.

The problem therefore reduces to proving local well-posedness for data with $\mathcal{I}_0 < \infty$, where

$$\mathcal{I}_0 = \|\mathbf{A}|_{t=0}\|_{\dot{H}^1} + \|\partial_t \mathbf{A}|_{t=0}\|_{L^2} + \|\phi|_{t=0}\|_{H^1} + \|\partial_t \phi|_{t=0}\|_{L^2}.$$

This is essentially what is proved in [8, section 4], and the argument there is easily modified to handle the massive case. Let us give the details. As in [8], we set $c = 1$. Let $m > 0$ be the rest mass. Then we have to add the linear term $m^2\phi$ to the right hand side of [8, Eq. (4.1b)], which then corresponds to our equation (5) (but with $c = 1$). Then Propositions 4.1–4.4 in [8] hold as stated. The proofs only require a few extra lines to treat the term $m^2\phi$.

Consider Proposition 4.1. It is reduced to an inequality (see [8, Eq. (4.3)]) which reads, in our notation,

$$f(T) \leq \sigma(T) P(\mathcal{I}_0 + f(T)),$$

where $f(T) = \|\Box_c \mathbf{A}\|_{L_t^1 L_x^2(S_T)} + \|\Box_c \phi\|_{L_t^1 L_x^2(S_T)}$. To extend this to the massive case, we only have to verify $\|m^2\phi\|_{L_t^1L_x^2(S_T)} \leq \sigma(T)P(\mathcal{I}_0 + f(T))$. To this end, write $\|m^2\phi\|_{L_t^1L_x^2(S_T)} \leq m^2T\,\|\phi\|_{L_t^\infty H^1(S_T)}$ and use the energy inequality

$$\|\phi\|_{L_t^\infty H^1(S_T)} \lesssim \|\phi|_{t=0}\|_{H^1} + (1+T)\,\|\partial_t\phi|_{t=0}\|_{L^2} + (1+T)\,\|\Box_c\phi\|_{L_t^1L_x^2(S_T)}.$$

Proposition 4.2 is a corresponding estimate for a difference of two solutions, and since we have only added a linear term, the same changes apply there. Finally, Propositions 4.3 and 4.4 are corollaries of Proposition 4.2.

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