Delooping Moravian Maps

Stable and Unstable Operations in the Morava K–theories

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Questions and Answers

1. When is an unstable cohomology operation a component of a stable one?

2. If we have a component of a stable operation, can we construct the other components?

Outline

Preliminaries

Let $E^*(-)$ be a graded, generalised cohomology theory.

Contravariant functor $E^*(-) : \mathrm{Top} \to \mathrm{GAb}$

- X topological space $\rightsquigarrow$ $E^*(X)$, graded abelian group.
- $f : X \to Y$ continuous $\rightsquigarrow$ $f^* : E^*(Y) \to E^*(X)$ of graded abelian groups (degree zero), with $(fg)^* = g^* f^*$.
- $E^*(-)$ intertwines suspensions: $E^k(\Sigma X) \cong (\Sigma E^*(X))^k = E^{k-1}(X)$, natural in X.

Forgetfulness

Three views of $E^*(-)$:

- One functor, $E^*(-)$, into graded abelian groups.
- A family of functors, $\{E^k(-)\}$, into abelian groups.
- A family of functors, $\{E^k(-)\}$, into sets.

Operations

An operation is a natural transformation between functors.
$F, G : \mathcal{C} \to \mathcal{D}$ contravariant.
$\nu : F \to G$ is:
for every $\mathcal{C}$–object X, $\nu_X : F(X) \to G(X)$ such that:

$$\begin{tikzcd}
F(X) \arrow[r, "\nu_X"] & G(X) \\
F(Y) \arrow[u, "F(f)"] \arrow[r, "\nu_Y"] \arrow[ur, phantom, "\circlearrowleft"] & G(Y) \arrow[u, "G(f)"']
\end{tikzcd}$$

Operations

There are three types of operation:

- Stable: $r : E^*(-) \to E^*(-)$ of graded abelian groups, respecting suspension. $\mathcal{S}^h$

- Additive: $r : E^k(-) \to E^l(-)$ of abelian groups. $\mathcal{A}^l_k$

- Unstable: $r : E^k(-) \to E^l(-)$ of sets. $\mathcal{U}^l_k$

$$\mathcal{S}^h \to \mathcal{A}^{k+h}_k \subseteq \mathcal{U}^{k+h}_k$$

Examples

- Coefficient operations on $E^*(-)$: $n(x) = nx$;
- Multiplication operations on $H^*(-)$: $x \mapsto x^k$
- Steenrod squares on $H^*(-;\mathbb{F}_2)$.
- Bott periodicity in K–theory: $\beta : K^{k+2}(X) \xrightarrow{\cong} K^k(X)$;
- Adams operations in K–theory: for $k \in \mathbb{Z}$,
  $\Psi^k : K^0(X) \to K^0(X)$.
  $\Psi^k(L) = L^{\otimes k}$, $\Psi^k(V \oplus W) = \Psi^k(V) \oplus \Psi^k(W)$.

Questions and Answers

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Looping

Consider an unstable operation:

$$r : E^k(-) \to E^l(-)$$

Define a new operation:

$$\Omega r : E^{k-1}(-) \to E^{l-1}(-)$$

by:

$$(\Omega r)_X : E^{k-1}(X) \cong E^k(\Sigma X) \xrightarrow{r_{\Sigma X}} E^l(\Sigma X) \cong E^{l-1}(X)$$

Looping

Proposition

1. *Ωr is an unstable (additive) operation;*
2. *If r is the kth component of a stable operation, $(-1)^{l-k}\Omega r$ is the $(k-1)$th component.*

Lower components are easy.
Higher components are the hard part.
That is, to deloop the operation r.
Mild help: often have a uniqueness theorem.

Example: K–theory

K–theory: 2–periodic.

$$
\begin{tikzcd}
K^{-2}(X) \arrow[d, "\Omega^2 r"] & K^{-1}(X) \arrow[d, "\Omega r"] & K^0(X) \arrow[d, "r"] & K^1(X) \arrow[d, "\Omega(\beta^{-1} r \beta)"] & K^2(X) \arrow[d, "\beta^{-1} r \beta"] \arrow[ll, bend right, "\beta"'] \\
K^{-2}(X) & K^{-1}(X) & K^0(X) \arrow[rr, bend right, "\beta^{-1}"'] & K^1(X) & K^2(X)
\end{tikzcd}
$$

Question: $r = \Omega^2(\beta^{-1} r \beta)$?

Adams Operations

$$\Omega^2(\beta^{-1}\Psi^k\beta) = k\Psi^k$$

$$\begin{array}{ccccc}
K^{-2}(X) & & K^0(X) & & K^2(X) \\
\downarrow{\scriptstyle k\Psi^k} & & \downarrow{\scriptstyle \Psi^k} & & \downarrow{\scriptstyle \frac{1}{k}\Psi^k} \\
K^{-2}(X) & & K^0(X) & & K^2(X)
\end{array}$$

$\frac{1}{k}\Psi^k$ not an operation on $K^0(-)$
(unless $k = 1$ or $k = -1$)

Coefficients

How to divide by k: introduce coefficients.
R a commutative, unital ring
$K(-;R)$ K–theory with coefficients in R.

Examples

1. $R = \mathbb{Q}$, but $K^*(-;\mathbb{Q}) \cong H^{\pm}(-;\mathbb{Q})$
2. $R = \mathbb{Z}_{(p)}$ retains p–typical information
   Ψ^k is stable if $p \nmid k$
   (see work of Clarke, Crossley, and Whitehouse)

Mod p

Warning: $K(X;\mathbb{F}_p) \neq K(X)/(p)$.

$$p = 3,\ k = 11$$

$$\begin{tikzcd}
K^0(X;\mathbb{F}_p) \arrow[d, "\Psi^{11}"] & K^2(X;\mathbb{F}_p) \arrow[d, "\frac{1}{11}\Psi^{11}=2\Psi^{11}"] & K^4(X;\mathbb{F}_p) \arrow[d, "\frac{2}{11}\Psi^{11}=\Psi^{11}"] \\
K^0(X;\mathbb{F}_p) & K^2(X;\mathbb{F}_p) & K^4(X;\mathbb{F}_p)
\end{tikzcd}$$

Answers

Ψ^{11} repeats with period $4 = 2(3-1)$
In $K^*(-;\mathbb{F}_p)$, for $p \nmid k$, Ψ^k repeats with period $2(p-1)$ (Fermat)

Proposition

In $K^*(-;\mathbb{F}_p)$:

1. Ψ^k *is stable if and only if* $p \nmid k$*;*
2. *If* Ψ^k *is stable the (even) components are blocks of:*

$$(\Psi^k, k^{p-2}\Psi^k, k^{p-3}\Psi^k, k^{p-4}\Psi^k, \ldots, k\Psi^k).$$

All Operations

Theorem (S – Whitehouse)

The components of a stable operation on $K^*(-;\mathbb{F}_p)$ repeat with periodicity $2(p-1)$.

Corollary

If $\Omega^{2(p-1)}(\beta^{-(p-1)} r \beta^{p-1}) = r$ then r is a component of a stable operation.

Reconstruction

Start with a component of a stable operation.

$$\begin{tikzcd}
K^i(X;\mathbb{F}_p) \arrow[d, "\Omega^j r_k"] & K^k(X;\mathbb{F}_p) \arrow[d, "r_k"] & K^l(X;\mathbb{F}_p) \arrow[ll, bend right, "\beta^{m(p-1)}"'] \arrow[d, "r_l"', "=\beta^{-m(p-1)}(\Omega^j r_k)\beta^{m(p-1)}"] \\
K^{i+h}(X;\mathbb{F}_p) \arrow[rr, bend right, "\beta^{-m(p-1)}"'] & K^{k+h}(X;\mathbb{F}_p) & K^{l+h}(X;\mathbb{F}_p)
\end{tikzcd}$$

Recognition Improvement

Corollary

If $\Omega^{2(p-1)}(\beta^{-(p-1)} r \beta^{p-1}) = r$ then r is a component of a stable operation.

Theorem (S – Whitehouse)

Let r be an unstable operation on $K^*(-;\mathbb{F}_p)$. Then r is a component of a stable operation if (and only if) there is an unstable operation s with $r = \Omega s$.

That is, if r deloops once then it deloops as many times as we like.

Morava K–theories

For each prime p, a sequence of cohomology theories $\{K(n)^*(-)\}$ – the chromatic filtration.

- $K(0)^*(-) = H^*(-;\mathbb{Q})$
- $K(1)^*(-)$ summand of $K^*(-;\mathbb{F}_p)$
- $K(n)^*(-)$:
  - is periodic, period $2(p^n - 1)$
  - has coefficients $\mathbb{F}_p[v_n, {v_n}^{-1}]$, $|v_n| = -2(p^n - 1)$
  - has Künneth formula and duality

Recognition and Reconstruction

Theorem (S – Whitehouse)

1. *The components of a stable operation in* $K(n)^*(-)$ *repeat with periodicity* $2(p^n - 1)$*;*
2. *If* r *is an unstable operation such that there is another unstable operation* s *with* $r = \Omega s$ *then* r *is a component of a (unique) stable operation.*

Notes

1. The periodicity has changed to reflect the periodicity of the cohomology theory.
2. The periodicity is always that of the cohomology theory.
   (Compare with $K^*(-; \mathbb{F}_p)$)
3. The “delooping” condition has not changed: if we can deloop once we can deloop as many times as we like.

Remarks

1. Projection $P : \mathcal{U}_k^l \to \mathcal{U}_k^l$ via:

$$Pr = \Omega^{2(p^n-1)}({v_n}^{-1} r v_n)$$

   such that r is a component of a stable operation if and only if $r = Pr$.
2. Closely linked to the Bousfield–Kuhn functor.
3. Reconstruction is easy using periodicity.
4. Proof is a straightforward analysis of the p–series of the formal group law.

Questions and Answers

For the Morava K–theories:

1. When is an unstable cohomology operation a component of a stable one?

   If it can be delooped once.

2. If we have a component of a stable operation, can we construct the other components?

   Yes; easily, using the periodicity.

Further Reading

J. Michael Boardman, David Copeland Johnson, and W. Stephen Wilson.
Unstable operations in generalized cohomology.
In *Handbook of algebraic topology*, pages 687–828. North-Holland, Amsterdam, 1995.

J. Michael Boardman.
Stable operations in generalized cohomology.
In *Handbook of algebraic topology*, pages 585–686. North-Holland, Amsterdam, 1995.

A. K. Bousfield.
Uniqueness of infinite deloopings for K-theoretic spaces.
Pacific J. Math., 129(1):1–31, 1987.

Francis Clarke, M. D. Crossley, and Sarah Whitehouse.
Bases for cooperations in K-theory.
K-Theory, 23(3):237–250, 2001.

Francis Clarke, Martin Crossley, and Sarah Whitehouse.
Algebras of operations in K-theory.
Topology, 44(1):151–174, 2005.

Further Reading

Takuji Kashiwabara, Neil Strickland, and Paul Turner.
The Morava K-theory Hopf ring for BP.
In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 209–222. Birkhäuser, Basel, 1996.

Nicholas J. Kuhn.
Morava K-theories and infinite loop spaces.
In *Algebraic topology (Arcata, CA, 1986)*, volume 1370 of *Lecture Notes in Math.*, pages 243–257. Springer, Berlin, 1989.

Douglas C. Ravenel and W. Stephen Wilson.
The Hopf ring for complex cobordism.
J. Pure Appl. Algebra, 9(3):241–280, 1976/77.

Andrew Stacey and Sarah Whitehouse.
Stable and Unstable Operations in mod p Cohomology Theories, arXiv:math.AT/0605471.

W. Stephen Wilson.
The Hopf ring for Morava K-theory.
Publ. Res. Inst. Math. Sci., 20(5):1025–1036, 1984.