Category Theory
Making Mathematical Pearls Since the 1940s

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I Am Not An Expert

I Am Not A Historian

Disclaimer
Making Pearls: A Recipe

1. Find something irritating

2. Cover it with something disgusting

3. Leave, for someone else to find beautiful
Making Mathematical Pearls

1. Find something irritating

\[ A = \{ B : B \notin B \}, \quad A \in A? \]

2. Cover it with something disgusting

**Axiom:** There is an inaccessible cardinal

3. Leave, for someone else to find beautiful

Hey look! \( \pi_2(X) \) is abelian! And I know why!
Size Matters

First Rule of Set Theory
You don’t talk about the “Set of all sets”.

Obvious Question
Who would want to, and why?

Non-obvious Answer
Algebraic topologists, that’s who.
Question

Is a doughnut a donut?

Simultaneous Translation

Er en smultring en berlinerbolle?
Cohomology

Solution

Use Cohomology

\[ \mathbb{Z} \oplus \mathbb{Z}_3 \]
Cohomology

Top Spaces \rightarrow Ab Groups

Cts Functs \rightarrow Grp Homs

Looks like \ldots Functions!

But they aren't

Domains, codomains not sets

So what?
Naturally Speaking

Cohomology is okay on a case by case basis
But that’s not enough!

- $H^1(\text{Doughnut}) = \mathbb{Z}^2$
- $H^1(\text{Donut}) = \{0\}$

$\implies \text{Doughnut} \neq \text{Donut}$

$H^k(\text{Doughnut})$

But not the same
Naturally Speaking

Cohomology is highly structured:

1. $H^k(X)$ abelian group
   Doughnut $\neq$ Donut

2. $H^*(X)$ graded commutative ring
   Doughnut $\neq$ Donut + String

3. $H^*(X)$ action of operations
   $\Sigma$ Doughnut $\neq \Sigma(\text{Donut + String})$
Cohomology Operations

Definition

To each topological space $X$

a group homomorphism

$\theta_X: H^k(X) \rightarrow H^l(X)$

such that

for every continuous function $f: X \rightarrow Y$

\[ H^k(X) \xrightarrow{\theta_X} H^l(X) \]

\[ H^k(Y) \xrightarrow{\theta_Y} H^l(Y) \]
“for every”
Making a Pearl

The Grit
To say “for every” means sets form a set
\[\implies\text{paradox}\]

The Disgusting Part
Isolate the bits of Set Theory that lead to the paradox and contain them

The Pearl
Can talk of “functions” between “big sets”.

The Deal

A Category has:

- Objects, \( X \). Possibly lots.
- Arrows, \( X \rightarrow Y \). Possibly lots.
  But not too many between two objects.
  Act a bit like functions.

The Deal

Don’t examine objects too closely.

The Consequence

A thing is not what it is, nor what it does,
but how it relates to other things.
More Grit

Question

Why is $\pi_1(X)$ a group?

Answer

Because it is.

$$f, g: S^1 \to X, \quad (f \ast g)(t) = \begin{cases} 
  f(2t) & 0 \leq t \leq \frac{1}{2} \\
  g(2t - 1) & \frac{1}{2} \leq t \leq 1 
\end{cases}$$

Question

But Why? And why not $\pi_0(X)$?
Question

What is a group?

Old Answer

Set: $G$

Operations: $\mu : G \times G \rightarrow G$

$\iota : G \rightarrow G$

$1 \in G$

(plus identities)

New Answer?

An object of Groups
Groups In Nature

$\text{Gr}(G, \mu, \iota, 1)$

$\mathbb{CP}^\infty$

$\text{Aut}(X)$ (e.g. $\Sigma_n$)

$E_8$

$\pi_k(X)$

$\text{Sym}(\square)$

Platonic Group?
The Platonic Group

Theorem (Lawvere)

There is a Platonic Group!

It’s a CATEGORY!

- Objects: \( \mathbb{N} \)
- Arrows: \( m \to n \)
  All ways of producing \( n \)-elements from \( m \)-elements in all groups.

Generated by:

- photocopy
- forget
- multiply
- invert
- identity
Group Objects

Definition

A group object in a category is a shadow of the Platonic group.

A co-group object in a category is a shadow of the Platonic group in a mirror.
They Do It With Mirrors, Y’Know

Theorem

- A shadow of a shadow of the Platonic group is a shadow of the Platonic group.
- Mirrors cancel in pairs.
- A shadow of the Platonic group in Set is a group.
Shadows of the Circle

$S^1$: shadow of Platonic group in mirror

$\rightarrow$ mirrored shadows of $S^1$ form groups
The Answer From the Shadows

- $\pi_1(X)$ group: $S^1$ shadow of Platonic group
- $\pi_0(X)$ not group: $S^0$ not shadow of Platonic group
Aside

Remark

Nothing special about groups!

Same works for

- Abelian groups
- Rings
- Algebras
- Lattices
- Compact Hausdorff Spaces
- Banach Spaces (almost)
- Banach Algebras (almost)
- $C^*$–Algebras (exactly!)
Making a Pearl

The Grit
Group-like things appear everywhere.

The Disgusting Part
Constructing a “Platonic Group” seems far from intuition of “a group”.

The Pearl
Can prove Big Theorems in one go.
Can link ideas in group theory, ring theory, $C^*$—algebra theory . . .
Yet More Grit

- $\pi_1(X)$ group: $S^1$ shadow of Platonic group
- $\pi_2(X)$ abelian: $S^2$ shadow of Platonic abelian group

Question

Why is $\pi_2(X)$ abelian?

Answer

Because it is.
Three Views

\[ \pi_2(X) \cong \pi_1(\Omega X) \cong \pi_0(\Omega^2 X) \]

Question

Which is the best description?

- \( S^1 \) is a (mirror of) a shadow of a group
- \( \Omega X \) is a shadow of a group

So \( \pi_2(\Omega X) \) is a group for TWO reasons!

\[
(a \circ b) \ast (c \circ d) = (a \ast c) \circ (b \ast d)
\]

\[ \implies \text{same multiplication and commutative} \]
Making a Pearl

The Grit

\[ \pi_2(X) \] is abelian, but why?

The Disgusting Part

Tracing arrows round a diagram.

The Pearl

The Eckmann–Hilton Dance!

\[
(a \circ b) \ast (c \circ d) = (a \ast c) \circ (b \ast d)
\]
Summary

- Categories provide context
- Categories link concepts
- Categories are everywhere
  - “Large” categories, Set, Top
  - “Small” categories:
    - Partially ordered sets
    - Groups
    - Metric spaces (!)
    - Projective spaces (!)
- Key concept:
  Things are defined by their relationship to other things
- Categories contain beautiful pearls