

# Category Theory

## Making Mathematical Pearls Since the 1940s

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12th February 2010

## Disclaimer

I Am Not A Historian  
I Am Not An Expert

## Making Pearls: A Recipe

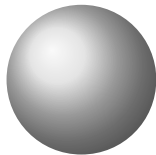
1. Find something irritating



2. Cover it with something disgusting



3. Leave, for someone else to find beautiful



## Making Mathematical Pearls

1. Find something irritating

$$A = \{B : B \notin B\}, \quad A \in A?$$

2. Cover it with something disgusting

**Axiom:** There is an inaccessible cardinal

3. Leave, for someone else to find beautiful

Hey look!  $\pi_2(X)$  is abelian! **And I know why!**

# Size Matters

## First Rule of Set Theory

You don't talk about the "Set of all sets".

## Obvious Question

Who would want to, and why?

## Non-obvious Answer

Algebraic topologists, that's who.

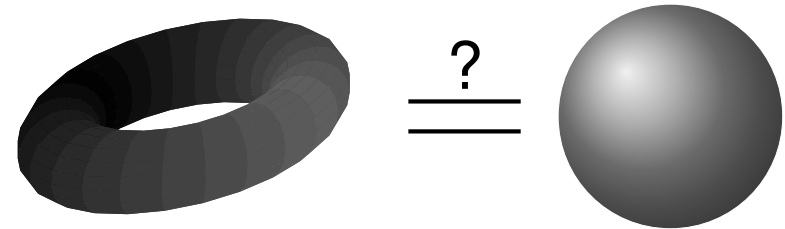
# Fundamental Problem of Algebraic Topology

## Question

Is a doughnut a donut?

## Simultaneous Translation

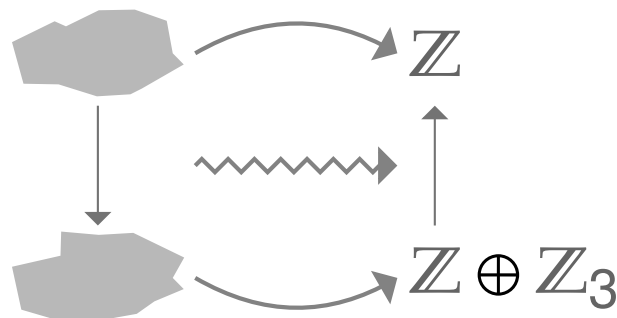
Er en smultring en berlinerbolle?



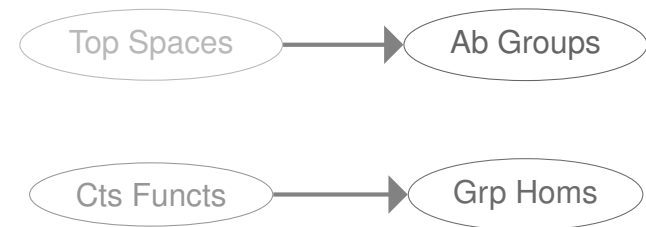
# Cohomology

## Solution

Use Cohomology



# Cohomology



Looks like ... Functions!

But they **aren't**

Domains, codomains not **sets**

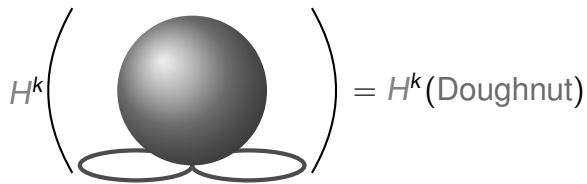
So what?

# Naturally Speaking

Cohomology is okay on a **case by case** basis  
But that's not enough!

- ▶  $H^1(\text{Doughnut}) = \mathbb{Z}^2$
- ▶  $H^1(\text{Donut}) = \{0\}$

$\implies \text{Doughnut} \neq \text{Donut}$

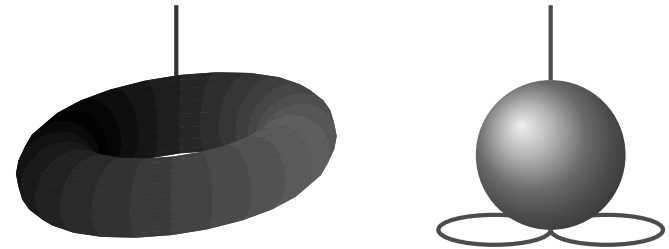


But **not** the same

# Naturally Speaking

Cohomology is highly structured:

1.  $H^k(X)$  abelian group  
Doughnut  $\neq$  Donut
2.  $H^*(X)$  graded commutative ring  
Doughnut  $\neq$  Donut + String
3.  $H^*(X)$  action of **operations**  
 $\Sigma \text{ Doughnut} \neq \Sigma(\text{Donut} + \text{String})$



# Cohomology Operations

## Definition

To each *topological space*  
 $X$   
a *group homomorphism*  
 $\theta_X: H^k(X) \rightarrow H^l(X)$   
such that  
for every *continuous function*  
 $f: X \rightarrow Y$

$$\begin{array}{ccc}
 H^k(X) & \xrightarrow{\theta_X} & H^l(X) \\
 \uparrow H^k(f) & & \uparrow H^l(f) \\
 H^k(Y) & \xrightarrow{\theta_Y} & H^l(Y)
 \end{array}$$

“for every”

## Making a Pearl

### The Grit

To say “for every” means **sets form a set**  
 $\implies$  paradox

### The Disgusting Part

Isolate the bits of Set Theory that lead to the paradox and **contain them**

### The Pearl

Can talk of “functions” between “big sets”.

## The Deal

### A Category has:

- Objects,  $X$ . Possibly **lots**.
- Arrows,  $X \rightarrow Y$ . Possibly **lots**.  
But not **too** many between two objects.  
Act a bit like functions.

### The Deal

Don't examine **objects** too closely.

### The Consequence

A thing is not what it **is**,  
nor what it **does**,  
but how it **relates** to other things.

## More Grit

### Question

Why is  $\pi_1(X)$  a group?

### Answer

Because it is.

$$f, g: S^1 \rightarrow X, \quad (f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

### Question

But **Why?** And why not  $\pi_0(X)$ ?

## Groups

### Question

What is a group?

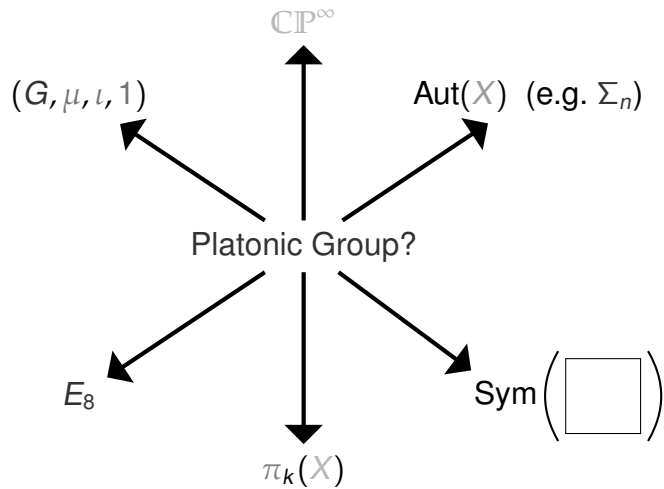
### Old Answer

Set:  $G$   
 $\mu: G \times G \rightarrow G$   
Operations:  $\iota: G \rightarrow G$   
 $1 \in G$   
(plus identities)

### New Answer?

An object of Groups

# Groups In Nature



# The Platonic Group

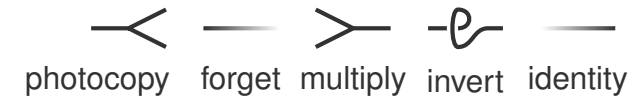
## Theorem (Lawvere)

*There is a Platonic Group!*

It's a **CATEGORY!**

- ▶ Objects:  $\mathbb{N}$
- ▶ Arrows:  $m \rightarrow n$   
All ways of producing  $n$ -elements from  $m$ -elements in all groups.

Generated by:



# Group Objects

## Definition

A  
group object  
in a category  
is a **shadow** of the Platonic group.

A  
co-group object  
in a category  
is a **shadow** of the Platonic group in a **mirror**.

# They Do It With Mirrors, Y'Know

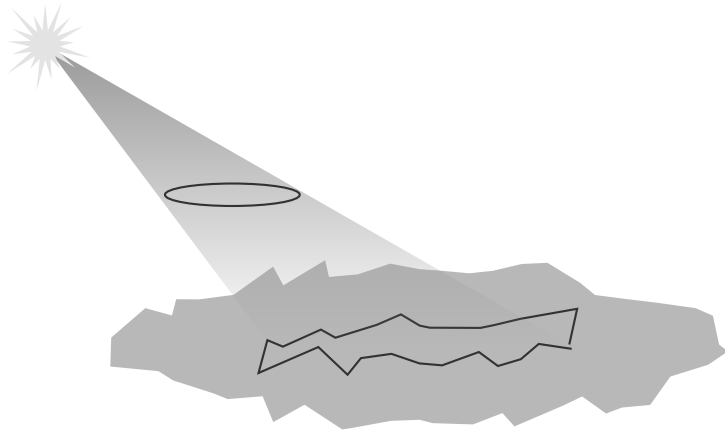
## Theorem

- ▶ A **shadow** of a **shadow** of the Platonic group is a **shadow** of the Platonic group.
- ▶ **Mirrors** cancel in pairs.
- ▶ A shadow of the Platonic group in **Set** is a group.

## Shadows of the Circle

$S^1$ : shadow of Platonic group in mirror

$\implies$  mirrored shadows of  $S^1$  form groups



## The Answer From the Shadows

- ▶  $\pi_1(X)$  group:  $S^1$  shadow of Platonic group
- ▶  $\pi_0(X)$  not group:  $S^0$  not shadow of Platonic group

## Aside

### Remark

Nothing special about groups!

Same works for

- ▶ Abelian groups
- ▶ Rings
- ▶ Algebras
- ▶ Lattices
- ▶ Compact Hausdorff Spaces
- ▶ Banach Spaces (almost)
- ▶ Banach Algebras (almost)
- ▶  $C^*$ -Algebras (exactly!)

## Making a Pearl

### The Grit

Group-like things appear everywhere.

### The Disgusting Part

Constructing a “Platonic Group” seems far from intuition of “a group”.

### The Pearl

Can prove Big Theorems in one go.

Can link ideas in group theory, ring theory,  $C^*$ -algebra theory ...

## Yet More Grit

- ▶  $\pi_1(X)$  group:  $S^1$  shadow of Platonic group
- ▶  $\pi_2(X)$  abelian:  $S^2$  shadow of Platonic abelian group

### Question

Why is  $\pi_2(X)$  abelian?

### Answer

Because it is.

## Three Views

$$\pi_2(X) \cong \pi_1(\Omega X) \cong \pi_0(\Omega^2 X)$$

### Question

Which is the best description?

- ▶  $S^1$  is a (mirror of) a shadow of a group
- ▶  $\Omega X$  is a shadow of a group

So  $\pi_2(\Omega X)$  is a group for **TWO** reasons!

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$$

$\implies$  same multiplication **and** commutative

## Making a Pearl

### The Grit

$\pi_2(X)$  is abelian, but **why**?

### The Disgusting Part

Tracing arrows round a diagram.

### The Pearl

The Eckmann–Hilton Dance!

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$$

## Summary

- ▶ Categories provide context
- ▶ Categories link concepts
- ▶ Categories are **everywhere**
  - ▶ “Large” categories, *Set*, *Top*
  - ▶ “Small” categories:
    - ▶ Partially ordered sets
    - ▶ Groups
    - ▶ Metric spaces (!)
    - ▶ Projective spaces (!)
- ▶ Key concept:  
Things are defined by their **relationship** to other things
- ▶ Categories contain beautiful pearls