Delooing Moravian Maps
Stable and Unstable Operations in the Morava
K–theories
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21st British Topology Meeting

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Questions

1. When is an unstable cohomology operation a component of a stable one?
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2. If we have a component of a stable operation, can we construct the other components?
Let $E^*(-)$ be a graded, generalised cohomology theory.
Preliminaries

Let $E^*(-)$ be a graded, generalised cohomology theory.

Contravariant functor $E^*(-) : \text{Top} \rightarrow \text{GAb}$

- $X$ topological space $\mapsto E^*(X)$, graded abelian group.
- $f : X \rightarrow Y$ continuous $\mapsto f^* : E^*(Y) \rightarrow E^*(X)$ of graded abelian groups (degree zero), with $(fg)^* = g^*f^*$. 
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- $E^*(-)$ intertwines suspensions: $E^k(\Sigma X) \cong (\Sigma E^*(X))^k = E^{k-1}(X)$, natural in $X$. 
Forgetfulness

Three views of $E^*(-)$:
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- One functor, $E^*(-)$, into graded abelian groups.
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- A family of functors, $\{E^k(-)\}$, into abelian groups.
Forgetfulness

Three views of $E^*(-)$:

- One functor, $E^*(-)$, into graded abelian groups.
- A family of functors, $\{E^k(-)\}$, into abelian groups.
- A family of functors, $\{E^k(-)\}$, into sets.
An operation is a natural transformation between functors.
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\( F, G : \mathcal{C} \to \mathcal{D} \) contravariant.
\( \nu : F \to G \) is:
for every \( \mathcal{C} \)-object \( X \), \( \nu_X : F(X) \to G(X) \) such that:

\[
\begin{align*}
F(X) \xrightarrow{\nu_X} G(X) \\
F(f) \uparrow & \quad \Uparrow \\
F(Y) \xrightarrow{\nu_Y} G(Y)
\end{align*}
\]
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- **Stable**: \( r : E^*(-) \to E^*(-) \) of graded abelian groups, respecting suspension.

\[ S^h \]
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- **Stable:** $r : E^*(-) \rightarrow E^*(-)$ of graded abelian groups, respecting suspension.

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- **Unstable:** $r : E^k(-) \rightarrow E^l(-)$ of sets.
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\[ S^h \to A_{k}^{k+h} \subseteq U_{k}^{k+h} \]
Examples

- Coefficient operations on $E^*(-)$: $n(x) = nx$;
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- Bott periodicity in K-theory: $\beta : K^{k+2}(X) \xrightarrow{\simeq} K^k(X)$.
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- Multiplication operations on $H^*(-)$: $x \mapsto x^k$;
- Steenrod squares on $H^*(-; \mathbb{F}_2)$.
- Bott periodicity in $K$–theory: $\beta : K^{k+2}(X) \xrightarrow{\cong} K^k(X)$;
- Adams operations in $K$–theory: for $k \in \mathbb{Z}$, $\Psi^k : K^0(X) \to K^0(X)$.
  $\Psi^k(L) = L \otimes^k$, $\Psi^k(V \oplus W) = \Psi^k(V) \oplus \Psi^k(W)$. 
Questions

1. When is an unstable cohomology operation a component of a stable one?

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Looping

Consider an unstable operation:

\[ r : E^k(-) \rightarrow E^l(-) \]
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\[ r : E^k(\_) \to E^l(\_) \]

Define a new operation:

\[ \Omega r : E^{k-1}(\_) \to E^{l-1}(\_) \]

by:

\[ (\Omega r)_X : E^{k-1}(X) \cong E^k(\Sigma X) \xrightarrow{r_{\Sigma X}} E^l(\Sigma X) \cong E^{l-1}(X) \]
Proposition

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2. If $r$ is the $k$th component of a stable operation, $(-1)^{l-k}\Omega r$ is the $(k - 1)$th component.
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Higher components are the hard part.
That is, to deloop the operation $r$.
Mild help: often have a uniqueness theorem.
Example: K–theory

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\[
\begin{array}{cccccc}
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
\downarrow r & & & & \\
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X)
\end{array}
\]
Example: K–theory

K–theory:

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\begin{array}{cccccc}
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
\downarrow \Omega^2r & \downarrow \Omega r & \downarrow r & & & \\
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X)
\end{array}
\]
Example: \( \text{K-theory} \)

K-theory: 2–periodic.

\[
\begin{array}{cccccc}
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
\Omega^2 r & \Omega r & r & & \\
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
\beta^{-1} & \beta & & & \\
\end{array}
\]
Example: $K$–theory


$$
\begin{array}{cccccc}
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
\downarrow \Omega^2 r & \downarrow \Omega r & \downarrow r & \downarrow \beta^{-1} r \beta & \\
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X)
\end{array}
$$

$$
\begin{array}{c}
\xymatrix{K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
\downarrow \Omega^2 r & \downarrow \Omega r & \downarrow r & \downarrow \beta^{-1} r \beta & \\
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X)}
\end{array}
$$

Question: $r = \Omega^2 r (\beta^{-1} r \beta) ?$
Example: K–theory


\[ K^{-2}(X) \quad K^{-1}(X) \quad K^0(X) \quad K^1(X) \quad K^2(X) \]

\[ \xrightarrow{\Omega^2 r} \quad \xrightarrow{\Omega r} \quad \xrightarrow{r} \quad \xrightarrow{\Omega(\beta^{-1} r\beta)} \quad \xrightarrow{\beta^{-1} r\beta} \]

\[ \xleftarrow{\beta^{-1}} \]
Example: K–theory


\[
\begin{array}{cccccc}
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\Omega^2 r & \Omega r & r & \Omega(\beta^{-1} r \beta) & \beta^{-1} r \beta \\
K^{-2}(X) & K^{-1}(X) & K^0(X) & K^1(X) & K^2(X) \\
& & \beta^{-1} & & \\
& \beta & & & \\
\end{array}
\]

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K^{-2}(X) & K^0(X) & K^2(X) \\
\downarrow k\Psi^k & \downarrow \Psi^k & \\
K^{-2}(X) & K^0(X) & K^2(X)
\end{array} \]
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\[ \begin{array}{ccc}
K^{-2}(X) & K^0(X) & K^2(X) \\
\downarrow k\Psi^k & \downarrow \Psi^k & \downarrow \frac{1}{k}\Psi^k \\
K^{-2}(X) & K^0(X) & K^2(X)
\end{array} \]
Adams Operations

$$\Omega^2(\beta^{-1}\Psi^k\beta) = k\Psi^k$$

$\begin{align*}
K^{-2}(X) & \quad K^0(X) & \quad K^2(X) \\
\downarrow k\Psi^k & \quad \downarrow \Psi^k & \quad \downarrow \frac{1}{k}\Psi^k \\
K^{-2}(X) & \quad K^0(X) & \quad K^2(X)
\end{align*}$

$\frac{1}{k}\Psi^k$ not an operation on $K^0(-)$ (unless $k = 1$ or $k = -1$)
Coefficients

How to divide by $k$: introduce coefficients. $R$ a commutative, unital ring $K(−; R)$ K–theory with coefficients in $R$. 

Examples:
1. $R = \mathbb{Q}$, but $K^∗(−; \mathbb{Q}) \cong H^∗(−; \mathbb{Q})$
2. $R = \mathbb{Z}(p)$ retains $p$–typical information $\Psi_k$ is stable if $p \nmid k$ (see work of Clarke, Crossley, and Whitehouse)
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Coefficients

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Examples

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2. \( R = \mathbb{Z}_p \) retains \( p \)–typical information
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Mod $p$

**Warning:** $K(X; \mathbb{F}_p) \neq K(X)/(p)$.
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$p = 3, \ k = 11$

\[
\begin{array}{ccc}
K^0(X; \mathbb{F}_p) & K^2(X; \mathbb{F}_p) & K^4(X; \mathbb{F}_p) \\
\downarrow^{\Psi^{11}} & & \\
K^0(X; \mathbb{F}_p) & K^2(X; \mathbb{F}_p) & K^4(X; \mathbb{F}_p)
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\downarrow^{\Psi^1} & \downarrow^{\frac{1}{11}\Psi^1} & \\
K^0(X; \mathbb{F}_p) & K^2(X; \mathbb{F}_p) & K^4(X; \mathbb{F}_p)
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\begin{array}{ccc}
K^0(X; \mathbb{F}_p) & \xrightarrow{\Psi^{11}} & K^2(X; \mathbb{F}_p) \\
\psi^{11} & & \frac{1}{11} \psi^{11} = 2 \psi^{11} \\
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K^0(X; \mathbb{F}_p) & K^2(X; \mathbb{F}_p) & K^4(X; \mathbb{F}_p)
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K^0(X; \mathbb{F}_p) & K^2(X; \mathbb{F}_p) & K^4(X; \mathbb{F}_p)
\end{array}
\]
Answers

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In \( K^*(-; \mathbb{F}_p) \), for \( p \nmid k \), \( \Psi^k \) repeats with period \( 2(p - 1) \) (Fermat)
Answers

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Proposition

In \( K^*(-; \mathbb{F}_p) \):

1. \( \Psi^k \) is stable if and only if \( p \nmid k \);
$\Psi^{11}$ repeats with period 4 = $2(3 - 1)$

In $K^*(-; \mathbb{F}_p)$, for $p \nmid k$, $\Psi^k$ repeats with period $2(p - 1)$ (Fermat)

**Proposition**

In $K^*(-; \mathbb{F}_p)$:

1. $\Psi^k$ is stable if and only if $p \nmid k$;
2. If $\Psi^k$ is stable the (even) components are blocks of:

   $(\Psi^k, k^{p-2}\Psi^k, k^{p-3}\Psi^k, k^{p-4}\Psi^k, \ldots, k\Psi^k)$. 

All Operations

Theorem (S – Whitehouse)

The components of a stable operation on $K^*(-; \mathbb{F}_p)$ repeat with periodicity $2(p - 1)$. 
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Corollary

If $\Omega^{2(p-1)}(\beta^{-(p-1)}r\beta^{p-1}) = r$ then $r$ is a component of a stable operation.
Reconstruction

Start with a component of a stable operation.

\[
K^k(X; \mathbb{F}_p) \xrightarrow{r_k} K^{k+h}(X; \mathbb{F}_p)
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Reconstruction

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\[ K^k(X; \mathbb{F}_p) \xrightarrow{r_k} K^{k+h}(X; \mathbb{F}_p) \]
\[ K^l(X; \mathbb{F}_p) \xrightarrow{r_l} K^{l+h}(X; \mathbb{F}_p) \]
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Theorem (S – Whitehouse)
Let $r$ be an unstable operation on $K^*(-; \mathbb{F}_p)$. Then $r$ is a component of a stable operation if (and only if) there is an unstable operation $s$ with $r = \Omega s$. 
Corollary

If $\Omega^2(p-1)(\beta^{-1}(p-1)r\beta^{p-1}) = r$ then $r$ is a component of a stable operation.

Theorem (S – Whitehouse)

Let $r$ be an unstable operation on $K^*(-; \mathbb{F}_p)$. Then $r$ is a component of a stable operation if (and only if) there is an unstable operation $s$ with $r = \Omega s$.

That is, if $r$ deloops once then it deloops as many times as we like.
Morava K–theories

For each prime $p$, a sequence of cohomology theories $\{K(n)^*(-)\}$ – the chromatic filtration.
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- $K(0)^*(-) = H^*(-; \mathbb{Q})$
- $K(1)^*(-)$ summand of $K^*(-; \mathbb{F}_p)$
Morava K–theories

For each prime $p$, a sequence of cohomology theories $\{K(n)^*(-)\}$ – the chromatic filtration.

- $K(0)^*(-) = H^*(-; \mathbb{Q})$
- $K(1)^*(-)$ summand of $K^*(-; \mathbb{F}_p)$
- $K(n)^*(-)$:
  - is periodic, period $2(p^n - 1)$
  - has coefficients $\mathbb{F}_p[v_n, v_n^{-1}], |v_n| = -2(p^n - 1)$
  - has Künneth formula and duality
Theorem (S – Whitehouse)

1. The components of a stable operation in $K(n)^*(-)$ repeat with periodicity $2(p^n - 1)$;
Theorem (S – Whitehouse)

1. The components of a stable operation in $K(n)^*(-)$ repeat with periodicity $2(p^n - 1)$;
2. If $r$ is an unstable operation such that there is another unstable operation $s$ with $r = \Omega s$ then $r$ is a component of a (unique) stable operation.
Notes

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2. The periodicity is always that of the cohomology theory.
(Compare with $K^*(-; \mathbb{F}_p)$)
Notes

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2. The periodicity is always that of the cohomology theory.
   (Compare with $K^*(-; \mathbb{F}_p)$)

3. The “delooping” condition has not changed: if we can deloop once we can deloop as many times as we like.
Remarks

1. Projection $P : \mathcal{U}_k^l \rightarrow \mathcal{U}_k^l$ via:

$$Pr = \Omega^{2(p^n - 1)}(v_n^{-1}rv_n)$$

such that $r$ is a component of a stable operation if and only if $r = Pr$. 

Closely linked to the Bousfield–Kuhn functor.

Reconstruction is easy using periodicity.

Proof is a straightforward analysis of the $p$–series of the formal group law.
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Questions

For the Morava K–theories:

1. When is an unstable cohomology operation a component of a stable one?
   
   If it can be delooped once.

2. If we have a component of a stable operation, can we construct the other components?
   
   Yes; easily, using the periodicity.
Further Reading

Unstable operations in generalized cohomology.

J. Michael Boardman.
Stable operations in generalized cohomology.

A. K. Bousfield.
Uniqueness of infinite deloopings for $K$-theoretic spaces.

Francis Clarke, M. D. Crossley, and Sarah Whitehouse.
Bases for cooperations in $K$-theory.

Francis Clarke, Martin Crossley, and Sarah Whitehouse.
Algebras of operations in $K$-theory.
Further Reading

The Morava $K$-theory Hopf ring for $BP$.

Nicholas J. Kuhn.
Morava $K$-theories and infinite loop spaces.

The Hopf ring for complex cobordism.

Andrew Stacey and Sarah Whitehouse.

W. Stephen Wilson.
The Hopf ring for Morava $K$-theory.