

Category Theory

Making Mathematical Pearls Since the 1940s

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Disclaimer

I Am Not A Historian
I Am Not An Expert

Making Pearls: A Recipe

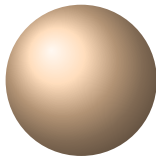
1. Find something irritating



2. Cover it with something disgusting



3. Leave, for someone else to find beautiful



Making Mathematical Pearls

1. Find something irritating

$$A = \{B : B \notin B\}, \quad A \in A?$$

2. Cover it with something disgusting

Axiom: There is an inaccessible cardinal

3. Leave, for someone else to find beautiful

Hey look! $\pi_2(X)$ is abelian! **And I know why!**

Size Matters

First Rule of Set Theory

You don't talk about the "Set of all sets".

Obvious Question

Who would want to, and why?

Non-obvious Answer

Algebraic topologists, that's who.

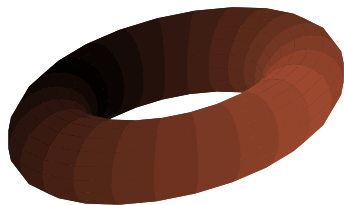
Fundamental Problem of Algebraic Topology

Question

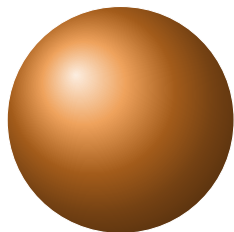
Is a doughnut a donut?

Simultaneous Translation

Er en smultring en berlinerbolle?



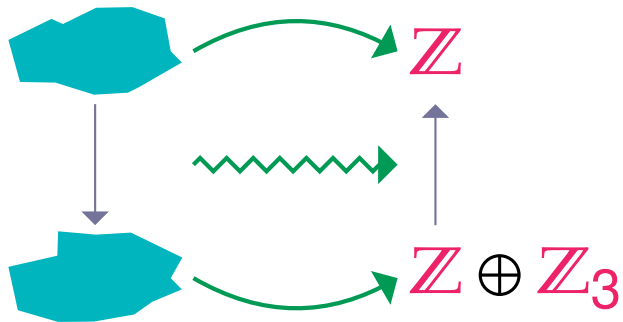
?



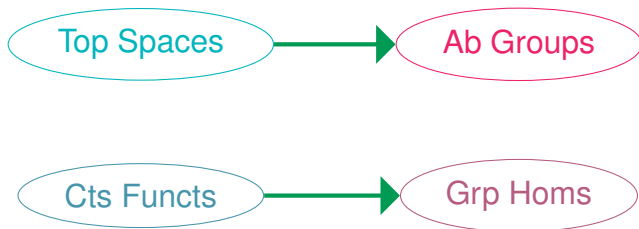
Cohomology

Solution

Use Cohomology



Cohomology



Looks like ... Functions!

But they **aren't**

Domains, codomains not **sets**

So what?

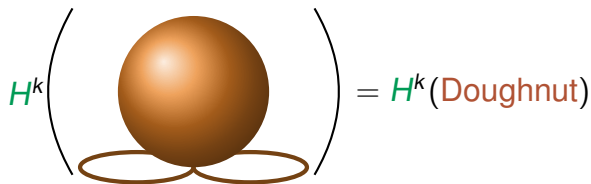
Naturally Speaking

Cohomology is okay on a **case by case** basis
But that's not enough!

▶ $H^1(\text{Doughnut}) = \mathbb{Z}^2$

▶ $H^1(\text{Donut}) = \{0\}$

$\implies \text{Doughnut} \neq \text{Donut}$

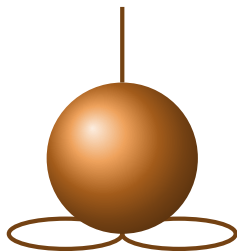
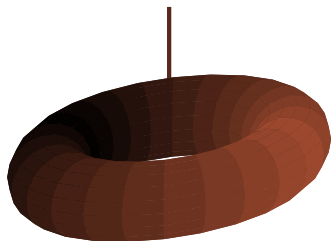


But **not** the same

Naturally Speaking

Cohomology is highly structured:

1. $H^k(X)$ abelian group
Doughnut \neq Donut
2. $H^*(X)$ graded commutative ring
Doughnut \neq Donut + String
3. $H^*(X)$ action of **operations**
 Σ Doughnut $\neq \Sigma$ (Donut + String)



Cohomology Operations

Definition

To each *topological space*
 X

a *group homomorphism*

$$\theta_X: H^k(X) \rightarrow H^l(X)$$

such that

for every *continuous function*

$$f: X \rightarrow Y$$

$$\begin{array}{ccc} H^k(X) & \xrightarrow{\theta_X} & H^l(X) \\ \uparrow H^k(f) & & \uparrow H^l(f) \\ H^k(Y) & \xrightarrow{\theta_Y} & H^l(Y) \end{array}$$

“for every”

Making a Pearl

The Grit

To say “for every” means **sets form a set**
 \implies paradox

The Disgusting Part

Isolate the bits of Set Theory that lead to the paradox and
contain them

The Pearl

Can talk of “functions” between “big sets”.

The Deal

A Category has:

- ▶ Objects, X . Possibly **lots**.
- ▶ Arrows, $X \rightarrow Y$. Possibly **lots**.
But not **too** many between two objects.
Act a bit like functions.

The Deal

Don't examine **objects** too closely.

The Consequence

A thing is not what it **is**,
nor what it **does**,
but how it **relates** to other things.

More Grit

Question

Why is $\pi_1(X)$ a group?

Answer

Because it is.

$$f, g: S^1 \rightarrow X, \quad (f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

Question

But **Why?** And why not $\pi_0(X)$?

Groups

Question

What is a group?

Old Answer

Set: G

$$\mu: G \times G \rightarrow G$$

Operations: $\iota: G \rightarrow G$

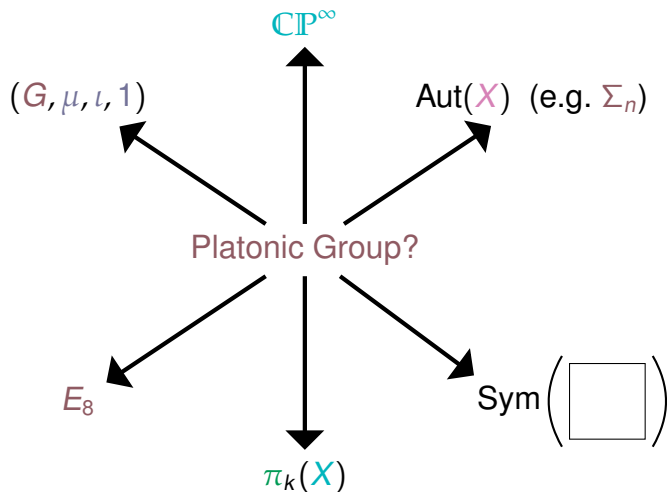
$$1 \in G$$

(plus identities)

New Answer?

An object of **Groups**

Groups In Nature



The Platonic Group

Theorem (Lawvere)

There is a Platonic Group!

It's a **CATEGORY!**

- ▶ Objects: \mathbb{N}
- ▶ Arrows: $m \rightarrow n$
All ways of producing n -elements from m -elements in all groups.

Generated by:



Group Objects

Definition

A
group object
in a category
is a *shadow* of the Platonic group.

A
co-group object
in a category
is a *shadow* of the Platonic group in a *mirror*.

They Do It With Mirrors, Y'Know

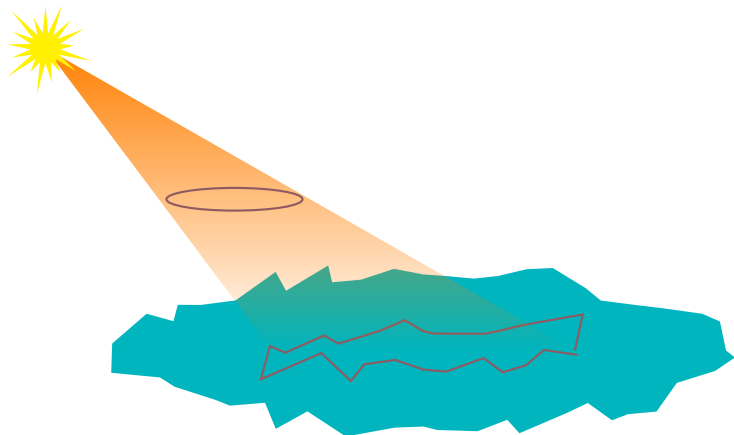
Theorem

- ▶ A *shadow* of a *shadow* of the *Platonic group* is a *shadow* of the *Platonic group*.
- ▶ *Mirrors* cancel in pairs.
- ▶ A shadow of the *Platonic group* in *Set* is a *group*.

Shadows of the Circle

S^1 : shadow of Platonic group in mirror

\implies mirrored shadows of S^1 form groups



The Answer From the Shadows

- ▶ $\pi_1(X)$ group: S^1 shadow of Platonic group
- ▶ $\pi_0(X)$ not group: S^0 not shadow of Platonic group

Aside

Remark

Nothing special about groups!

Same works for

- ▶ Abelian groups
- ▶ Rings
- ▶ Algebras
- ▶ Lattices
- ▶ Compact Hausdorff Spaces
- ▶ Banach Spaces (almost)
- ▶ Banach Algebras (almost)
- ▶ C^* -Algebras (exactly!)

Making a Pearl

The Grit

Group-like things appear everywhere.

The Disgusting Part

Constructing a “Platonic Group” seems far from intuition of “a group”.

The Pearl

Can prove Big Theorems in one go.

Can link ideas in group theory, ring theory, C^* -algebra theory . . .

Yet More Grit

- ▶ $\pi_1(X)$ group: S^1 shadow of Platonic group
- ▶ $\pi_2(X)$ abelian: S^2 shadow of Platonic abelian group

Question

Why is $\pi_2(X)$ abelian?

Answer

Because it is.

Three Views

$$\pi_2(X) \cong \pi_1(\Omega X) \cong \pi_0(\Omega^2 X)$$

Question

Which is the best description?

- ▶ S^1 is a (mirror of) a shadow of a group
- ▶ ΩX is a shadow of a group

So $\pi_2(\Omega X)$ is a group for **TWO** reasons!

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$$

\implies same multiplication **and** commutative

Making a Pearl

The Grit

$\pi_2(X)$ is abelian, but why?

The Disgusting Part

Tracing arrows round a diagram.

The Pearl

The Eckmann–Hilton Dance!

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$$

Summary

- ▶ Categories provide context
- ▶ Categories link concepts
- ▶ Categories are **everywhere**
 - ▶ “Large” categories, **Set**, **Top**
 - ▶ “Small” categories:
 - ▶ Partially ordered sets
 - ▶ Groups
 - ▶ Metric spaces (!)
 - ▶ Projective spaces (!)
- ▶ Key concept:
Things are defined by their **relationship** to other things
- ▶ Categories contain beautiful pearls