REMARK ON THE OPTIMAL REGULARITY FOR EQUATIONS OF WAVE MAPS TYPE

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1 Introduction

The goal of this paper is to review the estimates proved in [3] and extend them to all dimensions, in particular to the harder case of space dimension 2. As in [3], the main application we have in view is to equations of Wave Maps type, namely systems of equations of the form

$$\Box\phi^I + \Gamma^I_{JK}(\phi) Q_0(\phi^J, \phi^K) = 0. \tag{1}$$

Here, $\Box = -\partial_t^2 + \Delta$ denotes the standard D'Alembertian in $\mathbb{R}^{n+1}$, and Q_0 is the null form $Q_0(\phi, \psi) = \partial_\alpha\phi \cdot \partial^\alpha\psi = -\partial_t\phi\partial_t\psi + \sum_{i=1}^{n} \partial_i\phi\partial_i\psi$.

The main estimates of the paper are described in Theorems 2 and 3. It is easy to see, using the arguments in [3] and [5], that these estimates imply the following result:

Theorem 1. *Assume that the functions $\Gamma(\phi)$ are real analytic. Then the initial value problem for the equations* (1)*, in $\mathbb{R}^{n+1}$, subject to the initial conditions*

$$\phi(0,x) = f_0(x), \quad \partial_t\phi(0,x) = g_0(x), \tag{2}$$

is well posed for $f_0 \in H^s(\mathbb{R}^n)$, $g_0 \in H^{s-1}(\mathbb{R}^n)$ and any $s > \frac{n}{2}$.

In particular, in two space dimensions, our result comes very close to proving "well posedness" in the "energy norm" H^1. In [1] the above Theorem was proved in the special case of dimension $n = 3$. The proof in higher dimensions does not require any new idea. The case of dimension $n = 2$ was studied by Y. Zhou [6], who showed well posedness for H^s, $s \geq 1 + \frac{1}{8}$. Here we rely on a slight modification of a new version of Strichartz type estimates proved in [4].

As in [3], the proof relies on estimates for the null quadratic form Q_0 in the spaces $H^{s,\delta}$. The estimates diverge logarithmically for the critical exponents

$s = \frac{n}{2}$, $\delta = \frac{1}{2}$. In this paper we keep track of the precise divergences by using the modified spaces $H^{[s,\delta]}$ which will be discussed below.

Estimates involving the homogeneous [1] version of these norms, for the optimal exponents $\delta = \pm\frac{1}{2}$ in $\mathbb{R}^{3+1}$, have first appeared in [2]. For example section 6 in [2] proves, essentially, the estimates included here in Theorem 4 for the particular case of space dimension 3, and applies them to derive the following sharper version of estimate (4) of Theorem 2 in the case when $\Box\phi = \Box\psi = 0$ with initial data bounded in $H^{\frac{3}{2}}(\mathbb{R}^3)$:

$$\left\| (|\tau| + |\xi|)^{\frac{1}{2}} \left||\tau| - |\xi|\right|^{-\frac{1}{2}} \widetilde{Q_0(\phi,\psi)} \right\|_{L^2(\mathbb{R}^{3+1})} < \infty.$$

The inhomogeneous version of the norms, which have appeared in [3] and [5], was inspired from Bourgain's work [1]. The inhomogeneous norms appear naturally in connection with Bourgain's time cut-off idea, which allows one, essentially, to replace the symbol $\tau^2 - |\xi|^2$ of $\Box$ by $w_+ \cdot w_-(\tau,\xi)$ and thus circumvent the difficulties which appear when trying to treat the sharp local well posedness in $H^{\frac{n}{2}}$. That problem remains in fact open and probably requires to go back, in some way, to homogeneous estimates.

2 Notation

We define the space $H^{[s,\delta]}(\mathbb{R}^{1+n})$ to be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{1+n})$ w.r.t. the norm

$$\|\phi\|_{H^{[s,\delta]}} = \left\| w_+^s L(w_+)(\tau,\xi) w_-^\delta L(w_-)(\tau,\xi) \widetilde{\phi}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},$$

where $\widetilde{\phi}$ is the space-time Fourier transform of ϕ, the weights $w_\pm$ are given by

$$w_\pm(\tau,\xi) = 2 + \left||\tau| \pm |\xi|\right|$$

and $L : (0,\infty) \to (0,\infty)$ is a function with the properties

- L is increasing,
- $L(2x) \leq CL(x)$ for all $x \geq 2$,
- $\int_2^\infty \frac{dx}{xL^2(x)} < \infty$.

Note that the first two properties imply that for any $a > 1$,

$$L(ax) \leq C_a L(x) \quad \text{for all } x \geq 2. \tag{3}$$

Examples of such a function L are the following:

- $L(x) = x^\varepsilon$, for any $\varepsilon > 0$,

[1] i.e., with $w_\pm(\tau,\xi)$ replaced by $\left||\tau| \pm |\xi|\right|$.

- $L(x) = \log x$,
- $L(x) = (\log x)^{\frac{1}{2}} \log\log x$.

Introducing the notation

$$w_+^{[s]} = w_+^s L(w_+)$$
$$w_-^{[\delta]} = w_-^\delta L(w_-),$$

we have

$$\|\phi\|_{H^{[s,\delta]}} = \left\| w_+^{[s]} w_-^{[\delta]} \widetilde{\phi} \right\|_{L^2}.$$

The symbol $\lesssim$ will mean $\leq$ up to a multiplicative constant, which may depend on the space dimension n and other fixed parameters, but not on any variable quantities. Similarly, $\simeq$ means $=$ up to a constant.

3 The Main Estimates

Theorem 2. *Assume that* $s \geq \frac{n}{2}$ *and* $n \geq 2$. *Then*

(4)
$$\|Q_0(\phi,\psi)\|_{H^{[s-1,-\frac{1}{2}]}} \lesssim \|\phi\|_{H^{[s,\frac{1}{2}]}} \|\psi\|_{H^{[s,\frac{1}{2}]}}.$$

Also, if P *is a polynomial in* k *variables,* $P(x) = \sum_{|\alpha|\leq N} a_\alpha x^\alpha$, *then*

(5)
$$\|P(\phi_1,\dots,\phi_k)Q_0(\phi,\psi)\|_{H^{[s-1,-\frac{1}{2}]}} \leq Cp(C\|\phi_1\|_{H^{[s,\frac{1}{2}]}},\dots,C\|\phi_k\|_{H^{[s,\frac{1}{2}]}}) \|\phi\|_{H^{[s,\frac{1}{2}]}} \|\psi\|_{H^{[s,\frac{1}{2}]}},$$

where $p(x) = \sum_{|\alpha|\leq N} |a_\alpha| x^\alpha$ *and* C *is a constant independent of* P.

Theorem 3. *Assume that* $s \geq \frac{n}{2}$ *and* $n \geq 2$. *Then*

(6)
$$\|\phi\psi\|_{H^{[s,\frac{1}{2}]}} \lesssim \|\phi\|_{H^{[s,\frac{1}{2}]}} \|\psi\|_{H^{[s,\frac{1}{2}]}}$$

and

(7)
$$\|\phi\psi\|_{H^{[s-1,-\frac{1}{2}]}} \lesssim \|\phi\|_{H^{[s-1,-\frac{1}{2}]}} \|\psi\|_{H^{[s,\frac{1}{2}]}}.$$

We will show that Theorem 2 follows from Theorem 3, which in turn can be reduced to the following two bilinear estimates for solutions of the homogeneous wave equation.

Theorem 4. *Let* ϕ *and* ψ *be solutions of the homogeneous wave equation on* $\mathbb{R}^{1+n}, n \geq 2$,*with data* $\phi(0,\cdot) = f, \partial_t\phi(0,\cdot) = 0$ *and* $\psi(0,\cdot) = g, \partial_t\psi(0,\cdot) = 0$. *Then the following estimates hold:*

(8)
$$\left\| \big||\tau| - |\xi|\big|^{\frac{1}{2}} \widetilde{\phi\psi}(\tau,\xi) \right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)}$$

(9)
$$\left\| \frac{\big||\tau| - |\xi|\big|^{\frac{1}{2}}}{\big(|\tau| + |\xi|\big)^{\frac{n-2}{2}}} \widetilde{\phi\psi}(\tau,\xi) \right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{\dot{H}^1(\mathbb{R}^n)}.$$

Here, $\|f\|_{\dot{H}^s} = \left\| |\xi|^s \hat{f}(\xi) \right\|_{L^2_\xi}$ *is the homogeneous Sobolev norm of exponent* s*.*

To see that Theorem 2 follows from Theorem 3, first notice that

$$\widetilde{Q_0(\phi,\psi)}(\tau,\xi) \simeq \int q_0(\tau-\lambda,\xi-\eta;\lambda,\eta)\widetilde{\phi}(\tau-\lambda,\xi-\eta)\widetilde{\psi}(\lambda,\eta)\,d\lambda\,d\eta, \tag{10}$$

where q_0 is the symbol of the null form Q_0, i.e.,

$$q_0(\tau,\xi;\lambda,\eta) = \tau\lambda - \xi\cdot\eta.$$

From the identity $\tau\lambda - \xi\cdot\eta = \frac{1}{2}\left[(\tau+\lambda)^2 - |\xi+\eta|^2 - \tau^2 + |\xi|^2 - \lambda^2 + |\eta|^2\right]$, it follows that

$$|q_0(\tau,\xi;\lambda,\eta)| \le w_+w_-(\tau+\lambda,\xi+\eta) + w_+w_-(\tau,\xi) + w_+w_-(\lambda,\eta).$$

Combined with (10), and since we may assume w.l.o.g. that $\widetilde{\phi}$ and $\widetilde{\psi}$ are both non-negative, this shows that

$$\begin{aligned} &\|Q_0(\phi,\psi)\|_{H^{[s-1,-\frac{1}{2}]}} \\ &\qquad \le \|\phi\psi\|_{H^{[s,\frac{1}{2}]}} + \|(\Lambda_+\Lambda_-\phi)\psi\|_{H^{[s-1,-\frac{1}{2}]}} + \|\phi(\Lambda_+\Lambda_-\psi)\|_{H^{[s-1,-\frac{1}{2}]}}, \end{aligned}$$

where the operators $\Lambda_\pm$ are defined by $\widetilde{\Lambda_\pm\phi} = w_\pm\widetilde{\phi}$. Thus, (4) follows from Theorem 3, while (5) follows by an induction argument using (4) and (7).

4 Proof of Theorem 4

We will need two lemmas.

Lemma 1. *If* η_1 *and* η_2 *are two points on the ellipsoid* $|\eta| + |\xi-\eta| = \tau$*, where* $\tau > |\xi|, \xi \in \mathbb{R}^n$*, then*

$$|\eta_1-\eta_2| - \big||\eta_1| - |\eta_2|\big| \lesssim \left(\tau^2 - |\xi|^2\right)^{\frac{1}{2}}.$$

Proof. If $|\eta_1-\eta_2| \le \left(\tau^2-|\xi|^2\right)^{\frac{1}{2}}$, this is clearly true, so we assume $|\eta_1-\eta_2| \ge \left(\tau^2-|\xi|^2\right)^{\frac{1}{2}}$. Then

$$\begin{aligned} |\eta_1-\eta_2| - \big||\eta_1|-|\eta_2|\big| &= \frac{|\eta_1-\eta_2|^2 - \big||\eta_1|-|\eta_2|\big|^2}{|\eta_1-\eta_2| + \big||\eta_1|-|\eta_2|\big|} \\ &\le \frac{2\big(|\eta_1|\,|\eta_2| - \eta_1\cdot\eta_2\big)}{|\eta_1-\eta_2|} \le \frac{2\big(|\eta_1|\,|\eta_2| - \eta_1\cdot\eta_2\big)}{\left(\tau^2-|\xi|^2\right)^{\frac{1}{2}}}. \end{aligned}$$

But in [4] it was proved that $|\eta_1|\,|\eta_2| - \eta_1\cdot\eta_2 \lesssim \tau^2 - |\xi|^2$, and we are done. □

Lemma 2. *Assume $n \geq 2$ and $a, b \in \mathbb{R}$. The estimate*

$$\left(\tau^2 - |\xi|^2\right)^{-\frac{n-3}{2}} \int \frac{\delta\left(\tau - |\eta| - |\xi - \eta|\right)}{|\eta|^a \, |\xi - \eta|^b} \, d\eta \lesssim \tau^{2-a-b}$$

holds for all (τ, ξ) with $\tau > |\xi|$ if and only if $a, b < \frac{n+1}{2}$.

Proof. We denote the above integral by I. Introducing polar coordinates $\eta = \rho\omega$, we calculate

$$I \simeq \left(\tau^2 - |\xi|^2\right)^{-\frac{n-3}{2}} \int_{S^{n-1}} \frac{\rho^2}{\tau^2 - |\xi|^2} \rho^{n-3} \rho^{1-a} (\tau - \rho)^{1-b} \, d\sigma_{S^{n-1}}(\omega),$$

where $\rho = \rho(\tau, \xi, \omega) = \frac{\tau^2 - |\xi|^2}{2(\tau - \xi \cdot \omega)}$. Introducing the notation $\lambda = \frac{\tau}{|\xi|}, y = \frac{\xi}{|\xi|} \cdot \omega$ and $x = \frac{\rho}{|\xi|} = \frac{\lambda^2 - 1}{2(\lambda - y)}$, we have $\frac{dx}{dy} = \frac{\lambda^2 - 1}{2(\lambda - y)^2}$, and hence

$$\begin{aligned} I &\simeq \left(\tau^2 - |\xi|^2\right)^{-\frac{n-3}{2}} |\xi|^{2-a-b+n-3} \int_{S^{n-1}} \frac{dx}{dy} x^{n-3} x^{1-a} (\lambda - x)^{1-b} \, d\sigma_{S^{n-1}}(\omega) \\ &\simeq \left(\tau^2 - |\xi|^2\right)^{-\frac{n-3}{2}} |\xi|^{2-a-b+n-3} \int_{-1}^{1} \frac{dx}{dy} x^{n-3} x^{1-a} (\lambda - x)^{1-b} (1 - y^2)^{\frac{n-3}{2}} \, dy, \end{aligned}$$

where the last step follows from the formula

$$\int_{S^{n-1}} f\left(\frac{\xi}{|\xi|} \cdot \omega\right) d\sigma_{S^{n-1}}(\omega) = \int_{S^{n-2}} d\sigma_{S^{n-2}}(\omega') \int_{-1}^{1} f(y)(1 - y^2)^{\frac{n-3}{2}} \, dy.$$

Noting that $x^2 - x^2 y^2 = (\lambda^2 - 1)\left(\frac{1}{4} - (x - \frac{\lambda}{2})^2\right)$, we finally get

$$\begin{aligned} I &\simeq |\xi|^{2-a-b} \int_{\frac{\lambda-1}{2}}^{\frac{\lambda+1}{2}} x^{1-a} (\lambda - x)^{1-b} \left(\frac{1}{4} - \left(x - \frac{\lambda}{2}\right)^2\right)^{\frac{n-3}{2}} dx \\ &\simeq |\xi|^{2-a-b} \int_{-1}^{1} (\lambda + x)^{1-a} (1 + x)^{\frac{n-3}{2}} (\lambda - x)^{1-b} (1 - x)^{\frac{n-3}{2}} \, dx, \end{aligned}$$

and it is clear that

$$\int_{-1}^{1} (\lambda + x)^{1-a} (1 + x)^{\frac{n-3}{2}} (\lambda - x)^{1-b} (1 - x)^{\frac{n-3}{2}} \, dx \lesssim \lambda^{2-a-b}$$

for all $\lambda > 1$ if and only if $a, b < \frac{n+1}{2}$. $\square$

To prove (8), it is enough to prove that it holds with $\widetilde{\phi\psi}$ replaced by $\widetilde{\phi_+ \psi_\pm} = \widetilde{\phi_+} * \widetilde{\psi_\pm}$, where

$$\begin{aligned} \widetilde{\phi_+}(\tau, \xi) &\simeq \delta\left(\tau - |\xi|\right) \hat{f}(\xi) \\ \widetilde{\psi_\pm}(\lambda, \eta) &\simeq \delta\left(\lambda \mp |\eta|\right) \hat{g}(\eta). \end{aligned}$$

Note that

$$\widetilde{\phi_+\psi_\pm}(\tau,\xi) = \widetilde{\phi_+} * \widetilde{\psi_\pm}(\tau,\xi) \simeq \int \delta\big(\tau \mp |\eta| - |\xi-\eta|\big) F(\xi-\eta) \frac{G(\eta)}{|\eta|^{\frac{n}{2}}}\, d\eta,$$

where $F(\xi) = \hat{f}(\xi)$ and $G(\eta) = |\eta|^{\frac{n}{2}}\, \hat{g}(\eta)$.

We estimate $\phi_+\psi_+$ first. Applying the Cauchy-Schwarz inequality w.r.t. the measure $\delta\big(\tau - |\eta| - |\xi-\eta|\big)\, d\eta$, we get

$$\begin{aligned} \big||\tau| - |\xi|\big| \left|\widetilde{\phi_+\psi_+}(\tau,\xi)\right|^2 \le \big||\tau| - |\xi|\big| \int \frac{\delta\big(\tau - |\eta| - |\xi-\eta|\big)}{|\eta|^n}\, d\eta \\ \times \int \delta\big(\tau - |\eta| - |\xi-\eta|\big)\, |F|^2(\xi-\eta)\, |G|^2(\eta)\, d\eta, \end{aligned}$$

so it suffices to show that

$$\big(\tau - |\xi|\big) \int \frac{\delta\big(\tau - |\eta| - |\xi-\eta|\big)}{|\eta|^n}\, d\eta \lesssim 1 \tag{11}$$

when $\tau > |\xi|$. Introducing polar coordinates, (11) can be written

$$\frac{\tau - |\xi|}{\tau^2 - |\xi|^2} \int_{S^{n-1}} \rho^{n-1} \frac{\rho(\tau-\rho)}{\rho^n}\, d\sigma_{S^{n-1}}(\omega) \tag{12}$$

where $\rho = \frac{\tau^2 - |\xi|^2}{2(\tau - \xi\cdot\omega)}$. Noting that $0 < \rho < \tau$, we see that (12) is indeed bounded independently of (τ,ξ).

To estimate $\phi_+\psi_-$, we write[2] $\widetilde{\phi_+\psi_-} = A + B$, where

$$\begin{aligned} A(\tau,\xi) &= \int_{|\eta|<2|\xi|} \delta\big(\tau + |\eta| - |\xi-\eta|\big) F(\xi-\eta) \frac{G(\eta)}{|\eta|^{\frac{n}{2}}}\, d\eta \\ B(\tau,\xi) &= \int_{|\eta|\ge 2|\xi|} \delta\big(\tau + |\eta| - |\xi-\eta|\big) F(\xi-\eta) \frac{G(\eta)}{|\eta|^{\frac{n}{2}}}\, d\eta. \end{aligned}$$

By applying the Cauchy-Schwarz inequality as above, the estimate for A is reduced to showing that

$$\big(|\xi| - |\tau|\big) \int_{|\eta|<2|\xi|} \frac{\delta\big(\tau + |\eta| - |\xi-\eta|\big)}{|\eta|^n}\, d\eta \lesssim 1$$

when $|\tau| < |\xi|$. This integral can be written

$$\frac{|\xi| - |\tau|}{|\xi|^2 - \tau^2} \int_{S^{n-1}\cap\{0<\rho<2|\xi|\}} \rho^{n-1} \frac{\rho(\tau+\rho)}{\rho^n}\, d\sigma_{S^{n-1}}(\omega),$$

where now $\rho = \frac{|\xi|^2 - \tau^2}{2(\tau + \xi\cdot\omega)}$, and this integral is obviously bounded independently of (τ,ξ).

[2] See [3].

To estimate B, note that if $|\eta| \geq 2|\xi|$, then $|\xi - \eta| \sim |\eta|$. Thus, we have

$$|B(\tau,\xi)| \lesssim \int \delta\big(\tau + |\eta| - |\xi - \eta|\big) \frac{|F|(\xi - \eta)\,|G|(\eta)}{|\xi - \eta|^{\frac{n}{4}}\,|\eta|^{\frac{n}{4}}}\,d\eta.$$

Denoting this last integral by I, what we have to show is that

$$\int \big(|\xi| - |\tau|\big)\,|I|^2\,d\tau\,d\xi \lesssim \|F\|_{L^2}^2\,\|G\|_{L^2}^2\,. \tag{13}$$

To prove this, we use an argument introduced in [4]. First, we write

$$|I|^2 = \prod_{i=1}^{2} \int \delta\big(\tau + |\eta_i| - |\xi - \eta_i|\big) \frac{|F|(\xi - \eta_i)\,|G|(\eta_i)}{|\xi - \eta_i|^{\frac{n}{4}}\,|\eta_i|^{\frac{n}{4}}}\,d\eta_i.$$

Multiplying by $|\xi| - |\tau|$ and integrating w.r.t. $d\tau\,d\xi$, we get

$$\begin{aligned}
&\int \big(|\xi| - |\tau|\big)\,|I|^2\,d\tau\,d\xi \\
&\quad\leq \int \Big(|\xi| - \big||\xi - \eta_1| - |\eta_1|\big|\Big)\,\delta\big(|\xi - \eta_1| - |\eta_1| + |\eta_2| - |\xi - \eta_2|\big) \\
&\qquad\qquad \times \frac{|F|(\xi - \eta_1)\,|G|(\eta_1)\,|F|(\xi - \eta_2)\,|G|(\eta_2)}{|\xi - \eta_1|^{\frac{n}{4}}\,|\eta_1|^{\frac{n}{4}}\,|\xi - \eta_2|^{\frac{n}{4}}\,|\eta_2|^{\frac{n}{4}}}\,d\eta_1\,d\eta_2\,d\xi.
\end{aligned}$$

Now we change variables

$$\begin{gathered}
(\xi, \eta_1, \eta_2) \longrightarrow (\xi', \eta_1', \eta_2') \\
\xi' = \xi - \eta_1 - \eta_2, \quad \eta_1' = -\eta_2, \quad \eta_2' = -\eta_1.
\end{gathered}$$

This gives

$$\begin{aligned}
\xi &= \xi' - \eta_1' - \eta_2' \\
\xi - \eta_1 &= \xi' - \eta_1' \\
\xi - \eta_2 &= \xi' - \eta_2',
\end{aligned}$$

so the last integral is transformed to, when we drop the primes on the new variables,

$$\begin{aligned}
&\int \Big(|\xi - \eta_1 - \eta_2| - \big||\xi - \eta_1| - |\eta_2|\big|\Big)\,\delta\big(|\xi - \eta_1| + |\eta_1| - |\eta_2| - |\xi - \eta_2|\big) \\
&\qquad\qquad \times \frac{|F|(\xi - \eta_1)\,|G|(-\eta_1)\,|F|(\xi - \eta_2)\,|G|(-\eta_2)}{|\xi - \eta_1|^{\frac{n}{4}}\,|\eta_1|^{\frac{n}{4}}\,|\xi - \eta_2|^{\frac{n}{4}}\,|\eta_2|^{\frac{n}{4}}}\,d\eta_1\,d\eta_2\,d\xi.
\end{aligned}$$

Applying Cauchy-Schwarz w.r.t. the measure

$$\delta\big(|\xi - \eta_1| + |\eta_1| - |\eta_2| - |\xi - \eta_2|\big)\,d\eta_1\,d\eta_2\,d\xi,$$

we bound this by

$$\Bigg(\int \Big(|\xi-\eta_1-\eta_2| - \big||\xi-\eta_1|-|\eta_2|\big|\Big)\delta\big(|\xi-\eta_1|+|\eta_1|-|\eta_2|-|\xi-\eta_2|\big) \times \frac{|F|^2(\xi-\eta_1)\,|G|^2(-\eta_1)}{|\xi-\eta_2|^{\frac{n}{2}}\,|\eta_2|^{\frac{n}{2}}}\,d\eta_1\,d\eta_2\,d\xi\Bigg)^{\frac{1}{2}}$$

$$\times \Bigg(\int \Big(|\xi-\eta_1-\eta_2| - \big||\xi-\eta_1|-|\eta_2|\big|\Big)\delta\big(|\xi-\eta_1|+|\eta_1|-|\eta_2|-|\xi-\eta_2|\big) \times \frac{|F|^2(\xi-\eta_2)\,|G|^2(-\eta_2)}{|\xi-\eta_1|^{\frac{n}{2}}\,|\eta_1|^{\frac{n}{2}}}\,d\eta_1\,d\eta_2\,d\xi\Bigg)^{\frac{1}{2}}.$$

Thus, the estimate (13) has been reduced to the following lemma, which is an immediate consequence of Lemma 1 and Lemma 2.

Lemma 3. *If $n \geq 2$, the integral*

$$I(\tau,\xi,\eta_0) = \int \frac{\Big(|\eta-\eta_0| - \big||\eta|-|\eta_0|\big|\Big)\delta\big(\tau-|\eta|-|\xi-\eta|\big)}{|\xi-\eta|^{\frac{n}{2}}\,|\eta|^{\frac{n}{2}}}\,d\eta$$

is bounded uniformly in (τ,ξ,η_0), where $\tau > |\xi|$ and $|\eta_0| + |\xi-\eta_0| = \tau$.

Finally, we consider (9). If $n = 2$, the estimates (8) and (9) coincide, so we assume $n \geq 3$. Proceeding as in the proof of (8), the proof of (9) can be reduced to an estimate

$$\int \frac{\big(|\xi|-|\tau|\big)}{\big(|\tau|+|\xi|\big)^{n-2}}\,|I|^2\,d\tau\,d\xi \lesssim \|F\|_{L^2}^2\,\|G\|_{L^2}^2\,,$$

where now

$$I = \int \delta\big(\tau+|\eta|-|\xi-\eta|\big)\frac{F(\xi-\eta)}{|\xi-\eta|^{\frac{1}{2}}}\frac{G(\eta)}{|\eta|^{\frac{1}{2}}}\,d\eta.$$

In other words, we have to prove the estimate

$$\left\| \frac{\big(|\xi|-|\tau|\big)^{\frac{1}{2}}}{\big(|\tau|+|\xi|\big)^{\frac{n-2}{2}}}\widetilde{\phi_+\psi_-}\right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)}\,\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)}\,.$$

In fact, something stronger is true. In [4] it was proved that

$$\left\| \frac{1}{\big(|\tau|+|\xi|\big)^{\frac{n-3}{2}}}\widetilde{\phi_+\psi_-}\right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)}\,\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)}$$

when $n \geq 3$.

5 Proof of Theorem 3

We first prove (6). Writing

$$F(\tau,\xi) = w_+^{[s]} w_-^{[\frac{1}{2}]}(\tau,\xi)\widetilde{\phi}(\tau,\xi)$$
$$G(\lambda,\eta) = w_+^{[s]} w_-^{[\frac{1}{2}]}(\lambda,\eta)\widetilde{\psi}(\lambda,\eta),$$

we have

$$\|\phi\psi\|_{H^{[s,\frac{1}{2}]}} = \left\| \int \frac{w_+^{[s]} w_-^{[\frac{1}{2}]}(\tau,\xi) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta)}{w_+^{[s]} w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+^{[s]} w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}},$$

so by the self-duality of L^2, it will be enough to prove that

$$\text{(14)} \qquad \int \frac{w_+^{[s]} w_-^{[\frac{1}{2}]}(\tau,\xi) \left|F\right|(\tau-\lambda,\xi-\eta) \left|G\right|(\lambda,\eta) \left|H\right|(\tau,\xi)}{w_+^{[s]} w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+^{[s]} w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi \lesssim \|F\|_{L^2(\mathbb{R}^{1+n})} \|G\|_{L^2(\mathbb{R}^{1+n})} \|H\|_{L^2(\mathbb{R}^{1+n})}$$

for all $F, G, H \in L^2(\mathbb{R}^{1+n})$. From now on, we drop the absolute values on F, G and H, and simply assume that $F, G, H \geq 0$. In order to estimate the integral (14), we split the domain of integration into several pieces. First, note that by symmetry, we may assume

$$w_+(\lambda,\eta) \leq w_+(\tau-\lambda,\xi-\eta).$$

Combining this with the inequality

$$\text{(15)} \qquad w_+(\tau,\xi) \leq w_+(\lambda,\eta) + w_+(\tau-\lambda,\xi-\eta),$$

and making use of the property (3) of the function L, we get

$$w_+^{[s]}(\tau,\xi) \lesssim w_+^{[s]}(\tau-\lambda,\xi-\eta),$$

so we are left with the integral

$$\text{(16)} \qquad I = \int \frac{w_-^{[\frac{1}{2}]}(\tau,\xi) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta) H(\tau,\xi)}{w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+^{[s]} w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi.$$

To deal with the factor $w_-^{[\frac{1}{2}]}(\tau,\xi)$ in the numerator, we need the following lemma, the proof of which can be found in [5].

Lemma 4. *We have*

$$\text{(17)} \qquad w_-(\tau,\xi) \leq w_-(\tau-\lambda,\xi-\eta) + w_-(\lambda,\eta) + d_\pm(\xi-\eta,\eta),$$

where

$$d_\pm(\xi-\eta,\eta) = \Big| \big| |\xi-\eta| \pm |\eta| \big| - |\xi| \Big|,$$

and we choose + or − according to whether $\tau - \lambda$ and λ have equal or opposite signs.

Using Lemma 4, we split the domain of integration of I into three parts, according to which of the three terms on the right hand side of (17) is largest. Using once more the property (3), we thus get $I \lesssim I_1 + I_2 + I_3$, where

$$I_1 = \int \frac{F(\tau-\lambda,\xi-\eta)G(\lambda,\eta)H(\tau,\xi)}{w_+^{[s]}w_-^{[\frac{1}{2}]}(\lambda,\eta)}\, d\lambda\, d\eta\, d\tau\, d\xi \tag{18}$$

$$I_2 = \int \frac{F(\tau-\lambda,\xi-\eta)G(\lambda,\eta)H(\tau,\xi)}{w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta)w_+^{[s]}(\lambda,\eta)}\, d\lambda\, d\eta\, d\tau\, d\xi \tag{19}$$

$$I_3 = \int \frac{d_\pm^{\frac{1}{2}}L(d_\pm)(\xi-\eta,\eta)F(\tau-\lambda,\xi-\eta)G(\lambda,\eta)H(\tau,\xi)}{w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta)w_+^{[s]}w_-^{[\frac{1}{2}]}(\lambda,\eta)}\, d\lambda\, d\eta\, d\tau\, d\xi. \tag{20}$$

Now we perform the change of variables

$$(\tau,\lambda) \longrightarrow (u,v) \tag{21a}$$

$$u = |\tau-\lambda| - |\xi-\eta|\,, \quad v = |\lambda| - |\eta|\,. \tag{21b}$$

Without loss of generality, we assume that $\tau - \lambda \geq 0$. Thus,

$$u = \tau - \lambda - |\xi-\eta|\,, \quad v = \pm\lambda - |\eta|\,,$$

and we have

$$I_1 \leq \int \frac{f_u(\xi-\eta)g_v(\eta)H_{u,v}\big(|\xi-\eta| \pm |\eta|\,,\xi\big)}{(2+|v|)^{\frac{1}{2}}L(2+|v|)(2+|\eta|)^sL(2+|\eta|)}\, du\, dv\, d\eta\, d\xi$$

$$I_2 \leq \int \frac{f_u(\xi-\eta)g_v(\eta)H_{u,v}\big(|\xi-\eta| \pm |\eta|\,,\xi\big)}{(2+|u|)^{\frac{1}{2}}L(2+|u|)(2+|\eta|)^sL(2+|\eta|)}\, du\, dv\, d\eta\, d\xi,$$

where

$$\begin{aligned} f_u(\xi) &= F\big(u+|\xi|\,,\xi\big) \\ g_v(\eta) &= G\big(\pm(v+|\eta|),\eta\big) \\ H_{u,v}(\tau,\xi) &= H(u \pm v + \tau,\xi). \end{aligned}$$

Therefore, using the following lemma first w.r.t. the (u,v) variables and then w.r.t. (ξ,η), we get the estimate $I_i \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}$ for $i = 1, 2$.

Lemma 5. *Assume $a \geq \frac{n}{2}, n \geq 1$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)h(x+y)}{(2+|x|)^aL(2+|x|)}\, dx\, dy \lesssim \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}\|h\|_{L^2(\mathbb{R}^n)}\,.$$

Proof. By the Cauchy-Schwarz inequality, the above integral is dominated by

$$\left(\int \frac{dx}{(2+|x|)^{2a}L^2(2+|x|)}\right)^{\frac{1}{2}} \int g(y)\left(\int |f|^2(x)\,|h|^2(x+y)\,dx\right)^{\frac{1}{2}} dy.$$

By the assumptions on L, the integral in front is bounded, so applying Cauchy-Schwarz once more finishes the proof. ☐

We now turn our attention to the difficult part, I_3. We obtained I_3 after using Lemma 4 and then restricting I to the region where

$$w_-(\tau-\lambda,\xi-\eta), w_-(\lambda,\eta) \leq d_\pm(\xi-\eta,\eta).$$

Note that $w_-(\tau-\lambda,\xi-\eta)$ and $w_-(\lambda,\eta)$ measure how far removed the points $(\tau-\lambda,\xi-\eta)$ and (λ,η) are from the light cone through the origin. Thus, we may think of I_3 as the part of I that results from concentrating F and G on the light cone.

To estimate I_3, note that by the triangle inequality, $d_\pm(\xi-\eta,\eta) \leq 2|\eta|$. Therefore, by the properties of L, we have $L(d_\pm)(\xi-\eta,\eta) \lesssim L(w_+)(\lambda,\eta)$, whence

$$I_3 \lesssim \int \frac{d_\pm^{\frac{1}{2}}(\xi-\eta,\eta)F(\tau-\lambda,\xi-\eta)G(\lambda,\eta)H(\tau,\xi)}{w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta)w_-^{[\frac{1}{2}]}(\lambda,\eta)(2+|\eta|)^{\frac{n}{2}}} d\lambda\, d\tau\, d\eta\, d\xi.$$

Performing the change of variables (21), we get

$$I_3 \lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{J_{u,v}^+ + J_{u,v}^-}{(2+|u|)^{\frac{1}{2}}L(2+|u|)(2+|v|)^{\frac{1}{2}}L(2+|v|)}\, du\, dv,$$

where

$$J_{u,v}^\pm = \int d_\pm^{\frac{1}{2}}(\xi-\eta,\eta)f_u(\xi-\eta)g_v(\eta)H_{u,v}\big(|\xi-\eta| \pm |\eta|, \xi\big)\frac{1}{|\eta|^{\frac{n}{2}}} d\eta\, d\xi.$$

Note that if we can show that

$$J_{u,v}^\pm \lesssim \|f_u\|_{L^2(\mathbb{R}^n)} \|g_v\|_{L^2(\mathbb{R}^n)} \|H\|_{L^2(\mathbb{R}^{1+n})}, \tag{22}$$

then the estimate

$$I_3 \lesssim \|F\|_{L^2} \|G\|_{L^2} \|H\|_{L^2}$$

follows after an application of the Cauchy-Schwarz inequality w.r.t. du and dv. To prove (22), we rewrite the integral $J_{u,v}^\pm$ as follows:

$$\int \frac{\big||\tau|-|\xi|\big|^{\frac{1}{2}} f_u(\xi-\eta)g_v(\eta)\delta\big(\tau \mp |\eta| - |\xi-\eta|\big)H_{u,v}(\tau,\xi)}{|\eta|^{\frac{n}{2}}} d\eta\, d\tau\, d\xi.$$

This is dominated by

$$\left\| \big||\tau|-|\xi|\big|^{\frac{1}{2}} \int \frac{f_u(\xi-\eta)g_v(\eta)\delta\big(\tau \mp |\eta| - |\xi-\eta|\big)}{|\eta|^{\frac{n}{2}}}\, d\eta \right\|_{L^2_{\tau,\xi}} \|H_{u,v}\|_{L^2(\mathbb{R}^{1+n})}.$$

Noting that (i) $\|H_{u,v}\|_{L^2} = \|H\|_{L^2}$, and (ii) by the proof of (8),

$$\left\| \big||\tau|-|\xi|\big|^{\frac{1}{2}} \int \frac{f_u(\xi-\eta)g_v(\eta)\delta\big(\tau \mp |\eta| - |\xi-\eta|\big)}{|\eta|^{\frac{n}{2}}}\, d\eta \right\|_{L^2_{\tau,\xi}} \lesssim \|f_u\|_{L^2} \|g_v\|_{L^2},$$

the proof of (6) is complete.

To prove (7), it suffices to prove that

$$\int \frac{w_+^{[s-1]} w_-^{[-\frac{1}{2}]}(\tau,\xi) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta) H(\tau,\xi)}{w_+^{[s-1]} w_-^{[-\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+^{[s]} w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi \lesssim \|F\|_{L^2} \|G\|_{L^2} \|H\|_{L^2} . \tag{23}$$

First, we use (15) to dominate the integral in (23) by the sum of the two integrals

$$\int \frac{w_-^{[-\frac{1}{2}]}(\tau,\xi) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta) H(\tau,\xi)}{w_-^{[-\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+^{[s]} w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi \tag{24}$$

$$\int \frac{w_-^{[-\frac{1}{2}]}(\tau,\xi) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta) H(\tau,\xi)}{w_+^{[s-1]} w_-^{[-\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+ w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi. \tag{25}$$

Next, using the fact that (since L is increasing)

$$\frac{w_-^{\frac{1}{2}}(\tau-\lambda,\xi-\eta) L(w_-)(\tau,\xi)}{w_-^{\frac{1}{2}}(\tau,\xi) L(w_-)(\tau-\lambda,\xi-\eta)} \le \begin{cases} \frac{w_-^{\frac{1}{2}} L(w_-)(\tau,\xi)}{w_-^{\frac{1}{2}} L(w_-)(\tau-\lambda,\xi-\eta)} & \text{if} \quad w_-(\tau-\lambda,\xi-\eta) \le w_-(\tau,\xi) \\ \frac{w_-^{\frac{1}{2}} L(w_-)(\tau-\lambda,\xi-\eta)}{w_-^{\frac{1}{2}} L(w_-)(\tau,\xi)} & \text{if} \quad w_-(\tau,\xi) \le w_-(\tau-\lambda,\xi-\eta), \end{cases}$$

we see that (24) reduces to the integral (16), which was estimated above, while (25) is dominated by the sum of the integrals

$$\int \frac{w_-^{[\frac{1}{2}]}(\tau,\xi) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta) H(\tau,\xi)}{w_+^{[s-1]} w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+ w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi \tag{26}$$

$$\int \frac{w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta) F(\tau-\lambda,\xi-\eta) G(\lambda,\eta) H(\tau,\xi)}{w_+^{[s-1]}(\tau-\lambda,\xi-\eta) w_-^{[\frac{1}{2}]}(\tau,\xi) w_+ w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi. \tag{27}$$

Integral (26) reduces to integrals of the type (16) when we consider separately the regions where $w_+(\lambda,\eta) \le w_+(\tau-\lambda,\xi-\eta)$ and $w_+(\tau-\lambda,\xi-\eta) < w_+(\tau,\xi)$, so we are left with the integral (27), which after a change of variables takes the form

$$\int \frac{w_-^{[\frac{1}{2}]}(\tau,\xi) F(\tau,\xi) G(\lambda,\eta) H(\tau-\lambda,\xi-\eta)}{w_+^{[s-1]}(\tau,\xi) w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta) w_+ w_-^{[\frac{1}{2}]}(\lambda,\eta)} \, d\lambda \, d\eta \, d\tau \, d\xi. \tag{28}$$

(Here we have replaced $G(\lambda,\eta)$ by $G(-\lambda,-\eta)$.)

We now consider the two regions given by

$$w_-(\tau-\lambda,\xi-\eta)+w_-(\lambda,\eta)\geq \frac{1}{2}d_\pm(\xi-\eta,\eta) \tag{29}$$

$$w_-(\tau-\lambda,\xi-\eta)+w_-(\lambda,\eta)< \frac{1}{2}d_\pm(\xi-\eta,\eta) \tag{30}$$

Using Lemma 4, we find that (28) restricted to the region (29) is dominated by the sum of the two integrals

$$\int \frac{F(\tau,\xi)G(\lambda,\eta)H(\tau-\lambda,\xi-\eta)}{w_+^{[s-1]}(\tau,\xi)w_+w_-^{[\frac{1}{2}]}(\lambda,\eta)}\,d\lambda\,d\eta\,d\tau\,d\xi$$

$$\int \frac{F(\tau,\xi)G(\lambda,\eta)H(\tau-\lambda,\xi-\eta)}{w_+^{[s-1]}(\tau,\xi)w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta)w_+(\lambda,\eta)}\,d\lambda\,d\eta\,d\tau\,d\xi.$$

Both these integrals reduce to integrals of the type (18) or (19) when we consider separately the regions $w_+(\tau,\xi)\leq w_+(\lambda,\eta)$ and $w_+(\lambda,\eta)<w_+(\tau,\xi)$.

Now assume that we are in the region (30). Then by Lemma 4,

$$w_-^{[\frac{1}{2}]}(\tau,\xi)\lesssim d_\pm^{\frac{1}{2}}L(d_\pm)(\xi-\eta,\eta). \tag{31}$$

In fact, we also have

$$w_+^{[s-1]}(\tau,\xi)\gtrsim D_\pm^{\frac{n-2}{2}}L(D_\pm)(\xi-\eta,\eta), \tag{32}$$

where $D_\pm(\xi-\eta,\eta)=\big||\xi-\eta|\pm|\eta|\big|+|\xi|$. This follows from the estimate

$$w_+(\tau,\xi)\geq|\tau|+|\xi|\geq\frac{1}{2}D_\pm(\xi-\eta,\eta),$$

which clearly holds in the $-$ case, and which in the $+$ case is a consequence of the following lemma and the requirement (30).

Lemma 6. *If $\tau-\lambda$ and λ have the same sign, then*

$$\Big||\tau|-\big(|\xi-\eta|+|\eta|\big)\Big|\leq w_-(\tau-\lambda,\xi-\eta)+w_-(\lambda,\eta).$$

The proof is obvious.

Using (31),(32) and the fact that $d_\pm\leq D_\pm$, and hence $L(d_\pm)\leq L(D_\pm)$, we have then to show that

$$\int \frac{d_\pm^{\frac{1}{2}}(\xi-\eta,\eta)F(\tau,\xi)G(\lambda,\eta)H(\tau-\lambda,\xi-\eta)}{D_\pm^{\frac{n-2}{2}}(\xi-\eta,\eta)w_-^{[\frac{1}{2}]}(\tau-\lambda,\xi-\eta)(2+|\eta|)w_-^{[\frac{1}{2}]}(\lambda,\eta)}\,d\lambda\,d\eta\,d\tau\,d\xi \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}.$$

Performing the change of variables (21), we find that the estimate can be reduced to proving that

$$J^\pm_{u,v}\lesssim\|F\|_{L^2}\|g_v\|_{L^2}\|h_u\|_{L^2}, \tag{33}$$

where $g_v(\eta) = G\big(\pm(v + |\eta|), \eta\big), h_u(\xi) = H\big(u + |\xi|, \xi\big)$ and

$$J^{\pm}_{u,v} = \int \frac{d_{\pm}^{\frac{1}{2}}(\xi - \eta, \eta)}{D_{\pm}^{\frac{n-2}{2}}(\xi - \eta, \eta)\,|\eta|} F\big(u \pm v + |\xi - \eta| \pm |\eta|, \xi\big) G\big(\pm(v + |\eta|), \eta\big)$$
$$H\big(u + |\xi - \eta|, \xi - \eta\big)\, d\eta\, d\xi.$$

Setting $F_{u,v}(\tau, \xi) = F(u \pm v + \tau, \xi)$, we rewrite $J^{\pm}_{u,v}$ as follows:

$$\int \frac{\big||\tau| - |\xi|\big|^{\frac{1}{2}} \delta\big(\tau \mp |\eta| - |\xi - \eta|\big) F_{u,v}(\tau, \xi) g_v(\eta) h_u(\xi - \eta)}{\big(|\tau| + |\xi|\big)^{\frac{n-2}{2}} |\eta|}\, d\eta\, d\tau\, d\xi,$$

Applying the Cauchy-Schwarz inequality w.r.t. the measure $d\tau\, d\xi$, we see that (33) follows from the estimate

$$\left\| \frac{\big||\tau| - |\xi|\big|^{\frac{1}{2}}}{\big(|\tau| + |\xi|\big)^{\frac{n-2}{2}}} \int \frac{g_v(\eta) h_u(\xi - \eta) \delta\big(\tau \mp |\eta| - |\xi - \eta|\big)}{|\eta|}\, d\eta \right\|_{L^2_{\tau,\xi}}$$
$$\lesssim \|g_v\|_{L^2} \|h_u\|_{L^2},$$

which holds by the proof of (9). This concludes the proof of Theorem 3.

References

[1] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II*, Geom. Funct. Anal. **3** (1993), 107–156, 209–262

[2] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), 1221–1268

[3] S. Klainerman and M. Machedon, *Smoothing estimates for null forms and applications*, Duke Math. J. **81** (1995), 99–133

[4] S. Klainerman and M. Machedon, *Remark on Strichartz type inequalities*, International Mathematics Research Notices 1996, No. 5

[5] S. Klainerman and M. Machedon, *On the regularity properties of a model problem related to wave maps* , Duke Math. J. (to appear)

[6] Y. Zhou, *Local regularity for two space dimensional equations of wave maps type*, Math. Z. (to appear)