Category Theory

Making Mathematical Pearls Since the 1940s

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

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Disclaimer

I A N A H

I A N A E

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I Am Not A Historian

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I Am Not An Expert

Making Pearls: A Recipe

1. Find something irritating

Making Pearls: A Recipe

1. Find something irritating

2. Cover it with something disgusting

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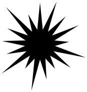

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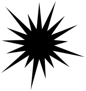

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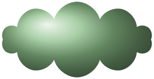

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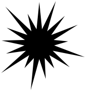

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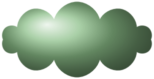

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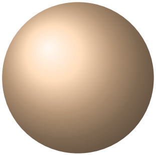

Making Mathematical Pearls

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Making Mathematical Pearls

1. Find something irritating
$$A = \left\{B : B \notin B\right\}, \quad A \in A?$$
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Making Mathematical Pearls

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   Axiom: There is an inaccessible cardinal

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Making Mathematical Pearls

1. Find something irritating

$$A = \left\{B : B \notin B\right\}, \quad A \in A?$$

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Axiom: There is an inaccessible cardinal

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Hey look! $\pi_2(X)$ is abelian! And I know why!

Size Matters

First Rule of Set Theory

You don’t talk about the “Set of all sets”.

Size Matters

First Rule of Set Theory

You don't talk about the "Set of all sets".

Obvious Question

Who would want to, and why?

Size Matters

First Rule of Set Theory

You don't talk about the "Set of all sets".

Obvious Question

Who would want to, and why?

Non-obvious Answer

Algebraic topologists, that's who.

Fundamental Problem of Algebraic Topology

Question

Is a doughnut a donut?

Fundamental Problem of Algebraic Topology

Question

Is a doughnut a donut?

Simultaneous Translation

Er en smultring en berlinerbolle?

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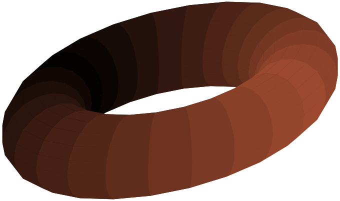
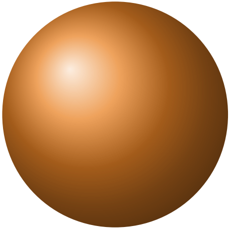


$$\overset{?}{=}$$



Cohomology

Solution

Use Cohomology

Cohomology

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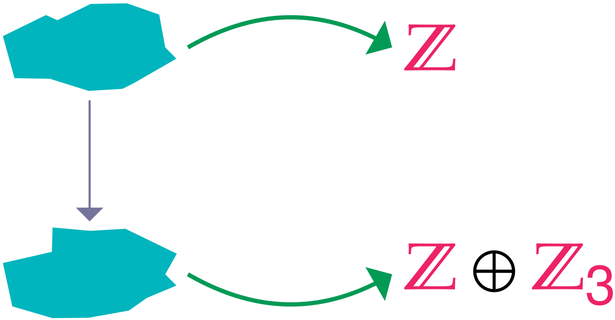

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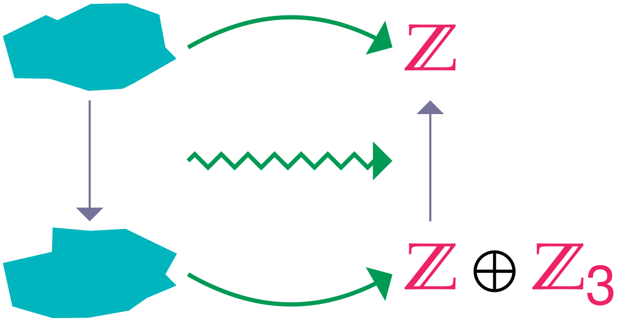

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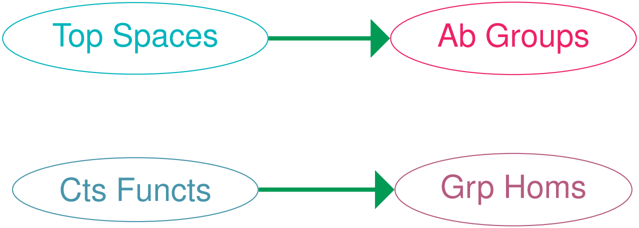

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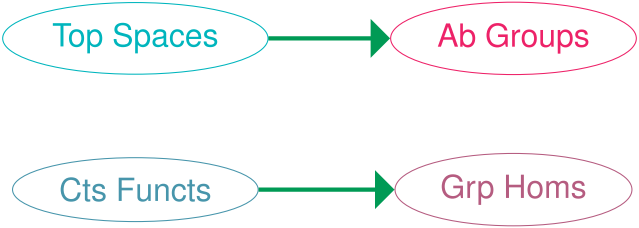
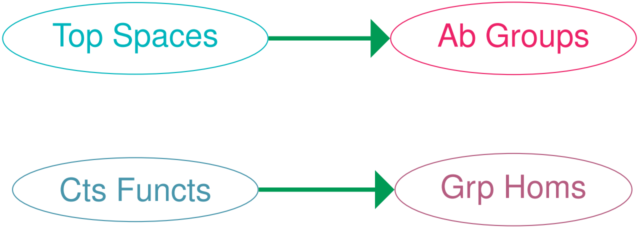


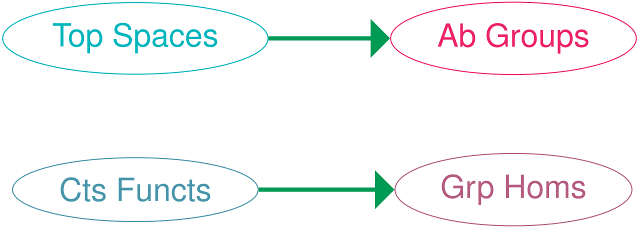
Looks like . . .

Cohomology



Looks like . . . Functions!

Cohomology



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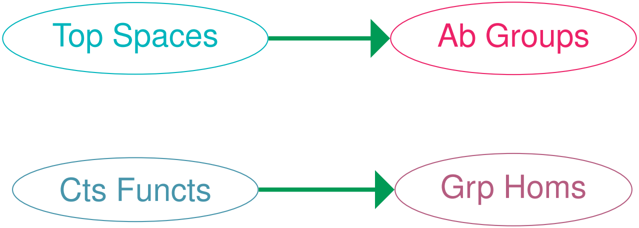
But they aren't

Cohomology



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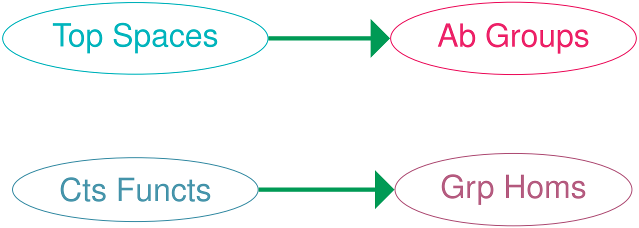
Domains, codomains not sets

Cohomology



Looks like . . . Functions!

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Domains, codomains not sets

So what?

Naturally Speaking

Cohomology is okay on a case by case basis

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But that’s not enough!

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- $H^1(\text{Doughnut}) = \mathbb{Z}^2$

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$\implies$ Doughnut $\neq$ Donut

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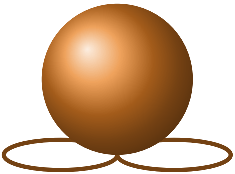
$H^k\Bigg($  $\Bigg) = H^k(\text{Doughnut})$

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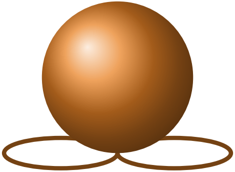

$\Big) = H^k(\text{Doughnut})$

But not the same

Naturally Speaking

Cohomology is highly structured:

1. $H^k(X)$ abelian group

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   Doughnut $\neq$ Donut

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2. $H^*(X)$ graded commutative ring

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Cohomology is highly structured:

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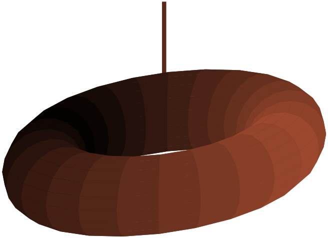
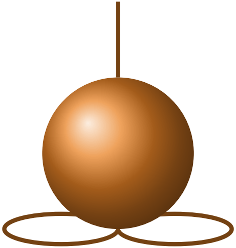
   Doughnut $\neq$ Donut
2. $H^*(X)$ graded commutative ring
   Doughnut $\neq$ Donut + String

Naturally Speaking

Cohomology is highly structured:

1. $H^k(X)$ abelian group
   Doughnut ≠ Donut
2. $H^*(X)$ graded commutative ring
   Doughnut ≠ Donut + String





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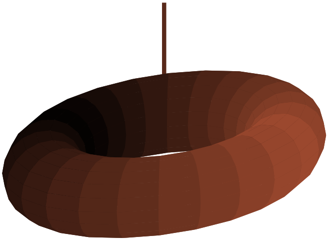
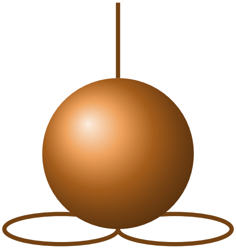
   Doughnut ≠ Donut + String
3. $H^*(X)$ action of operations





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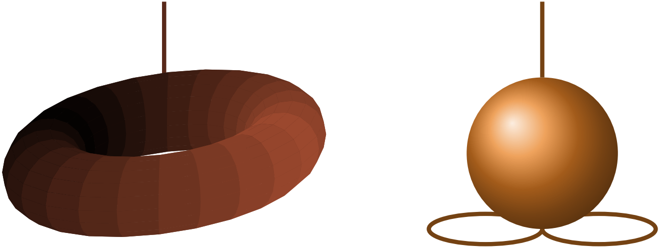
   Σ Doughnut $\neq$ $\Sigma($ Donut + String $)$



Cohomology Operations

Definition

To each topological space

X

a group homomorphism

$\theta_X : H^k(X) \to H^l(X)$

such that

for every continuous function

$f : X \to Y$

$$\begin{array}{ccc} H^k(X) & \xrightarrow{\theta_X} & H^l(X) \\ \uparrow {\scriptstyle H^k(f)} & & \uparrow {\scriptstyle H^l(f)} \\ H^k(Y) & \xrightarrow[\theta_Y]{} & H^l(Y) \end{array}$$

“for every”

Making a Pearl

The Grit

To say “for every” means sets form a set

$\Longrightarrow$ paradox

Making a Pearl

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To say “for every” means sets form a set
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The Disgusting Part

Isolate the bits of Set Theory that lead to the paradox and contain them

Making a Pearl

The Grit

To say “for every” means sets form a set

$\Longrightarrow$ paradox

Slightly Sanitised

Some sets are “big” and some are “small”.

Making a Pearl

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Slightly Sanitised

Some sets are “big” and some are “small”.
And some are very, very tall.

Making a Pearl

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To say “for every” means sets form a set
$\Longrightarrow$ paradox

Slightly Sanitised

Some sets are “big” and some are “small”.
And some are very, very tall.

The Pearl

Can talk of “functions” between “big sets”.

The Deal

A Category has:

- Objects, X.

The Deal

A Category has:

- Objects, X. Possibly lots.

The Deal

A Category has:

- Objects, X. Possibly lots.
- Arrows, $X \to Y$.

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  But not too many between two objects.

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Don't examine objects too closely.

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Don't examine objects too closely.

The Consquence

A thing is not what it is,
nor what it does,
but how it relates to other things.

More Grit

Question

Why is $\pi_1(X)$ a group?

More Grit

Question

Why is $\pi_1(X)$ a group?

Answer

Because it is.

More Grit

Question

Why is $\pi_1(X)$ a group?

Answer

Because it is.

$$f, g \colon S^1 \to X, \quad (f * g)(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1 \end{cases}$$

More Grit

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Question

But Why? And why not $\pi_0(X)$?

Groups

Question

What is a group?

Groups

Question

What is a group?

Old Answer

Set: G

Operations: $\mu\colon G \times G \to G$, $\iota\colon G \to G$, $1 \in G$

(plus identities)

Groups

Question

What is a group?

Old Answer

$$\begin{aligned} &\text{Set: } G \\ &\mu : G \times G \to G \\ \text{Operations: } &\iota : G \to G \\ &1 \in G \end{aligned}$$

(plus identities)

New Answer?

An object of Groups

Groups In Nature

$\mathbb{CP}^\infty$

$(G, \mu, \iota, 1)$

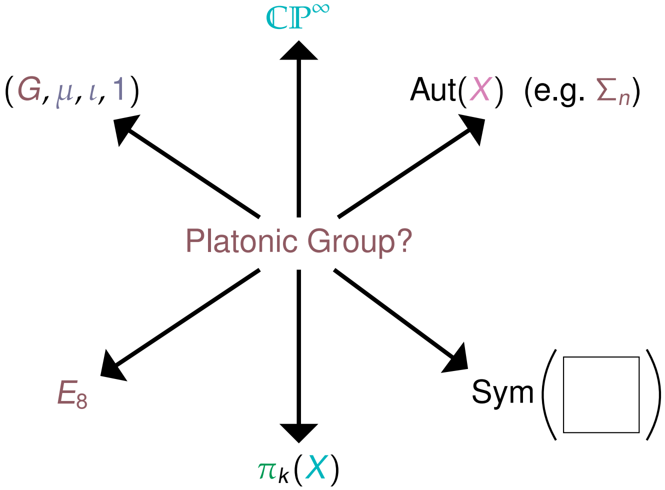
$\mathrm{Aut}(X)$ (e.g. Σ_n)

E_8

$\mathrm{Sym}\left(\square\right)$

$\pi_k(X)$

Groups In Nature



The Platonic Group

Theorem (Lawvere)

There is a Platonic Group!

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It's a **CATEGORY**!

The Platonic Group

Theorem (Lawvere)

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It's a **CATEGORY**!

- Objects: $\mathbb{N}$
- Arrows: $m \to n$
  All ways of producing n–elements from m–elements in all groups.

The Platonic Group

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It's a **CATEGORY**!

- Objects: $\mathbb{N}$
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  All ways of producing n–elements from m–elements in all groups.

Generated by:

Group Objects

Definition

A
group object
in a category
is a shadow of the Platonic group.

A
co-group object
in a category
is a shadow of the Platonic group in a mirror.

They Do It With Mirrors, Y'Know

Theorem

- *A shadow of a shadow of the Platonic group is a shadow of the Platonic group.*

They Do It With Mirrors, Y'Know

Theorem

- *A shadow of a shadow of the Platonic group is a shadow of the Platonic group.*
- *Mirrors cancel in pairs.*

They Do It With Mirrors, Y'Know

Theorem

- *A shadow of a shadow of the Platonic group is a shadow of the Platonic group.*
- *Mirrors cancel in pairs.*
- *A shadow of the Platonic group in* Set *is a group.*

Shadows of the Circle

S^1: shadow of Platonic group in mirror

$\Longrightarrow$ mirrored shadows of S^1 form groups

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S^1: shadow of Platonic group in mirror

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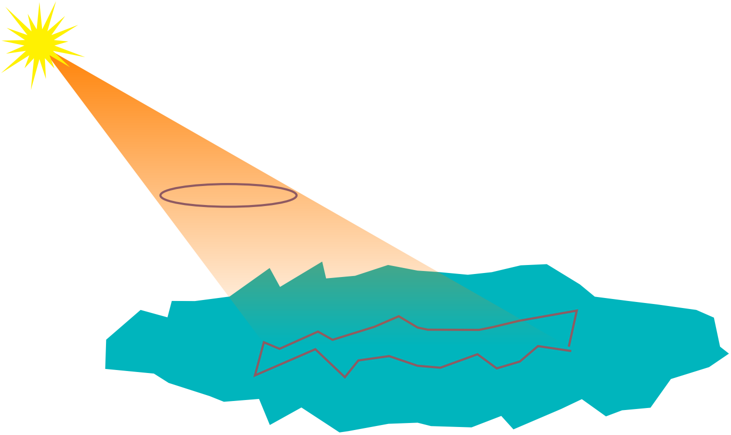

The Answer From the Shadows

- $\pi_1(X)$ group: S^1 shadow of Platonic group
- $\pi_0(X)$ not group: S^0 not shadow of Platonic group

Aside

Remark

Nothing special about groups!

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Same works for

- Abelian groups

Aside

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Same works for

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- Compact Hausdorff Spaces

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- Banach Algebras

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Remark

Nothing special about groups!

Same works for

- Abelian groups
- Rings
- Algebras
- Lattices
- Compact Hausdorff Spaces
- Banach Spaces (almost)
- Banach Algebras (almost)
- C^*–Algebras

Aside

Remark

Nothing special about groups!

Same works for

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- Rings
- Algebras
- Lattices
- Compact Hausdorff Spaces
- Banach Spaces (almost)
- Banach Algebras (almost)
- C^*–Algebras (exactly!)

Making a Pearl

The Grit

Group-like things appear everywhere.

Making a Pearl

The Grit

Group-like things appear everywhere.

The Disgusting Part

Constructing a “Platonic Group” seems far from intuition of “a group”.

Making a Pearl

The Grit

Group-like things appear everywhere.

The Disgusting Part

Constructing a “Platonic Group” seems far from intuition of “a group”.

The Pearl

Can prove Big Theorems in one go.
Can link ideas in group theory, ring theory, C^*–algebra theory . . .

Yet More Grit

- $\pi_1(X)$ group: S^1 shadow of Platonic group
- $\pi_2(X)$ abelian: S^2 shadow of Platonic abelian group

Yet More Grit

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Question

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Answer

Because it is.

Three Views

$$\pi_2(X) \quad \cong \quad \pi_1(\Omega X) \quad \cong \quad \pi_0(\Omega^2 X)$$

Question

Which is the best description?

Three Views

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Which is the best description?

- S^1 is a (mirror of) a shadow of a group
- ΩX is a shadow of a group

Three Views

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So $\pi_2(\Omega X)$ is a group for TWO reasons!

Three Views

$$\pi_2(X) \qquad \cong \qquad \pi_1(\Omega X) \qquad \cong \qquad \pi_0(\Omega^2 X)$$

Question

Which is the best description?

- S^1 is a (mirror of) a shadow of a group
- ΩX is a shadow of a group

So $\pi_2(\Omega X)$ is a group for TWO reasons!

$$\left(a \circ b\right) * \left(c \circ d\right) = \left(a * c\right) \circ \left(b * d\right)$$

$\implies$ same multiplication and commutative

Making a Pearl

The Grit

$\pi_2(X)$ is abelian, but **why**?

Making a Pearl

The Grit

$\pi_2(X)$ is abelian, but why?

The Disgusting Part

Tracing arrows round a diagram.

Making a Pearl

The Grit

$\pi_2(X)$ is abelian, but why?

The Disgusting Part

Tracing arrows round a diagram.

The Pearl

The Eckmann–Hilton Dance!

$$\left(a \circ b\right) * \left(c \circ d\right) = \left(a * c\right) \circ \left(b * d\right)$$

Summary

- Categories provide context

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- Categories link concepts

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    - Groups

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- Categories provide context
- Categories link concepts
- Categories are everywhere
  - “Large” categories, Set, Top
  - “Small” categories:
    - Partially ordered sets
    - Groups
    - Metric spaces (!)

Summary

- Categories provide context
- Categories link concepts
- Categories are everywhere
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