

The Differential Topology of Loop Spaces

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Abstract

This is an introduction to the subject of the differential topology of the space of smooth loops in a finite dimensional manifold. It began as background notes to a series of seminars given at NTNU and subsequently at Sheffield. The topics covered are: the smooth structure of the space of smooth loops; constructions involving vector bundles; submanifolds and tubular neighbourhoods; and a short introduction to the geometry and semi-infinite structure of loop spaces. It is meant to be readable by anyone with a good grounding in finite dimensional differential topology.

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1 Introduction

This document started out as an accompaniment to a series of seminars given at NTNU, and subsequently at Sheffield University, entitled: “The Differential Topology of the Loop Space”. The purpose of those seminars was to present an introduction to this topic leading up to the work contained in my preprint on the construction of a Dirac operator on loop spaces, [Sta]. This was intended to be accessible to anyone with knowledge of basic finite dimensional differential topology. Thus there was a reasonable amount of background material to be explained before the subject of Dirac operators was broached. The overall outline of this background material was divided as follows:

1. The differential topology of loop spaces.
2. Spinors in arbitrary dimension.
3. The Dirac operator and the Atiyah-Singer index theorem in finite dimensions.

The latter two topics are already superbly covered by books accessible to any differential topologist, perhaps with a little functional analysis. The book [LM89] is an excellent introduction to both topics in finite dimensions whilst [PR94] deals with spinors in arbitrary dimension. A group-centric viewpoint is presented in [PS86], which is required reading for anyone seriously thinking about loop spaces.

The main reference for the first topic is [KM97]. Whilst several of the standard texts on differential topology and geometry deal with infinite dimensional manifolds, they only do so rigorously with manifolds modelled on Banach spaces, which for loop spaces this usually means either continuous or H^1 -Sobolev loops. Often smooth and piece-wise smooth loops are used as a source of “useful” loops within these manifolds, but the space of smooth loops is not given an actual manifold structure. This is due to the technical difficulties in extending calculus outside the realm of Banach spaces.

The purpose of [KM97] is to deal with precisely this issue and to set up a theory of analysis in infinite dimensions in arbitrary topological vector spaces. It goes on to develop the theory of infinite dimensional manifolds in its broadest setting which includes smooth loop spaces. However, this means that whilst being an excellent book, its subject matter is perhaps too broad and too deep for someone who just wants to know about the differential topology of loop spaces. The statement that the smooth loop space is a smooth manifold appears in section 42, about two-thirds of the way through the book. Whilst the preceding four hundred pages are not all required reading to get to this point, it is still a somewhat daunting task to extract what one needs for loop spaces.

Therefore I started writing this document to fill in the details of what I did not have time to say in my seminars. The intention was to provide a more gentle introduction than [KM97] but still include the required detail to understand the differential topology of the space of smooth loops. It subsequently grew beyond that remit as I thought of more topics that could be included without increasing the technical difficulty. A brief description of the topics covered is as follows:

- §3 The smooth structure of the space of smooth loops in a finite dimensional manifold.

We start by discussing what it means to be smooth outside the realm of Banach spaces, focusing particularly on the model spaces for loop spaces. Using this we exhibit an atlas for the space of smooth loops and show that the transition functions are smooth. Also in this section we show that various maps involving loop spaces are smooth.

- §4 The basic theory of vector bundles and associated principal and gauge bundles on the loop space defined by looping vector bundles on the original manifold.

The main theme of this section is that looping something on the original manifold gives a corresponding object on the loop space. Thus the loop space of the tangent space is (diffeomorphic to) the tangent space of the loop space; vector bundles and their frame bundles loop to vector bundles and their frame bundles (in a sense) as do connections. We conclude this section with an important example where this loopy behaviour fails: the loop space of the cotangent bundle is *not* the cotangent bundle of the loop space.

- §5 Some important submanifolds with tubular neighbourhoods.

We show that various submanifolds of a loop space have tubular neighbourhoods. In particular, we consider submanifolds defined by imposing some condition on values that the loops can take at certain times – for example, the based loop space – and we consider fixed point submanifolds coming from the circle action. One consequence of this is that the fundamental fibration $\Omega M \rightarrow LM \rightarrow M$ is locally trivial. We conclude with a submanifold that does not have a tubular neighbourhood.

- §6 Basic introductions to two topics of further interest: geometry on loop spaces and the semi-infinite nature of loop spaces.

We begin with a discussion of some of the issues in infinite dimensional geometry and prove some simple results concerning the Levi-Civita connection and geodesics. We also give a very basic introduction to semi-infinite theory.

This document began life as notes from talks given at NTNU and at Sheffield so I would like to thank the topologists at those institutions, and in particular Nils Baas, for letting me talk about my favourite mathematical subject. I would also like to thank Ralph Cohen and the “loop group” at Stanford.

This is by no means a finished document, as an example it is somewhat sparse on references. Any comments, suggestions, and constructive criticism will be welcomed.

2 Loopy Notation

We first establish a little notation and some basic results for maps between loop spaces that have nothing to do with calculus. For a (finite dimensional smooth) manifold M , we denote $C^\infty(S^1, M)$ by LM . If M is a space over S^1 , so that we have a smooth map $\pi : M \rightarrow S^1$, we denote the subspace of LM consisting of sections of π by $\Gamma_{S^1}(M)$.

Definition 2.1 Let $f : M \rightarrow N$ be a smooth map between manifolds. Define $f^L : LM \rightarrow LN$ by $f^L(\alpha) = f \circ \alpha$.

If M is a space over S^1 , f^L restricts to a map $\Gamma_{S^1}(M) \rightarrow LN$. If, in addition, N is also a space over S^1 and f is a map of spaces over S^1 then f^L restricts to a map $\Gamma_{S^1}(M) \rightarrow \Gamma_{S^1}(N)$. We shall use the notation f^L for all three of these maps. Our convention will be that, unless otherwise stated, f^L refers to the most restrictive function that is applicable.

The most obvious type of space over S^1 is the product: $S^1 \times M$. In this case, the projection $p : S^1 \times M \rightarrow M$ defines a bijection $p^L : \Gamma_{S^1}(S^1 \times M) \rightarrow LM$.

Definition 2.2 Let $f : X \rightarrow M$ be a map from a set X to the manifold of a manifold M . Let $f^\vee : S^1 \times X \rightarrow M$ be the adjoint of f : $f^\vee(t, x) = f(x)(t)$.

Definition 2.3 Let M be a smooth manifold. Define the evaluation map, $e : S^1 \times LM \rightarrow M$ by $e(t, \gamma) = \gamma(t)$. We also write $e_t : LM \rightarrow M$ for the evaluation at t map, $e_t(\gamma) = \gamma(t)$.

Lemma 2.4 1. The adjoint of $f : X \rightarrow LM$ is the composition:

$$f^\vee : S^1 \times X \xrightarrow{1 \times f} S^1 \times LM \xrightarrow{e} M.$$

2. The adjoint of the identity on LM is the evaluation map.

3. Let $f : M \rightarrow N$ be a smooth map. The adjoint of f^L is the map:

$$S^1 \times LM \xrightarrow{e} M \xrightarrow{f} N$$

4. Looping respects composition (i.e. is a functor) in that $(f \circ g)^L = f^L \circ g^L$. It also maps products to products in that $(f \times g)^L = f^L \times g^L$.

5. Let $f : M \rightarrow N$, $g : X \rightarrow LM$, and $h : Y \rightarrow X$ be maps with f smooth. The adjoint of $f^L \circ g \circ h$ is $f \circ g^\vee \circ (1 \times h)$.

Proof. 1. We compute the composition as:

$$(t, x) \xrightarrow{1 \times f} (t, f(x)) \xrightarrow{e} f(x)(t) = f^\vee(t, x).$$

2. The follows from the above.

3. We have:

$$(f^L)^\vee(t, \alpha) = f^L(\alpha)(t) = (f \circ \alpha)(t) = f(\alpha(t)) = f \circ e(t, \alpha).$$

Another way of writing this identity is: $e \circ (1 \times f^L) = f \circ e$.

4. This is obvious.

5. Using the above, the adjoint of $f^L \circ g \circ h$ is:

$$\begin{aligned} e \circ (1 \times (f^L \circ g \circ h)) &= e \circ (1 \times f^L) \circ (1 \times g) \circ (1 \times h) \\ &= f \circ e \circ (1 \times g) \circ (1 \times h) \\ &= f \circ g^\vee \circ (1 \times h). \end{aligned}$$

□

3 Smoothly Does It

The aim of this section is to prove that the space of smooth loops in a finite dimensional manifold is again a smooth manifold. After doing this we go on to consider the topology of the space of smooth loops. It may seem slightly topsy-turvy to consider the smooth structure before the topology. The reason is that in infinite dimensions the link between continuity and smoothness is not as tight as it is in finite dimensions (there are smooth maps which are not continuous). After considering the topology we show that various obvious maps involving loop spaces are smooth. We conclude this section with a short discussion of how to go about changing the type of loop under consideration.

3.1 Curvaceous Calculus

The definitive reference for this section is the weighty tome [KM97]. Whilst a rewarding read for someone keen to learn about infinite dimensional analysis, it is rather intricate for someone just wanting to know the basics. Therefore, we have tried in this section to keep the material reasonably self-contained and focused solely on the case of loop spaces.

The maxim in infinite dimensional analysis is:

Smooth is as smooth does.

The problem is that once one strays outside the realm of Banach spaces then there are many different ways of defining the derivative. The *Historical Remarks* at the end of [KM97, ch I] is a good place to start investigating. For example, one reference is quoted as listing no less than *twenty-five* inequivalent definitions of the first derivative. Even for Fréchet spaces – the next “nicest” spaces after Banach spaces, of which the space $C^\infty(S^1, \mathbb{R})$ is an example – there are three inequivalent definitions of infinite differentiability. Therefore one needs to consider what one wants from calculus before one actually defines it.

For example, smoothness and continuity are no longer as tightly linked as they are in finite dimensions. Let E be an infinite dimensional locally convex topological vector space that is not normable. That is to say, the topology on E cannot be given by a single norm. Consider the evaluation map $E' \times E \rightarrow \mathbb{R}$, $(f, e) \rightarrow f(e)$. It is a straightforward result that with the product topology on $E' \times E$ then this map is not continuous. However, one feels that it ought to be included in any “reasonable” calculus on E .

Even in the theory of Hilbert spaces one finds that the topology on a topological vector space is not fixed. This becomes particularly obvious when dealing with duality: on the dual space one can discuss the strong topology, the weak topology, the simple topology, the compact topology, and many, many more. The guiding rule is to select the topology most appropriate for the job in hand.

Thus with open season declared on the topology and a feeling of freedom as to the definition of the derivative, one is left wondering what exactly is well-defined. The answer is: smooth curves.

It is simple to define smooth curves in any locally convex topological vector space: a curve $c : \mathbb{R} \rightarrow E$ is *differentiable* if, for all t , the derivative $c'(t)$ exists, where:

$$c'(t) := \lim_{s \rightarrow 0} \frac{1}{s} (c(t+s) - c(t)).$$

It is *smooth* if all iterated derivatives exist.

Providing E satisfies a mild completeness condition (called c^∞ -complete or convenient in [KM97]), then there is an incredibly simple recognition criterion for smoothness of curves:

Proposition 3.1 ([KM97, Theorem 2.14(4)]) *Let E be a convenient locally convex space. Then a curve $c : \mathbb{R} \rightarrow E$ is smooth if and only if the curves $l \circ c : \mathbb{R} \rightarrow \mathbb{R}$ are smooth for all $l \in E^*$, the continuous dual of E .*

This appears to connect the concept of smoothness firmly to that of continuity, but that is not so. The smoothness, or otherwise, of a curve depends only on what is called the *bornology* of the space E . That is, the family of *bounded sets*. The topology can vary considerably without changing the bornology.

As this is our fixed point in analysis, it seems reasonable to define everything else in terms of it. This leads to:

Definition 3.2 ([KM97, Definition 3.11]) *A function $f : E \supseteq U \rightarrow F$ defined on a c^∞ -open subset U of E is smooth if it takes smooth curves in U to smooth curves in F .*

I have sneaked in the term “ c^∞ -open” about which I don’t intend to go into great detail. Essentially, as we can vary the topology somewhat without changing the notion of smoothness, it makes sense to choose a particularly convenient topology. This topology is always at least as fine as the locally convex topology. For Fréchet spaces, it coincides with the standard topology.

To emphasise the point about smoothness being related to bornology rather than topology, we record the following result about linear maps:

Lemma 3.3 ([KM97, Corollary 2.11]) *A linear map $l : E \rightarrow F$ between locally convex vector spaces is bounded, i.e. bornological, if and only if it maps smooth curves in E to smooth curves in F , i.e. is a smooth map.*

Thus as the evaluation map $E' \times E \rightarrow \mathbb{R}$ is linear and bounded it is smooth, but as mentioned above is not continuous for the product topology.

The crucial tool in this calculus is the *exponential law*. One important corollary of this is that in finite dimensions this notion of smooth agrees with the standard one (originally proved in [Bom67]). To state the exponential law, we need first to note that the space of smooth functions $C^\infty(U, F)$ has a natural topology which is initial for the mappings $c^* : C^\infty(U, F) \rightarrow C^\infty(\mathbb{R}, F)$ for $c \in C^\infty(\mathbb{R}, U)$.

Theorem 3.4 (Exponential Law [KM97, Theorem 3.12]) *Let $U_i \subseteq E_i$ be c^∞ -open subsets. Then $C^\infty(U_1 \times U_2, F) \cong C^\infty(U_1, C^\infty(U_2, F))$.*

This is called the exponential law because it can be written as $F^{U_1 \times U_2} \cong (F^{U_2})^{U_1}$.

We have now defined smoothness without reference to the derivative. It is rather reassuring to discover that the derivative of a smooth map is still definable:

Theorem 3.5 ([KM97, Theorem 3.18]) *Let E and F be locally convex spaces, and let $U \subseteq E$ be c^∞ -open. Then the differentiation operator*

$$d : C^\infty(U, F) \rightarrow C^\infty(U, \mathcal{L}(E, F)),$$

$$df(x)(v) := \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t},$$

exists and is bounded (smooth). Also the chain rule holds:

$$d(f \circ g)(x)v = df(g(x))dg(x)v.$$

Here, $\mathcal{L}(E, F)$ is the space of all *bounded* linear maps from E to F . We differ from [KM97] in notation for this space as we have reserved the letter “ L ” for loops and thus use “ \mathcal{L} ” for linear maps. It is a closed subspace of $C^\infty(E, F)$ and is topologised as such. Essentially, this result says that the directional derivatives (which there is no problem *defining*, just problems with properties) fit together nicely to define the global derivative.

3.2 Euclidean Loop Spaces

Before we can consider an arbitrary loop space we need to examine what will become the model space, $L\mathbb{R}^n$. For those who like lists, here is a list of the functional analysis properties that $L\mathbb{R}^n$ satisfies:

1. $L\mathbb{R}^n$ is a Baire, barrelled, bornological, complete, convenient, Fréchet, infrabarrelled, Mackey, metrisable, Montel, nuclear, quasi-complete, reflexive, Schwartz, semi-reflexive, separable, space. Also, the bidual map is a topological isomorphism, the c^∞ topology referred to above is the standard topology, and it has many nice properties involving tensor products.
2. the dual space, $(L\mathbb{R}^n)^*$ (with the strong topology), is all of the same except that it is not a Fréchet space – and hence not metrisable – nor is it a Baire space. Also, the strong topology agrees with the Mackey topology and with the inductive topology.

We are more interested in the calculus properties of this space, although the “nice properties involving tensor products” are an important factor in [Sta]. We shall start with the topology on $L\mathbb{R}^n$. As we noted in the previous section, this is also the natural topology to consider when dealing with smooth maps on $L\mathbb{R}^n$. To define it, we first define the topology on the space $C(S^1, \mathbb{R}^n)$ of continuous maps.

Definition 3.6 *A subbasis for the topology on $C(S^1, \mathbb{R}^n)$ consists of the sets of the form:*

$$\{f \in C(S^1, \mathbb{R}^n) : f(K) \subseteq U\}$$

where $K \subseteq S^1$ is compact and $U \subseteq \mathbb{R}^n$ is open. This is called the compact-open topology.

In fact, $C(S^1, \mathbb{R}^n)$ is a Banach space with supremum norm:

$$\|\gamma\|_\infty = \sup\{\|\gamma(t)\| : t \in S^1\}.$$

The topology on the smooth loop space is then defined as the topology initial for the maps $L\mathbb{R}^n \rightarrow C(S^1, \mathbb{R}^{kn})$ given by:

$$\gamma \rightarrow (t \rightarrow (\gamma(t), \gamma'(t), \dots, \gamma^{(k-1)}(t))).$$

If we replace continuous functions by some other type, for example: square-integrable, we end up with the same topology on $L\mathbb{R}^n$. This is an example of how smoothness “smooths out” any initial irregularities in definition.

We now turn to the smooth structure of $L\mathbb{R}^n$. Ultimately, we need to know enough about the smooth maps on $L\mathbb{R}^n$ to decide whether or not the transition functions of an arbitrary loop space are smooth. These transition functions have a particularly simple structure and so the task is easier than that of trying to characterise *all* smooth maps between open subsets of $L\mathbb{R}^n$.

We start by generalising the exponential law. The problem with it as stated in theorem 3.4 is that the domains allowed are open subsets of some linear space. We wish to use S^1 as one of the domains. That is, we wish to show:

Proposition 3.7 *A curve $c : \mathbb{R} \rightarrow L\mathbb{R}^m$ is smooth if and only if its adjoint $c^\vee : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^m$ is smooth.*

Proof. Consider the quotient mapping $\mathbb{R} \rightarrow S^1$. This map completely determines the smooth structure of S^1 in that a map $S^1 \rightarrow \mathbb{R}^m$ is smooth if and only if the composite $\mathbb{R} \rightarrow S^1 \rightarrow \mathbb{R}^m$ is smooth. Similarly, $\mathbb{R} \times S^1 \rightarrow \mathbb{R}^m$ is smooth if and only if $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^m$ is smooth.

Thus given a curve $c : \mathbb{R} \rightarrow L\mathbb{R}^m$ with adjoint $c^\vee : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^m$ we get a curve $c_{\mathbb{R}} : \mathbb{R} \rightarrow C^\infty(\mathbb{R}, \mathbb{R}^m)$ and a map $c_{\mathbb{R}}^\vee : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^m$. As both use the same projection $\mathbb{R} \rightarrow S^1$, it is not hard to see that the adjoint of $c_{\mathbb{R}}$ is $c_{\mathbb{R}}^\vee$; that is, $c_{\mathbb{R}}^\vee = c_{\mathbb{R}}^\vee$. Thus by the standard exponential theorem, theorem 3.4, $c_{\mathbb{R}}$ is smooth if and only if $c_{\mathbb{R}}^\vee$ is smooth. By the above, $c_{\mathbb{R}}^\vee$ is smooth if and only if c^\vee is smooth. Thus it remains to show that c is smooth if and only if $c_{\mathbb{R}}$ is smooth.

Now the map $\mathbb{R} \rightarrow S^1$ exhibits $L\mathbb{R}^m$ as a linear subspace of $C^\infty(\mathbb{R}, \mathbb{R}^m)$ in that the induced map $L\mathbb{R}^m \rightarrow C^\infty(\mathbb{R}, \mathbb{R}^m)$ is a homeomorphism onto its image. The image is obviously a closed subspace and hence c^∞ -closed. Therefore, by [KM97, Lemma 3.8], a curve in $L\mathbb{R}^m$ is smooth if and only if it is smooth as a curve in $C^\infty(\mathbb{R}, \mathbb{R}^m)$. \square

The type of function that we shall examine is the following: let $V \subseteq S^1 \times \mathbb{R}^n$ and $W \subseteq S^1 \times \mathbb{R}^m$ be open subsets, and let $\psi : V \rightarrow W$ be a map of spaces over S^1 ; here, $S^1 \times \mathbb{R}^k$ has the obvious structure of a space over S^1 and a subspace of a space over S^1 has the obvious inherited structure.

Lemma 3.8 *Under the natural identification $\Gamma_{S^1}(S^1 \times \mathbb{R}^k) \cong L\mathbb{R}^k$, $\Gamma_{S^1}(V)$ and $\Gamma_{S^1}(W)$ are open subsets of the respective loop spaces and ψ^L is a smooth map.*

Note that $\Gamma_{S^1}(V)$ is empty unless the map $V \rightarrow S^1$ is surjective.

Proof. That $\Gamma_{S^1}(V)$ identifies with an open subset of $L\mathbb{R}^n$ is obvious as it is true for the topology induced by the inclusion into the space of continuous loops.

To show that ψ^L is smooth, it is sufficient to take the case where $W = S^1 \times \mathbb{R}^m$. This means that we are considering the target to be $L\mathbb{R}^m$ via the bijection $p^L : \Gamma_{S^1}(S^1 \times \mathbb{R}^m) \rightarrow L\mathbb{R}^m$. Thus we need to show that if $c : \mathbb{R} \rightarrow \Gamma_{S^1}(V)$ is smooth then $p^L \circ \psi^L \circ c : \mathbb{R} \rightarrow L\mathbb{R}^m$ is smooth.

By the exponential law, it is sufficient to consider the adjoint of this. Using lemma 2.4 we see that this is $p \circ \psi \circ c^\vee : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^m$. This is smooth and hence $p^L \circ \psi^L$ takes smooth maps to smooth maps. Thus ψ^L is smooth. \square

Let $d_v\psi : V \times \mathbb{R}^n \rightarrow W \times \mathbb{R}^m$ be the vertical derivative of ψ .

Lemma 3.9 $d(\psi^L) = (d_v\psi)^L$.

Proof. Recall that the derivative of a smooth function is determined by the directional derivatives. Using the identification of $\Gamma_{S^1}(S^1 \times \mathbb{R}^k)$ with $L\mathbb{R}^k$, the derivative of ψ^L is a smooth map $\Gamma_{S^1}(V) \rightarrow \mathcal{L}(L\mathbb{R}^n, L\mathbb{R}^m)$. So the derivative at a loop $\alpha \in \Gamma_{S^1}(V)$ in the direction of $\beta \in L\mathbb{R}^n$ is an element of $L\mathbb{R}^m$ which is the limit of the following net indexed by $s \in \mathbb{R}^+$:

$$\frac{\psi^L(\alpha + s\beta) - \psi^L(\alpha)}{s}.$$

Now evaluation at time t is a continuous linear map $L\mathbb{R}^n \rightarrow \mathbb{R}^n$ which takes this net to:

$$\frac{\psi^L(\alpha + s\beta)(t) - \psi^L(\alpha)(t)}{s} = \frac{\psi(t, \alpha(t) + s\beta(t)) - \psi(t, \alpha(t))}{s}.$$

This is precisely the difference quotient which tends to the vertical derivative of ψ at $\alpha(t)$ in the direction $\beta(t)$. Since a loop is completely determined by its values at each time, $d(\psi^L)(\alpha)\beta$ is the same loop as $t \rightarrow d_v\psi(t, \alpha(t))\beta(t)$. Hence:

$$d(\psi^L)(\alpha)\beta = (d_v\psi)^L(\alpha)\beta. \quad \square$$

Corollary 3.10 *The map $d(\psi^L)(\alpha) : L\mathbb{R}^n \rightarrow L\mathbb{R}^m$ is $L\mathbb{R}$ -linear.*

Proof. Since $d\psi^L(\alpha)$ is \mathbb{R} -linear, the part to prove in the statement about linearity is that $d\psi^L(\alpha)$ commutes with the action of $L\mathbb{R}$. Let $v \in L\mathbb{R}$, then:

$$\begin{aligned} d\psi^L(\alpha)(v\beta)(t) &= d_v\psi(t, \alpha(t))(v(t)\beta(t)) \\ &= v(t)d_v\psi(t, \alpha(t))\beta(t) \\ &= (vd\psi^L(\alpha)\beta)(t). \end{aligned}$$

Hence $d\psi^L(\alpha)(v\beta) = vd\psi^L(\alpha)\beta$ as required. \square

Lemma 3.11 *Now suppose that $n = m$ and that $\psi : V \rightarrow W$ is a diffeomorphism with inverse $\phi : W \rightarrow V$. Then $\psi^L : \Gamma_{S^1}(V) \rightarrow \Gamma_{S^1}(W)$ is a diffeomorphism with inverse ϕ^L .*

Proof. Both ψ^L and ϕ^L are smooth. By property 4 of lemma 2.4, the compositions $\psi^L \circ \phi^L$ and $\phi^L \circ \psi^L$ are the identities on their domains. Hence ψ^L is a diffeomorphism with inverse ϕ^L . \square

This has a few nice consequences that we shall exploit. Most are things that jolly well ought to be true, but given that the definition of smooth is probably unfamiliar it's as well to spell out the details.

Corollary 3.12 1. *Let $\alpha : S^1 \rightarrow \text{Diff}(\mathbb{R}^n)$ be a smooth map (the group $\text{Diff}(\mathbb{R}^n)$ inherits a smooth structure from $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$). Then the induced map $L\mathbb{R}^n \rightarrow L\mathbb{R}^n$ given by $\gamma \rightarrow (t \rightarrow \alpha(t)\gamma(t))$ is a diffeomorphism.*

2. *Let $E \rightarrow S^1$ be a smooth, orientable vector bundle. Then the space of sections, $\Gamma_{S^1}(E)$, has a natural smooth structure.*

3. The results above generalise to orientable smooth vector bundles as follows: let $E, F \rightarrow S^1$ be orientable smooth vector bundles. Let $V \subseteq E$ and $W \subseteq F$ be open subsets. Let $\psi : V \rightarrow W$ be a smooth map covering the identity on S^1 . Then $\psi^L : \Gamma_{S^1}(V) \rightarrow \Gamma_{S^1}(W)$ is smooth with derivative the loop of the vertical derivative of ψ . If ψ is a diffeomorphism with inverse ϕ then ψ^L is a diffeomorphism with inverse ϕ^L .

The key in the second is to choose a diffeomorphism of (the total space of) E with $S^1 \times \mathbb{R}^n$ and so identify the space of sections of E with $L\mathbb{R}^n$. This defines a smooth structure on this space of sections which is independent of the diffeomorphism chosen by the first result.

The same results hold for non-orientable bundles but are a little fiddly to prove. One has to consider what are called *twisted* loop spaces. These add no analytic difficulties, but perhaps add one or two conceptual ones so we shall avoid using them.

3.3 The Smooth Structure of an Arbitrary Loop Space

We are now able to consider the smooth loop space of a finite dimensional smooth manifold, M . We shall make two assumptions on the type of manifold that we consider: one necessary and one for convenience.

1. $\partial M = \emptyset$.

The loop space of a manifold with boundary is a complicated object. To put it simply, it is not true that $LM = L(M \setminus \partial M) \cup L(\partial M)$. The best description of LM in this case is as a stratified space, with strata indexed by closed subspaces of the circle. The layer corresponding to $F \subseteq S^1$ consists of those loops $\alpha : S^1 \rightarrow M$ such that $\alpha^{-1}(\partial M) = F$. The top level, corresponding to the empty set, is $L(M \setminus \partial M)$. The next level consists of all loops which intersect ∂M at one point, and so on.

2. M is orientable.

This allows us to stay out of the twisted realm. We shall not actually use this in the analysis, it is merely to make this account reasonably self-contained.

The key tool for defining the charts for the loop space is the notion of a *local addition* on M , cf [KM97, §42.4]:

Definition 3.13 A local addition on M consists of a smooth map $\eta : TM \rightarrow M$ such that

1. the composition of η with the zero section is the identity on M , and
2. there exists an open neighbourhood V of the diagonal in M such that the map $\pi \times \eta : TM \rightarrow M \times M$ is a diffeomorphism onto V .

In [KM97, §42.4], the above is called a *globally defined local addition* but the difference is not important for us. We shall later want to relax this further by replacing the tangent bundle by an arbitrary vector bundle (which must, *a fortiori*, be isomorphic to the tangent bundle) but for now we stick with the tangent bundle to keep things conceptually simple. The following result is contained in the discussion following the definition of a local addition in [KM97, §42.4]:

Proposition 3.14 *Any finite dimensional manifold without boundary admits a local addition.*

Proof. Without going into great detail, the essentials of the proof are that the exponential map coming from a Riemannian structure almost defines a local addition except that the domain of the diffeomorphism is not the whole tangent space (except in a few simple cases) but a neighbourhood of the zero section. The proof is completed by exhibiting a smooth fibre-preserving embedding of the total space of the tangent bundle into the domain of the diffeomorphism. The composition of this with the exponential map is the required local addition. \square

Let $\eta : TM \rightarrow M$ be a local addition on M . Let $V \subseteq M \times M$ be the image of the map $\pi \times \eta : TM \rightarrow M \times M$. Although as yet we know nothing about the topologies of LTM or of LV , we can at least say that the looped map, $(\pi \times \eta)^L$, is a bijection.

Lemma 3.15 *Let $\alpha \in LM$. Define the set $U_\alpha \subseteq LM$ by:*

$$U_\alpha := \{\beta \in LM : (\alpha, \beta) \in LV\}.$$

*Then the preimage of $\{\alpha\} \times U_\alpha$ under $(\pi \times \eta)^L$ is naturally identified with $\Gamma_{S^1}(\alpha^*TM)$. In particular, the zero section of α^*TM maps to $(\alpha, \alpha) \in \{\alpha\} \times U_\alpha$.*

Proof. We claim that there is a diagram:

$$\begin{array}{ccc} LTM & \xrightarrow{(\pi \times \eta)^L} & LV \\ \uparrow & & \uparrow \beta \rightarrow (\alpha, \beta) \\ \Gamma_{S^1}(\alpha^*TM) & & U_\alpha \end{array}$$

such that the bijection at the top takes the image of the left-hand vertical map to the image of the right-hand one. Both of the vertical maps are injective – the right-hand one obviously so, we shall investigate the left-hand one in a moment – and thus the bijection $(\pi \times \eta)^L$ induces a bijection from the lower left to the lower right.

The left-hand vertical map, $\Gamma_{S^1}(\alpha^*TM) \rightarrow LTM$, is defined as follows: the total space α^*TM is:

$$\{(t, v) \in S^1 \times TM : \alpha(t) = \pi(v)\}.$$

It is an embedded submanifold of $S^1 \times TM$. Therefore a map into α^*TM is smooth if and only if the compositions with the projections to S^1 and to TM are smooth. Now a map $S^1 \rightarrow \alpha^*TM$ is a section if and only if it projects to the identity on S^1 . Therefore there is a bijection (of sets):

$$\begin{aligned} \Gamma_{S^1}(\alpha^*TM) &\cong \{\beta \in LTM : (t, \beta(t)) \in \alpha^*TM \text{ for all } t \in S^1\} \\ &= \{\beta \in LTM : \alpha(t) = \pi\beta(t) \text{ for all } t \in S^1\} \\ &= \{\beta \in LTM : \pi^L\beta = \alpha\} =: L_\alpha TM. \end{aligned}$$

In particular, the map $\Gamma_{S^1}(\alpha^*TM) \rightarrow LTM$ is injective.

We apply $(\pi \times \eta)^L$ to the defining condition for $L_\alpha TM$ and see that $L_\alpha TM$ is the preimage under this map of everything of the form (α, γ) in LV . By

construction, $\gamma \in LM$ is such that $(\alpha, \gamma) \in LV$ if and only if $\gamma \in U_\alpha$. Hence $(\pi \times \eta)^L$ identifies $L_\alpha TM$ with $\{\alpha\} \times U_\alpha$.

Finally, note that the zero section of α^*TM maps to the image of α under the zero section of TM . Since η composed with the zero section of TM is the identity on M , the image of the zero section of α^*TM in V is (α, α) as required. \square

Definition 3.16 Let $\Psi_\alpha : \Gamma_{S^1}(\alpha^*TM) \rightarrow U_\alpha$ be the resulting bijection.

In detail, this map is as follows: let $\beta \in \Gamma_{S^1}(\alpha^*TM)$ and let $\tilde{\beta}$ be the corresponding loop in TM , so $\beta(t) = (t, \tilde{\beta}(t))$. Then $(\pi \times \eta)^L(\tilde{\beta}) = (\alpha, \eta^L(\tilde{\beta}))$ so $\Psi_\alpha(\beta) = \eta^L(\tilde{\beta})$.

The domains of these functions are sections over S^1 of smooth orientable finite dimensional vector bundles. We equip these with the smooth structure as in corollary 3.12. On the other side, since $\alpha \in U_\alpha$, the family $\{U_\alpha\}$ covers the loop space of M . Thus we have an open covering of the loop space by sets each of which we can identify with a linear space with a chosen smooth structure.

We now turn to the investigation of the transition functions. We shall prove a slightly stronger result than is needed at this moment in that we shall allow the local addition to vary. As well as showing that the transition functions are smooth this will show that the maximal atlas so defined contains all such charts with any choice of local addition.

Let $\eta_1, \eta_2 : TM \rightarrow M$ be local additions with corresponding neighbourhoods V_1, V_2 of the diagonal in $M \times M$. Let α_1, α_2 be smooth loops in M . Let $\Psi_1 : \Gamma_{S^1}(\alpha_1^*TM) \rightarrow U_1, \Psi_2 : \Gamma_{S^1}(\alpha_2^*TM) \rightarrow U_2$ be the corresponding charts.

Lemma 3.17 Let $W_{12} \subseteq \alpha_1^*TM$ be the set:

$$\{(t, v) \in \alpha_1^*TM : (\alpha_2(t), \eta_1(v)) \in V_2\}.$$

Then W_{12} is open and $\Psi_1^{-1}(U_1 \cap U_2) = \Gamma_{S^1}(W_{12})$.

Proof. The set W_{12} is open as it is the preimage of an open set via a continuous map. To show the second statement we need to prove that $\gamma \in \Gamma_{S^1}(\alpha_1^*TM)$ takes values in W_{12} if and only if $\Psi_1(\gamma) \in U_2$ (by construction we already have $\Psi_1(\gamma) \in U_1$).

So let $\gamma \in \Gamma_{S^1}(\alpha_1^*TM)$ and let $\tilde{\gamma}$ be the image of γ in TM . Thus $\gamma(t) = (t, \tilde{\gamma}(t))$. Now γ takes values in W_{12} if and only if:

$$(\alpha_2(t), \eta_1(\tilde{\gamma}(t))) \in V_2$$

for all $t \in S^1$. That is to say, if and only if $(\alpha_2, \eta_1^L(\tilde{\gamma})) \in LV_2$. By definition, this is equivalent to the statement that $\eta_1^L(\tilde{\gamma}) \in U_2$. Now $\eta_1^L(\tilde{\gamma}) = \Psi_1(\gamma)$ so γ takes values in W_{12} if and only if $\Psi_1(\gamma) \in U_1 \cap U_2$. \square

Proposition 3.18 The transition function:

$$\Phi_{12} := \Psi_1^{-1}\Psi_2 : \Psi_1^{-1}(U_{12}) \rightarrow \Psi_2^{-1}(U_{12})$$

is a diffeomorphism.

Proof. We define $W_{21} \subseteq \alpha_2^*TM$ as the set of $(t, v) \in \alpha_2^*TM$ with $(\alpha_1(t), \eta_2(v)) \in V_1$. As for W_{12} , $\Psi_2^{-1}(U_1 \cap U_2) = \Gamma_{S^1}(W_{21})$.

The idea of the proof is to set up a diffeomorphism between W_{12} and W_{21} . Corollary 3.12 says that the resulting map on sections is a diffeomorphism.

Finally, we show that this diffeomorphism is the transition function defined in the statement of this proposition.

Let $\theta_1 : W_{12} \rightarrow TM$ be the map:

$$\theta_1(t, v) = (\pi \times \eta_2)^{-1}(\alpha_2(t), \eta_1(v)).$$

The definition of W_{12} ensures that $(\alpha_2(t), \eta_1(v)) \in V_2$ for $(t, v) \in W_{12}$ and this is the image of $\pi \times \eta_2$. Hence θ_1 is well-defined. Define $\theta_2 : W_{21} \rightarrow TM$ similarly. These are both smooth maps.

Notice that $\pi(\pi \times \eta_i)^{-1} : V_i \subseteq M \times M \rightarrow M$ is the projection onto the first factor and $\eta_i(\pi \times \eta_i)^{-1} : V_i \rightarrow M$ is the projection onto the second. Thus $\pi\theta_1(t, v) = \alpha_2(t)$. Hence $\theta_1 : W_{12} \rightarrow TM$ is such that $(t, \theta_1(t, v)) \in \alpha_2^*TM$ for all $(t, v) \in W_{12}$. Then:

$$(\alpha_1(t), \eta_2(\theta_1(t, v))) = (\alpha_1(t), \eta_1(v)) \in V_1$$

so $(t, \theta_1(t, v)) \in W_{21}$. Hence we have a map $\phi_{12} : W_{12} \rightarrow W_{21}$ given by:

$$\phi_{12}(t, v) = (t, \theta_1(t, v)).$$

Similarly we have a map $\phi_{21} : W_{21} \rightarrow W_{12}$. These are both smooth since the composition with the inclusion into $S^1 \times TM$ is smooth.

Consider the composition $\phi_{21}\phi_{12}(t, v)$. Expanding this out yields:

$$\begin{aligned} \phi_{21}\phi_{12}(t, v) &= \phi_{21}(t, \theta_1(t, v)) \\ &= (t, \theta_2(t, \theta_1(t, v))) \\ &= (t, (\pi \times \eta_1)^{-1}(\alpha_1(t), \eta_2(\theta_1(t, v)))) \\ &= (t, (\pi \times \eta_1)^{-1}(\alpha_1(t), \eta_1(v))) \\ &= (t, (\pi \times \eta_1)^{-1}(\pi(v), \eta_1(v))) \\ &= (t, v). \end{aligned}$$

The penultimate line is because $(t, v) \in \alpha_1^*TM$ so $\pi(v) = \alpha_1(t)$.

Hence ϕ_{21} is the inverse of ϕ_{12} and so $\phi_{12} : W_{12} \rightarrow W_{21}$ is a diffeomorphism. Thus the map ϕ_{12}^L is a diffeomorphism from $\Psi_1^{-1}(U_1 \cap U_2)$ to $\Psi_2^{-1}(U_1 \cap U_2)$. We just need to show that this is the transition function. It is sufficient to show that $\Psi_2\phi_{12}^L = \Psi_2\Phi_{12}$. The right-hand side is, by definition, Ψ_1 , which satisfies:

$$\Psi_1(\gamma)(t) = \eta_1(\tilde{\gamma}(t))$$

where $\tilde{\gamma} : S^1 \rightarrow TM$ is such that $\gamma(t) = (t, \tilde{\gamma}(t))$. On the other side:

$$\begin{aligned} \phi_{12}^L(\gamma)(t) &= \phi_{12}(\gamma(t)) \\ &= (t, \theta_1(t, \tilde{\gamma}(t))) \\ &= (t, (\pi \times \eta_2)^{-1}(\alpha_2(t), \eta_1(\tilde{\gamma}(t)))) \end{aligned}$$

hence:

$$\begin{aligned} \Psi_2\phi_{12}^L(\gamma)(t) &= \eta_2(\pi \times \eta_2)^{-1}(\alpha_2(t), \eta_1(\tilde{\gamma}(t))) \\ &= \eta_1(\tilde{\gamma}(t)), \end{aligned}$$

as required. Thus $\phi_{12}^L = \Phi_{12}$ and so the transition functions are diffeomorphisms. \square

Remark 3.19 There is no *a priori* reason why we needed to assume that M was a finite dimensional manifold. All we needed was to know what the smooth structure of M was and to know that it had a local addition. Since the loop of a local addition is again a local addition, we could iterate this construction and show that any iterated loop space is a smooth manifold.

3.4 The Loop Space of the Tangent Space

The next topic that we wish to consider is the topology of the space LM . It is usually defined, by analogy with the topology of $L\mathbb{R}^n$, as a projective limit of the maps $LM \rightarrow C(S^1, T^{(k)}M)$ where $T^{(k)}M$ is recursively defined as the tangent space of $T^{(k-1)}M$, and $T^{(1)}M = TM$. We shall not define the topology this way but shall show that our definition is equivalent. To do this, we need to know how the structure of LM relates to that of LTM . This is a useful piece of knowledge that we shall use again when we look at the tangent space of LM so we record it in its own section here.

We mentioned earlier that we wish to replace the source of the local addition by an arbitrary vector bundle, albeit one isomorphic to the tangent bundle. We can either repeat all of the above with arbitrary sources for the two local additions to see that the charts so defined lie in the same atlas, or we can simply observe that as the source must be isomorphic to the tangent bundle, we can use an isomorphism to transfer the local addition to the tangent bundle and also to define a linear diffeomorphism between the model spaces. Thus any chart defined using an arbitrary vector bundle factors through one that uses the tangent bundle and hence is in the same atlas.

The first application of this is to lift the charts defined for LM to charts for LTM . Let $T^{(2)}M$ be the tangent bundle of TM . Let $\eta : TM \rightarrow M$ be a local addition with corresponding open neighbourhood $V \subseteq M \times M$. Then $d\eta : T^{(2)}M \rightarrow TM$ is a local addition in this new sense but *not* in the strict sense.

This is because $d\pi \times d\eta : T^{(2)}M \rightarrow TV$ is a diffeomorphism and TV is an open neighbourhood in $TM \times TM$ of the diagonal. However while $d\pi : T^{(2)}M \rightarrow TM$ is a vector bundle projection it is not the projection of a tangent bundle onto its base. Rather we have a commutative diagram of vector bundle projections:

$$\begin{array}{ccc} T^{(2)}M & \xrightarrow{d\pi} & TM \\ \pi \downarrow & & \downarrow \pi \\ TM & \xrightarrow{\pi} & M \end{array}$$

where every map *except* $d\pi$ is the natural projection of a tangent bundle onto its base.

In all of the following, we consider $T^{(2)}M$ as a vector bundle over TM via the map $d\pi$.

Using this local addition we get a chart $\Phi_\alpha : \Gamma_{S^1}(\alpha^*T^{(2)}M) \rightarrow V_\alpha$ for any $\alpha \in LTM$. The first thing to observe about this chart is that $V_\alpha = (\pi^L)^{-1}(U_{\pi^L\alpha})$, where $\pi^L : LTM \rightarrow LM$ is the loop of the projection and $U_{\pi^L\alpha}$ is the codomain of the chart at $\pi^L\alpha$ defined using the local addition η . This is because $TV = (\pi \times \pi)^{-1}V$

and so:

$$\begin{aligned}
V_\alpha &= \{\beta \in LTM : (\alpha(t), \beta(t)) \in TV \text{ all } t \in S^1\} \\
&= \{\beta \in LTM : (\pi\alpha(t), \pi\beta(t)) \in V \text{ all } t \in S^1\} \\
&= \{\beta \in LTM : \pi^L\beta \in U_{\pi^L\alpha}\} \\
&= (\pi^L)^{-1}U_{\pi^L\alpha}.
\end{aligned}$$

Thus to get a cover of LTM , we start with a family of loops in M such that the corresponding charts cover LM and then we pick any-old lifts of these loops to TM . One of the most straightforward ways to choose such a lift is as follows: let $\tau : S^1 \rightarrow TS^1$ be a section. Define $\hat{\tau} : LM \rightarrow LTM$ by $\hat{\tau}\alpha = d\alpha \circ \tau : S^1 \rightarrow TS^1 \rightarrow TM$.

Proposition 3.20 *The map $\hat{\tau}$ induces a natural map on sections: $\hat{\tau} : \Gamma_{S^1}(\alpha^*TM) \rightarrow \Gamma_{S^1}((\hat{\tau}\alpha)^*T^{(2)}M)$. A trivialisation of α^*TM induces a trivialisation of $(\hat{\tau}\alpha)^*T^{(2)}M$ such that the following diagram commutes:*

$$\begin{array}{ccccccc}
\mathbb{L}\mathbb{R}^n & \longrightarrow & \Gamma_{S^1}(\alpha^*TM) & \xrightarrow{\Psi_\alpha} & U_\alpha & \xrightarrow{\subseteq} & LM \\
\hat{\tau} \downarrow & & \hat{\tau} \downarrow & & & & \hat{\tau} \downarrow \\
L\mathbb{T}\mathbb{R}^n & \longrightarrow & \Gamma_{S^1}((\hat{\tau}\alpha)^*T^{(2)}M) & \xrightarrow{\Phi_{\hat{\tau}\alpha}} & V_{\hat{\tau}\alpha} & \xrightarrow{\subseteq} & LTM.
\end{array}$$

Proof. We shall start by explaining the two maps induced by τ . The space α^*TM is an embedded submanifold of $S^1 \times TM$. It can be defined as the preimage of the diagonal in $M \times M$ under the map $(t, v) \rightarrow (\alpha(t), \pi(v))$. Its tangent space is therefore an embedded submanifold of $TS^1 \times T^{(2)}M$ and is the preimage of the diagonal in $TM \times TM$ under the map $d\alpha \times d\pi$. Hence $T(\alpha^*TM)$ is the pull-back of $d\pi : T^{(2)}M \rightarrow TM$ under $d\alpha$.

Now a section β of α^*TM is a special kind of map $\beta : S^1 \rightarrow \alpha^*TM$. It therefore differentiates to a map $d\beta : TS^1 \rightarrow T(\alpha^*TM) = (d\alpha)^*T^{(2)}M$. As β is a section, $\pi\beta$ is the identity on S^1 , hence $d\pi d\beta$ is the identity on TS^1 and so $d\beta$ is a section of $(d\alpha)^*T^{(2)}M$. We compose this with τ to get $d\beta \circ \tau : S^1 \rightarrow (d\alpha)^*T^{(2)}M$. As $d\beta$ is a section, $d\pi d\beta \tau = \tau$ and so $d\beta \tau$ is a section of $\tau^*(d\alpha)^*T^{(2)}M$. As $\tau^*(d\alpha)^* = (d\alpha\tau)^*$, $d\beta \tau$ is a section of $(\hat{\tau}\alpha)^*T^{(2)}M$. Thus we have the map $\hat{\tau} : \beta \rightarrow d\beta \tau$ from sections of α^*TM to sections of $(\hat{\tau}\alpha)^*T^{(2)}M$.

The other map works similarly: let $\phi : S^1 \times \mathbb{R}^n \rightarrow \alpha^*TM$ be a trivialisation. This differentiates to $d\phi : TS^1 \times T\mathbb{R}^n \rightarrow T(\alpha^*TM) = (d\alpha)^*T^{(2)}M$ which is a trivialisation over TS^1 . This pulls-back via τ to a trivialisation $S^1 \times T\mathbb{R}^n \rightarrow \tau^*(d\alpha)^*T^{(2)}M$. The left-hand side of this is $S^1 \times \mathbb{R}^{2n}$ and the right is $(\hat{\tau}\alpha)^*T^{(2)}M$.

To see how the commutative diagram works, we start with a loop in \mathbb{R}^n and trace how it becomes a loop in M . Let $\check{\beta} : S^1 \rightarrow S^1 \times \mathbb{R}^n$ be the map $t \rightarrow (t, \beta(t))$, then the loop in M is the composition:

$$S^1 \xrightarrow{\check{\beta}} S^1 \times \mathbb{R}^n \xrightarrow{\phi} \alpha^*TM \subseteq S^1 \times TM \rightarrow TM \xrightarrow{\eta} M.$$

We differentiate this, and use the identification of $T(\alpha^*TM)$ with $(d\alpha)^*T^{(2)}M \subseteq TS^1 \times T^{(2)}M$ to get the map:

$$TS^1 \xrightarrow{d\check{\beta}} TS^1 \times T\mathbb{R}^n \xrightarrow{d\phi} (d\alpha)^*TM \subseteq TS^1 \times T^{(2)}M \rightarrow T^{(2)}M \xrightarrow{d\eta} TM.$$

The first four terms in this sequence are fibre bundles over TS^1 (viewing TS^1 as a bundle over itself with one-point fibres) and all the maps are of fibre bundles. Therefore we can pull-back these terms over S^1 using the section $\tau : S^1 \rightarrow TS^1$. The map from the fourth to fifth terms is the projection of the product away from TS^1 so is unchanged under the pull-back. The third term is $(\hat{\tau}\alpha)^*T^{(2)}M$ and the second map is the induced trivialisation of this bundle. The first map, $\tau^*d\check{\beta}$, fits into the diagram:

$$\begin{array}{ccc} TS^1 & \xrightarrow{d\check{\beta}} & TS^1 \times T\mathbb{R}^n \\ \tau \uparrow & & \tau \times 1 \uparrow \\ S^1 & \xrightarrow{\tau^*d\check{\beta}} & S^1 \times T\mathbb{R}^n. \end{array}$$

Differentiating $\check{\beta}(t) = (t, \beta(t))$ yields $d\check{\beta} = (1, d\beta)$ and so $d\check{\beta} \circ \tau = (\tau, d\beta \circ \tau) = (\tau, \hat{\tau}\beta)$. Hence $(\tau^*d\check{\beta})(t) = (t, \hat{\tau}\beta(t))$. Thus $\tau^*d\check{\beta}$ is the map $S^1 \rightarrow T\mathbb{R}^n$ corresponding to the map $\hat{\tau}\beta \in L\mathbb{R}^n$. Write this as $\hat{\tau}\check{\beta}$.

We therefore have the commutative diagram:

$$\begin{array}{ccccccc} TS^1 & \xrightarrow{d\check{\beta}} & TS^1 \times T\mathbb{R}^n & \xrightarrow{d\phi} & (d\alpha)^*T^{(2)}M & \longrightarrow & T^{(2)}M & \xrightarrow{d\eta} & TM \\ \hat{\tau} \uparrow & & \hat{\tau} \uparrow & & \hat{\tau} \uparrow & & 1 \uparrow & & 1 \uparrow \\ S^1 & \xrightarrow{\hat{\tau}\check{\beta}} & S^1 \times T\mathbb{R}^n & \xrightarrow{\tau^*d\phi} & (\hat{\tau}\alpha)^*T^{(2)}M & \longrightarrow & T^{(2)}M & \xrightarrow{d\eta} & TM. \end{array}$$

Going first up and then all the way along results in $(d\Psi_\alpha(\beta))\tau = \hat{\tau}\Psi_\alpha(\beta)$. Going up at the third stage, $(\hat{\tau}\alpha)^*T^{(2)}M$, shows that this factors through the map $\hat{\tau}$ on sections, whilst going all the way along and then up shows that it factors through the map $\hat{\tau} : L\mathbb{R}^n \rightarrow L\mathbb{R}^n$. Hence the commutative diagram of the statement of this proposition is true. \square

3.5 The Topology of the Loop Space

We now turn to the topology on LM .

Definition 3.21 *The topology on LM is the topology such that $W \subseteq LM$ is open if and only if $\Psi_\alpha^{-1}(W)$ is open for all Ψ_α .*

Since the transition functions are diffeomorphisms and hence homeomorphisms, it is sufficient to check this condition using a family of charts which cover LM .

To investigate this topology we start with the topology of the continuous loop space, $C(S^1, M)$. The topology on this is similar to the ‘‘compact-open’’ topology on $C(S^1, \mathbb{R}^n)$. A basis consists of the sets:

$$N(K, U) := \{\alpha \in C(S^1, M) : \alpha(K) \subseteq U\}$$

where K runs through the family of compact subsets of S^1 and U through the open subsets of M .

With this topology, the space $C(S^1, M)$ has the following properties:

1. It is a separable metrisable space. An explicit metric is

$$d_{\text{cts}}(\alpha, \beta) := \sup\{d_M(\alpha(t), \beta(t)) : t \in S^1\}$$

where d_M is a metric on M compatible with its topology; for example, that which comes from a Riemannian structure.

2. The space of smooth loops is dense. The standard proof of this is to show that any continuous loop can be approximated to an arbitrary degree by a piecewise geodesic. Any piecewise smooth loop can be approximated (in the compact-open topology) by a smooth one by “rounding off the corners”.

Lemma 3.22 *The natural inclusion $LM \rightarrow C(S^1, M)$ is continuous and the codomains of the charts are open for the induced topology.*

Proof. We start with the statement on the codomains. Recall that the codomain of Ψ_α is the set U_α such that $\{\alpha\} \times U_\alpha = (\{\alpha\} \times LM) \cap LV$. Now a loop in V is smooth if and only if it is smooth as a loop in $M \times M$, so as α is smooth the right-hand side of this is:

$$\left((\{\alpha\} \times C(S^1, M)) \cap C(S^1, V) \right) \cap (LM \times LM).$$

Hence if $U_\alpha^0 \subseteq C(S^1, M)$ is the continuous version of U_α , then $U_\alpha = LM \cap U_\alpha^0$. As V is open in $M \times M$, $C(S^1, V)$ is open in $C(S^1, M)$ and so U_α^0 is open in $C(S^1, M)$. Hence U_α is open for the induced topology on LM .

The topology on LM is defined such that $W \subseteq LM$ is open if and only if $\Psi_\alpha^{-1}(W)$ is open for all α . Thus a map f from LM is continuous if and only if all the compositions $f \circ \Psi_\alpha$ are continuous.

A smooth trivialisation of α^*TM identifies $\Gamma_{S^1}(\alpha^*TM)$ with LR^n . It also identifies the space of continuous sections, $\Gamma_{S^1}^0(\alpha^*TM)$, with the continuous loop space, $C(S^1, \mathbb{R}^n)$. The obvious square commutes. As with smooth sections, a section of α^*TM is continuous if and only if the induced map $S^1 \rightarrow TM$ is continuous. Therefore there is an analogous map:

$$\Psi_\alpha^0 : \Gamma_{S^1}^0(\alpha^*TM) \rightarrow U_\alpha^0, \quad \Psi_\alpha^0(\beta)(t) = \eta(\tilde{\beta}(t)).$$

In this case, we have a topology on the target already. That the map Ψ_α^0 is continuous for this topology is a direct consequence of the fact that if $f : X \rightarrow Y$ is continuous then the induced map $f^L : C(S^1, X) \rightarrow C(S^1, Y)$ is continuous.

Consider the following diagram:

$$\begin{array}{ccccc} LR^n & \xrightarrow{\cong} & \Gamma_{S^1}(\gamma^*TM) & \xrightarrow{\Psi_\gamma} & LM \\ i_{\mathbb{R}^n} \downarrow & & i_\gamma \downarrow & & i_M \downarrow \\ C(S^1, \mathbb{R}^n) & \xrightarrow{\cong} & \Gamma_{S^1}^0(\gamma^*TM) & \xrightarrow{\Psi_\gamma^0} & C(S^1, M) \end{array}$$

The map i_M is continuous if and only if $i_M \circ \Psi_\gamma$ is continuous, for each γ . This is the same as $\Psi_\gamma^0 i_\gamma$. Then i_γ is continuous because it is equivalent to $i_{\mathbb{R}^n}$ which is the inclusion of smooth loops in continuous loops in Euclidean space. \square

This result has some useful corollaries:

Corollary 3.23 *With this topology, LM is:*

1. Hausdorff,

2. *regular,*
3. *second countable, and*
4. *paracompact.*

Proof. 1. Any topology finer than a Hausdorff topology is Hausdorff. Viewing LM as a (topological) subspace of $C(S^1, M)$, the induced topology is Hausdorff as $C(S^1, M)$ is metrisable. Hence the given topology on LM is Hausdorff.

2. With the compact-open topology, LM is a regular space. Therefore, as the codomains of charts are open for the compact-open topology, every point has a closed neighbourhood contained in the codomain of a chart.

Let $C \subseteq LM$ be a closed set and $\gamma \notin C$. Let D be a closed neighbourhood of γ that is contained in the codomain of a chart. The chart map induces a homeomorphism between D and a closed subset of a locally convex topological vector space, which is regular as a topological space.

Therefore, there is a set \widetilde{B} which is open in D such that $\gamma \in \widetilde{B}$ and the closure of \widetilde{B} does not meet $D \cap C$. Let B be the intersection of \widetilde{B} with the interior of D . As D is a neighbourhood of γ , this is still a non-empty set containing γ . Moreover, as an open subset of the interior of D , it is open in LM . Its closure in LM is the same as its closure in D . This does not meet $D \cap C$, whence – as it is contained within D – it does not meet C .

Hence LM is regular.

3. We start by observing that a countable number of codomains of charts will cover LM . This is because with the compact-open topology, i.e. with LM viewed as a subspace of $C(S^1, M)$, LM is second countable (as it is a subspace of a second countable space). It is therefore Lindelöf. Now the codomains of the charts form an open covering *with the compact-open topology*, hence a countable number of them, say the family $\{U_n\}$, will cover LM .

Each chart codomain is homeomorphic to an open subset of the separable, metrisable space $L\mathbb{R}^n$. It therefore is itself second countable. Thus for each codomain in our countable covering we choose a countable basis for the topology. This forms a countable family of open sets which we claim is a basis for the topology.

This is straightforward: for $W \subseteq LM$ open, as the countable family of codomains covers LM then W is the union of $W \cap U_n$. Each of these is then the union of sets in our basis. Hence W is the union of sets in our basis. Thus LM has a countable basis and so is second countable.

4. We use the following two results:

- (a) A second countable space is Lindelöf, and
- (b) A Lindelöf regular space is paracompact. □

From this, various topological results follow including the fact that LM is metrisable. We shall quote from [KM97] the result that LM is *smoothly paracompact* which means that it admits smooth partitions of unity subordinate to any

open cover. This implies that it is *smoothly Hausdorff* in that any two points can be separated by a smooth function.

We have now proved that:

Theorem 3.24 *The space LM is a smooth manifold.*

The topology on LM is usually given in a different way to that which we have used. The section $\tau : S^1 \rightarrow TS^1$, $\tau(t) = (t, \frac{\partial}{\partial t})$ defines a section $LM \rightarrow LTM$ and thus, by iteration, $LM \rightarrow LT^{(k)}M$. Composing this with the inclusion of smooth loops in continuous loops yields a family of maps $LM \rightarrow C(S^1, T^{(k)}M)$.

Proposition 3.25 *The topology on LM is the projective topology for this family of maps $LM \rightarrow C(S^1, T^{(k)}M)$.*

Proof. We apply proposition 3.20 with the section $\tau(t) = (t, \frac{\partial}{\partial t})$ to see that the following diagram commutes:

$$\begin{array}{ccc}
 L\mathbb{R}^n & \xrightarrow{\Psi_\alpha} & LM \\
 \beta \rightarrow \beta' \downarrow & & \downarrow \gamma \rightarrow \gamma' \\
 LTR^n & \xrightarrow{\Phi_\alpha} & LTM \\
 \downarrow & & \downarrow \\
 C(S^1, TR^n) & \longrightarrow & C(S^1, TM).
 \end{array}$$

The topology on $L\mathbb{R}^n$ is the projective limit of the obvious iteration of this diagram. Therefore we can construct a basis for the topology on LM out of sets which are the preimage of open sets in each $C(S^1, T^{(k)}M)$, which is precisely what is meant by the definition of the projective topology. \square

3.6 Loopy Maps

In this section we look at some maps involving loop spaces that ought to be smooth and show that indeed they are so.

One corollary of the proof of proposition 3.25 is that the canonical map $LM \rightarrow LTM$ given by $\alpha \rightarrow \hat{\alpha}$ is continuous. Further, we can show that it is smooth:

Lemma 3.26 *Let $\tau : S^1 \rightarrow TS^1$ be a section. The map $LM \rightarrow LTM$, $\alpha \rightarrow \hat{\alpha} = (d\alpha)\tau$, is smooth.*

Proof. Let $\alpha \in LM$. From the proof of proposition 3.25 we have charts for LM and LTM with chart maps Ψ_α and $\Phi_{\hat{\tau}\alpha}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 L\mathbb{R}^n & \xrightarrow{\Psi_\alpha} & LM \\
 \beta \rightarrow \hat{\tau}\beta \downarrow & & \downarrow \gamma \rightarrow \hat{\tau}\gamma \\
 LTR^n & \xrightarrow{\Phi_{\hat{\tau}\alpha}} & LTM.
 \end{array}$$

Hence if we can show that the statement of the lemma is true for $M = \mathbb{R}^n$ we can deduce that it holds for all M . However, this is obvious as it is a linear bounded map. \square

There is another class of maps that it will be useful to know are smooth:

Proposition 3.27 *Let $f : M \rightarrow N$ be a smooth map between smooth finite dimensional manifolds then $f^L : LM \rightarrow LN$ is smooth.*

Proof. This is essentially the same argument as when we showed that the transition functions were smooth. We need to show that the loop of f is smooth in charts. Let $\eta_M : TM \rightarrow M$ and $\eta_N : TN \rightarrow N$ be local additions with neighbourhoods V_M and V_N of the respective diagonals. Let $\alpha \in LM$. We therefore have charts $\Psi_M : \alpha^*TM \rightarrow U_M$ and $\Psi_N : (f^L\alpha)^*TN \rightarrow U_N$ at α and $(f^L\alpha)$ respectively.

Let $W \subseteq \alpha^*TM$ be the set:

$$\{(t, v) \in \alpha^*TM : (f(\alpha(t)), f(\eta_M(v))) \in V_N\}.$$

We claim that $\beta \in \Gamma_{S^1}(\alpha^*TM)$ takes values in W if and only if $f^L(\Psi_M(\beta)) \in U_N$. Let $\tilde{\beta} \in LTM$ be such that the section $t \rightarrow (t, \tilde{\beta}(t))$ is β . Then $\Psi_M(\beta) = \eta_M^L(\tilde{\beta})$. By the definition of U_N , $f^L(\Psi_M(\beta)) \in U_N$ if and only if $(f(\alpha(t)), f(\Psi_M(\beta))(t)) \in V_N$ for all t . As $f(\alpha(t)) = f(\alpha(t))$ and $f(\Psi_M(\beta))(t) = f(\eta_M(\tilde{\beta}(t)))$, this is equivalent to $(t, \tilde{\beta}(t)) \in W$ for all t , i.e. that β takes values in W .

As for the transition functions, let $\theta : W \rightarrow TN$ be the smooth function:

$$\theta(t, v) = (\pi_N \times \eta_N)^{-1}(f(\alpha(t)), f(\eta_M(v))).$$

Exactly as for the transition functions, $\pi_N\theta(t, v) = f(\alpha(t))$ and so the map $(t, v) \rightarrow (t, \theta(t, v))$ is a smooth map $W \rightarrow (f^L\alpha)^*TN$ (smooth as it is smooth into $S^1 \times TN$).

Hence we have a smooth map Θ from sections that take values in W into $\Gamma_{S^1}(f(\alpha)^*TN)$. It remains to show that this is the map induced by f ; that is, $f^L\Psi_M = \Psi_N\Theta$. This is straightforward. Let $\beta \in \Gamma_{S^1}(\alpha^*TM)$ take values in W and let $\tilde{\beta}$ be the related loop in TM . Then:

$$(f^L\Psi_M)(\beta)(t) = f(\eta_M(\tilde{\beta}(t)))$$

whilst:

$$(\Psi_N\Theta)(\beta) = \eta_N(\pi_N \times \eta_N)^{-1}(f(\alpha(t)), f(\eta_M(\tilde{\beta}(t)))) = f(\eta_M(\tilde{\beta}(t))).$$

Hence Θ is the map on charts corresponding to f^L . Thus $f^L : LM \rightarrow LN$ is smooth. \square

The other result about maps that we wish to prove is a variant of the exponential law:

Theorem 3.28 *Let M, N be smooth manifolds with M finite dimensional. Let $f : N \rightarrow LM$ be a map and let $f^\vee : S^1 \times N \rightarrow M$ be the adjoint: $f^\vee(t, x) = f(x)(t)$. Then f is smooth if and only if f^\vee is smooth. That is, the assignment $f \rightarrow f^\vee$ is an identification:*

$$C^\infty(N, LM) \cong C^\infty(S^1 \times N, M).$$

As stated, this is a corollary of [KM97, Theorem 42.14] with their $(\mathcal{M}, M, \mathcal{N})$ corresponding to our (N, S^1, M) . We give a self-contained proof here.

Proof. We start with the linear case: $M = \mathbb{R}^n$ and N is an open subset of a convenient vector space. Thus we have $f : N \rightarrow L\mathbb{R}^n$ and $f^\vee : S^1 \times N \rightarrow \mathbb{R}^n$.

Let $\pi : \mathbb{R} \rightarrow S^1$ be the standard covering map. Now $S^1 \times N$ is a smooth manifold modelled on $\mathbb{R} \times N$ using the usual charts for S^1 so $\pi \times 1 : \mathbb{R} \times N \rightarrow S^1 \times N$ is a local diffeomorphism. Hence f^\vee is smooth if and only if $f^\vee(\pi \times 1)$ is smooth.

On the other side, pre-composition with π defines a linear embedding of $L\mathbb{R}^n$ onto a closed subspace of $C^\infty(\mathbb{R}, \mathbb{R}^n)$ as in the proof of proposition 3.7. Therefore f is smooth if and only if the map $x \rightarrow f(x) \circ \pi$ is smooth.

The adjoint of $x \rightarrow f(x) \circ \pi$ is the map:

$$(t, x) \rightarrow (f(x) \circ \pi)(t) = f(x)(\pi(t)) = f^\vee(\pi(t), x) = f^\vee(\pi \times 1)(t, x).$$

Thus by the original exponential law, theorem 3.4, the map $x \rightarrow f(x) \circ \pi$ is smooth if and only if $f^\vee(\pi \times 1)$ is smooth. Hence f is smooth if and only if f^\vee is smooth.

In the general case, we use charts to move into the linear one. However, with things as they stand we find that we are only partially successful: we may assume that N is an open subset of a convenient vector space and that the codomain of f is contained in the codomain of a chart but it may not be possible to assume that the image of $f^\vee : S^1 \times N \rightarrow M$ is contained in the codomain of a chart. The easiest way around this is to enhance f^\vee to the map $f^\wedge : S^1 \times N \rightarrow S^1 \times M$, $(t, x) \rightarrow (t, f^\vee(t, x))$. This is smooth if and only if f^\vee is smooth.

Let $\Psi_\alpha : \Gamma_{S^1}(\alpha^*TM) \rightarrow U_\alpha$ be a chart for LM defined using a local addition, η , on M with neighbourhood V of the diagonal in $M \times M$. Let $V_\alpha \subseteq S^1 \times M$ be the open set $\{(t, x) : (\alpha(t), x) \in V\}$. We claim that the evaluation $S^1 \times U_\alpha \rightarrow S^1 \times M$ maps into V_α and moreover that there is a commutative diagram:

$$\begin{array}{ccc} S^1 \times \Gamma_{S^1}(\alpha^*TM) & \xrightarrow{1 \times \Psi_\alpha} & S^1 \times U_\alpha \\ \downarrow & & \downarrow \\ \alpha^*TM & \xrightarrow{\psi_\alpha} & V_\alpha \end{array} \quad (3.1)$$

where both vertical maps are evaluations and both horizontal maps are diffeomorphisms. The map ψ_α is defined as follows: there is a map

$$\alpha^*TM \subseteq S^1 \times TM \xrightarrow{1 \times \eta} S^1 \times M$$

which, we claim, has image V_α and is a diffeomorphism onto that image. To see this, consider the diagram:

$$\begin{array}{ccccc} S^1 \times TM & \xrightarrow{1 \times \pi \times \eta} & S^1 \times V & \xrightarrow{\subseteq} & S^1 \times M \times M \\ \uparrow & & \uparrow & & \uparrow \\ \alpha^*TM & & V_\alpha & \xrightarrow{\subseteq} & S^1 \times M, \end{array}$$

where the right-hand vertical map is $(t, x) \rightarrow (t, \alpha(t), x)$. Each vertical map is an embedding and the first upper horizontal map is a diffeomorphism. It is clear that α^*TM corresponds to V_α on the upper level and hence the map $\alpha^*TM \rightarrow V_\alpha$ is a diffeomorphism.

To see that diagram (3.1) commutes, let $\beta \in \Gamma_{S^1}(\alpha^*TM)$ and $\tilde{\beta} \in LTM$ be related as usual, so $\Psi_\alpha(\beta) = \eta(\tilde{\beta})$. Thus $\Psi_\alpha(\beta)(t) = \eta(\tilde{\beta})(t) = \eta(\tilde{\beta}(t))$ and so going

along and then down is the map:

$$(t, \beta) \rightarrow (t, \eta(\tilde{\beta}(t))).$$

Going down and then along, we get:

$$(1 \times \eta)(\beta(t)) = (1 \times \eta)(t, \tilde{\beta}(t)) = (t, \eta(\tilde{\beta}(t))).$$

Now f^\wedge is the composition of:

$$S^1 \times N \xrightarrow{1 \times f} S^1 \times U_\alpha \rightarrow V_\alpha \subseteq S^1 \times M$$

where the map $S^1 \times U_\alpha \rightarrow V_\alpha$ is the map $(t, \gamma) \rightarrow (t, \gamma(t))$. Thus the pair (f, f^\wedge) defines a pair (g, g^\wedge) with $g : N \rightarrow \Gamma_{S^1}(\alpha^*TM)$ and $g^\wedge : S^1 \times N \rightarrow \alpha^*TM$ such that g is smooth if and only if f is smooth and g^\wedge smooth if and only if f^\wedge is smooth.

A smooth trivialisation of α^*TM identifies that space with $S^1 \times \mathbb{R}^n$ and sections with $L\mathbb{R}^n$. Thus we can regard g and g^\wedge as maps into $L\mathbb{R}^n$ and $S^1 \times \mathbb{R}^n$ respectively. Moreover, $g^\wedge(t, x) = (t, g^\vee(t, x))$ so g^\wedge is smooth if and only if $g^\vee : S^1 \times N \rightarrow \mathbb{R}^n$ is smooth.

We are thus in the linear case and so g is smooth if and only if g^\vee is smooth. Hence f is smooth if and only if f^\vee is smooth. \square

Corollary 3.29 *Let $e : S^1 \times LM \rightarrow M$ be the evaluation map: $e(t, \gamma) = \gamma(t)$. Let $\iota : M \rightarrow LM$ be the inclusion of constant loops: $\iota(x) = (t \rightarrow x)$. Both e and ι are smooth.*

Proof. The map e is adjoint to the identity $LM \rightarrow LM$, hence is smooth, whereas ι adjoints to the projection map $S^1 \times M \rightarrow M$, hence is smooth. \square

For the second of those maps we can actually prove a stronger result:

Proposition 3.30 *The map $\iota : M \rightarrow LM$ is an embedding.*

Proof. Let $\eta : TM \rightarrow M$ be a local addition on M with $V \subseteq M \times M$ the corresponding neighbourhood of the diagonal. For $x \in M$ let $\eta_x : T_xM \rightarrow V_x \subseteq M$ be the restriction of η to T_xM , where $V_x = \eta(T_xM)$ is such that $\{x\} \times V_x = (\{x\} \times M) \cap V$. It is a simple result from finite dimensional differential topology that this family defines an atlas for M .

Let $\Psi_x : \Gamma_{S^1}(\gamma_x^*TM) \rightarrow U_x$ be the chart for LM at the constant loop γ_x . As γ_x is a constant loop, $\gamma_x^*TM = S^1 \times T_xM$. Define $T_xM \rightarrow \Gamma_{S^1}(\gamma_x^*TM)$ by $v \rightarrow \beta_v$ where $\beta_v(t) = (t, v)$. This is smooth since it is linear and bounded.

Now $\{x\} \times V_x = (\{x\} \times M) \cap V$ and $\{\gamma_x\} \times U_x = (\{\gamma_x\} \times LM) \cap LV$. Under the inclusion $M \rightarrow LM$ and $V \rightarrow LV$ we thus have $V_x = M \cap U_x$. We claim that the following diagram commutes:

$$\begin{array}{ccc} T_xM & \xrightarrow{\eta_x} & V_x \\ \downarrow & & \downarrow \\ \Gamma_{S^1}(\gamma_x^*TM) & \xrightarrow{\Psi_x} & U_x. \end{array}$$

To see this, let $\beta \in \Gamma_{S^1}(\gamma_x^*TM)$ and $\tilde{\beta} \in LTM$ be related as usual. Thus $\tilde{\beta}$ satisfies $\pi\tilde{\beta} = \gamma_x$ and so $\tilde{\beta} \in LT_xM$. By definition, $\Psi_x(\beta) = \eta(\tilde{\beta}) = \eta_x(\tilde{\beta})$. Now $\tilde{\beta}_v$ is the

constant map at $v \in T_x M$ and so $\Psi_x(\beta_v)$ is the constant map at $\eta_x(v)$. Hence the above diagram commutes.

Thus these charts satisfy the requirements for exhibiting M as a submanifold of LM . \square

3.7 Vector Space Idol

In proving the topological results about the space of smooth loops we used the fact that the chart maps extend naturally to continuous loops. In fact, the chart maps can be used to construct a manifold structure on the spaces of many different types of loop. The key ingredients are:

1. The type of loop is at least continuous (with topology at least as fine as the compact-open topology) and at most smooth, and
2. The smooth structure is diffeomorphism-invariant. That is to say, for $V, W \subseteq S^1 \times \mathbb{R}^n$ open subsets and $\psi : V \rightarrow W$ a diffeomorphism, the induced map ψ^L on the appropriate type of sections is a diffeomorphism.

With these two properties, we can adapt the work of section 3.3 to this new type of loop. To keep the alterations as straightforward as possible, we still anchor our charts at genuinely smooth loops: the first condition above ensures that these charts still cover the loop space. If we allowed other anchors, we would have to expand the second condition above to homeomorphisms that are fibrewise diffeomorphisms and satisfy some sort of condition amongst the fibres.

Where we may run into trouble is with the topological properties of the new loop space. In particular, we need the model space to be second countable to prove that the loop space is second countable. If this fails, paracompactness is also thrown into doubt. The space will always be Hausdorff and regular, though.

Thus, for example, the space of continuous loops is a smooth manifold, as is the space of H^1 -Sobolev loops. However, this technique is not going to define the space of L^2 -loops.

4 Looping Bundles

In this section we consider the general construction of looping a bundle. That is, given a bundle $\pi : E \rightarrow M$ we consider the resulting triple $\pi^L : LE \rightarrow LM$. Our central theme is vector bundles, but we take in principal bundles, gauge bundles, and connections on the way. We start by considering the tangent space of LM , showing that it is diffeomorphic to the loop of the tangent space of M . We conclude by commenting that this does not hold for the cotangent bundle.

4.1 The Tangent Space

Tangent spaces in infinite dimensions can be problematic: there is the *kinematic* tangent space consisting of the derivatives of short curves and the *operational* tangent space consisting of derivations of functions. In finite dimensions these are the same, but that need not hold in infinite dimensions. The kinematic always satisfies $TE \cong E \times E$ for a convenient vector space E , but the operational tangent space may be much larger – it at least contains the bidual ([KM97, 28.3]). We shall not go into a detailed discussion here since for $L\mathbb{R}^n$, [KM97, Theorem 28.7] implies that the two notions coincide. For more on the types of tangent vector see [KM97, §28].

We start with a straightforward result on the structure of the tangent bundle.

Proposition 4.1 *The tangent bundle of LM , TLM , has the structure of a bundle of $L\mathbb{R}$ -modules.*

Proof. We have local charts $U_\alpha \cong L\mathbb{R}^n$ and so locally $TU_\alpha \cong L\mathbb{R}^n \times L\mathbb{R}^n$. Thus we can attempt to transfer the natural $L\mathbb{R}$ -module structure on $L\mathbb{R}^n$ to the tangent spaces of LM . Lemma 3.9 shows that we can do this as from it we deduce that the derivatives of the transition functions are $L\mathbb{R}$ -linear. \square

Our main theorem is the following characterisation of the tangent space.

Theorem 4.2 *There is a natural diffeomorphism $TLM \rightarrow LTM$ covering the identity on LM .*

The idea is that a small perturbation of a loop involves a small perturbation of each of its points and so we have a loop of small perturbations of points.

Proof. The evaluation map $S^1 \times LM \rightarrow M$ differentiates to a smooth map: $TS^1 \times TLM \rightarrow TM$. We compose this with the zero section $S^1 \rightarrow TS^1$ to get a smooth map $S^1 \times TLM \rightarrow TM$. This is the adjoint of a map $TLM \rightarrow LTM$ which is thereby shown to be smooth. As the map $S^1 \times TLM \rightarrow TM$ projects down to the evaluation map $S^1 \times LM \rightarrow M$, the map $TLM \rightarrow LTM$ projects down to the identity on LM .

In the case of Euclidean space, we have a natural identifications of $TL\mathbb{R}^n$ with $L\mathbb{R}^n \times L\mathbb{R}^n$ and of $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$. It is simple to show that under these identifications, the map $S^1 \times TL\mathbb{R}^n \rightarrow T\mathbb{R}^n$ described above becomes $(t, \alpha, \beta) \rightarrow (\alpha(t), \beta(t))$. The adjoint of this is the canonical isomorphism $L\mathbb{R}^n \times L\mathbb{R}^n \rightarrow L(\mathbb{R}^n \times \mathbb{R}^n)$, which is a diffeomorphism. Hence the result is true for Euclidean space.

For the more general case we consider the local situation. We choose a chart at a loop $\alpha \in LM$. Using a trivialisation of α^*TM we extend the chart map to $\Psi_\alpha : L\mathbb{R}^n \rightarrow U_\alpha$. This trivialisation also extends diagram (3.1) of the proof of

theorem 3.28 in the obvious way. Hence the evaluation map $S^1 \times U_\alpha \rightarrow M$ factors as:

$$S^1 \times U_\alpha \xrightarrow{1 \times \Psi_\alpha^{-1}} S^1 \times LR^n \rightarrow S^1 \times \mathbb{R}^n \rightarrow TM \xrightarrow{\eta} M,$$

We differentiate this and evaluate on the zero section to get:

$$S^1 \times TU_\alpha \rightarrow S^1 \times TLR^n \rightarrow S^1 \times T\mathbb{R}^n \rightarrow T^{(2)}M \xrightarrow{d\eta} TM.$$

We now 'de-adjointise' this. As the second map is $(t, \alpha) \rightarrow (t, \alpha(t))$, the image of TLR^n in $L(S^1 \times T\mathbb{R}^n)$ is $\Gamma_{S^1}(S^1 \times T\mathbb{R}^n)$ which we identify in the usual way with LTR^n . Thus we get:

$$TU_\alpha \rightarrow TLR^n \rightarrow LTR^n \rightarrow LT^{(2)}M \rightarrow LTM.$$

From proposition 3.20, the induced map $LTR^n \rightarrow LTM$ is a chart map, hence a diffeomorphism onto its image. The first map is the derivative of a chart map for LM , hence is a diffeomorphism. We have already shown that the second map is a diffeomorphism.

Thus the map $TLM \rightarrow LTM$ is a local diffeomorphism where by "local" we mean in LM . As both are spaces over LM and the map is of spaces over LM we therefore have a diffeomorphism. \square

Corollary 4.3 *There is a canonical identification as LR -modules of $T_\alpha LM$ with $\Gamma_{S^1}(\alpha^*TM)$.*

Proof. As the isomorphism $TLM \rightarrow LTM$ covers the identity on LM , the tangent space at $\alpha \in LM$ identifies with the set of loops in TM which project down to α . As we have already seen, this is naturally $\Gamma_{S^1}(\alpha^*TM)$.

For the LR -module structure, we observe that a trivialisation of α^*TM identifies $T_\alpha LM$ with T_0LR^n and $\Gamma_{S^1}(\alpha^*TM)$ with LR^n . The identification between these is the canonical identification of T_0LR^n with LR^n which is LR -linear. \square

This needs a word of explanation. The chart map at $\alpha \in LM$, $\Psi_\alpha : \Gamma_{S^1}(\alpha^*TM) \rightarrow U_\alpha$ identifies the tangent space at a point of U_α with $\Gamma_{S^1}(\alpha^*TM)$ and so in particular we have an identification of the tangent space at α with $\Gamma_{S^1}(\alpha^*TM)$. The problem with this is that it may depend on the local addition η used to define the chart map. Moreover, we have also relaxed the definition of a local addition so that the vector bundle did not need to be TM , though *a fortiori* it has to lie in the same isomorphism class. In this case, the chart map identifies the tangent space of α with $\Gamma_{S^1}(\alpha^*E)$ which, although isomorphic to $\Gamma_{S^1}(\alpha^*TM)$ does not seem to be canonically so.

The solution is that differentiating the diffeomorphism $(\pi \times \eta) : E \rightarrow V$ defines an isomorphism of bundles $\pi^*E \cong \ker d\pi \rightarrow (\pi \times \eta)^*TM$, where the TM is the second factor in $T(M \times M)$. Over the zero section of E we thus get an isomorphism $E \rightarrow TM$ defined canonically from the local addition. The isomorphism $T_\alpha LM \rightarrow \Gamma_{S^1}(\alpha^*TM)$ uses this isomorphism of bundles. One caveat of this is that if it so happens that our local addition was defined on TM then the canonical identification of the above corollary may not be the simple identification coming from the chart map.

Now that we have this identification of the tangent bundle, we can extend proposition 3.27.

Proposition 4.4 *Let $f : M \rightarrow N$ be a smooth map between finite dimensional smooth manifolds. Under the identifications $TLM \cong LTM$ and $TLN \cong LTN$ we have: $d(f^L) = (df)^L$.*

Proof. This is a local result so it is sufficient to work in LR^n , where it is obvious. \square

4.2 Vector Bundles

We start by considering various loop spaces associated to a vector bundle. We shall formulate our results for a general finite dimensional vector bundle $\xi := E \xrightarrow{\pi_\xi} M$. To avoid dealing with the twisted situation, we assume that E is orientable.

The fibrewise vector space structure on E can be thought of as consisting of two maps: $\mathbb{R} \times E \rightarrow E$ and $E \times_M E \rightarrow E$, satisfying certain relations. As before, $E \times_M E$ is an embedded submanifold of $E \times E$ and so a map into $E \times_M E$ is smooth if and only if the composition with the two projections to E are smooth.

We can loop both of these maps and get: $L\mathbb{R} \times LE \rightarrow LE$ and $L(E \times_M E) \rightarrow LE$. These satisfy similar relations to the ones satisfied by the maps on E . To identify $L(E \times_M E)$, from the remark above this consists of pairs of loops in E which, time-by-time, project down to the same point on M . That is:

$$L(E \times_M E) = \{(\alpha, \beta) \in LE \times LE : \pi_\xi(\alpha(t)) = \pi_\xi(\beta(t)) \text{ for all } t \in S^1\}.$$

Now $\pi_\xi(\alpha(t)) = \pi_\xi(\beta(t))$ for all t if and only if $\pi_\xi^L(\alpha) = \pi_\xi^L(\beta)$. Thus $L(E \times_M E) = LE \times_{LM} LE$.

Hence we have a structure on the fibres of $\pi_{L\xi} := \pi_\xi^L : LE \rightarrow LM$ similar to that on the fibres of $\pi_\xi : E \rightarrow M$. That is, the fibres of $LE \rightarrow LM$ naturally have the structure of LR -modules. Note that this implies that they are vector spaces as there is an inclusion $\mathbb{R} \rightarrow LR$. What we wish to show is that this structure is locally trivial and thus defines an LR -module bundle over LM .

We start by noting that, as for the case $E = TM$, the fibre of LE above a loop $\alpha \in LM$ can be naturally considered to be $\Gamma_{S^1}(\alpha^*E)$. Moreover, the LR -module structure defined above on the fibres of $LE \rightarrow LM$ is the natural LR -module structure on $\Gamma_{S^1}(\alpha^*E)$. Thus once we have established that LE is a locally trivial bundle of LR -modules over LM , the isomorphism of theorem 4.2 will be of LR -module bundles.

The mainstay of the analysis of the loop space of E in terms of that of M is that we do not need to use a local addition on the whole of E to define the charts for LE . All we need is the local addition on M together with a connection on E .

The bundle $TM \oplus E$ can be realised by the *Whitney sum*. This identifies the total space of $TM \oplus E$ with the pull-back of $TM \times E \rightarrow M \times M$ by the diagonal map $M \rightarrow M \times M$. Thus the total space of $TM \oplus E$ is:

$$TM \times_M E := \{(u, v) \in TM \times E : \pi(u) = \pi_\xi(v)\}.$$

The obvious projection maps $TM \times_M E$ to TM and to E fit together to give a commuting diagram:

$$\begin{array}{ccc} TM \times_M E & \longrightarrow & E \\ \downarrow & & \downarrow \pi_\xi \\ TM & \xrightarrow{\pi} & M. \end{array}$$

Thus we can think of $TM \times_M E$ as the total space of any of: $TM \oplus E \rightarrow M$, $\pi^*E \rightarrow TM$, or $\pi_\xi^*TM \rightarrow E$.

Let $\eta : TM \rightarrow M$ be a local addition on M with $V \subseteq M \times M$ the corresponding open neighbourhood of the diagonal. For $v \in TM$, define $P_u : E_{\pi(u)} \rightarrow E_{\eta(u)}$ to be the operation of parallel transport along the path $t \rightarrow \eta(tv)$ (recall that $\eta(0u) = \pi(u)$). Using this, we define a variant of local addition. Define $\mu : TM \times_M E \rightarrow E$ by $\mu(u, v) = P_u(v)$. Note that this is well-defined since $\pi_\xi(v) = \pi(u)$ so $v \in E_{\pi(u)}$. Observe that $\pi_\xi \mu(u, v) = \eta(u)$ so we have a commutative diagram:

$$\begin{array}{ccc} TM \times_M E & \xrightarrow{\mu} & E \\ \downarrow & & \downarrow \pi_\xi \\ TM & \xrightarrow{\eta} & M. \end{array}$$

Consider the map $\pi \times \mu : TM \times_M E \rightarrow M \times E$ where, by abuse of notation, the map $\pi : TM \times_M E \rightarrow M$ is $(u, v) \rightarrow \pi(u) = \pi_\xi(v)$. Let $E_V := \{(x, v) \in M \times E : (x, \pi_\xi(v)) \in V\}$.

We claim that $\pi \times \mu : TM \times_M E \rightarrow E_V$ is a diffeomorphism. The most geometric way to see this is to observe that there is another space hidden in the background, namely:

$$TM \times_\eta E := \{(u, v) \in TM \times E : \eta(u) = \pi_\xi(v)\}.$$

This is the pull-back of E over TM via η , so the following diagram commutes:

$$\begin{array}{ccc} TM \times_\eta E & \xrightarrow{\tilde{\mu}} & E \\ \downarrow & & \downarrow \pi_\xi \\ TM & \xrightarrow{\eta} & M \end{array}$$

The map $\pi \times \eta : TM \rightarrow V \subseteq M \times M$ is a diffeomorphism, so $\pi \times \tilde{\mu} : TM \times_\eta E \rightarrow E_V \subseteq M \times E$ is also a diffeomorphism.

Now $TM \times_\eta E$ is isomorphic to $TM \times_M E$ as bundles over TM since $\pi : TM \rightarrow M$ and $\eta : TM \rightarrow M$ are homotopic maps. An explicit homotopy is $h(v, t) = \eta(tv)$. This defines an explicit isomorphism between the bundles by parallel transporting the bundle along the fibres of the homotopy, i.e. along the paths $t \rightarrow \eta(tv)$. This isomorphism is $(u, v) \rightarrow (u, P_u(v))$. Hence the composition of this with $\tilde{\mu}$ is μ . Thus $\pi \times \mu : TM \times_M E \rightarrow E_V$ is a diffeomorphism.

We wish to play the same game for $\mu : TM \times_M E \rightarrow E$ that we had for $\eta : TM \rightarrow M$. That is, we wish to construct maps that have the effect of sending a section β of something to the loop $t \rightarrow \mu(\beta(t))$ in E . Clearly, we must be able to think of β as a loop in $TM \times_M E$, just as in the case of M we thought of a section of α^*TM as a loop in TM . We have three candidates for what the section is of since $TM \times_M E$ has the structure of a vector bundle over each of TM , E , and M . To determine the correct one, we recall from the story for M that the two components of the map $\pi \times \eta : TM \rightarrow V \subseteq M \times M$ were used as follows: the second defined the chart map, the first told us which chart we were in.

Therefore, as we have $\pi \times \mu : TM \rightarrow E_V \subseteq M \times E$, we should try to index the charts by loops in M , *not* by loops in E . Thus the domain of the chart map will be sections of $\alpha^*(TM \times_M E)$ for $\alpha \in LM$. We can write this as $\alpha^*(TM \oplus E)$ to

indicate this without constantly needing to remind ourselves that α is a loop in M .

The codomain of the chart map is the set of loops in E with the property that $(\alpha(t), \beta(t)) \in E_V$, for all $t \in S^1$. Now β satisfies this condition if and only if $(\alpha(t), \pi_\xi(\beta(t))) \in V$ for all $t \in S^1$. Thus the codomain is $\{\beta \in LE : \pi_\xi^L(\beta) \in U_\alpha\}$. Write this as LE_{U_α} . Note that this is the pull-back of $LE \rightarrow LM$ under the inclusion $U_\alpha \rightarrow LM$. These cover LE because the U_α cover LM .

One can alter $\mu : TM \times_M E \rightarrow E$ to a full local addition on E by taking the pull-back of $TM \oplus E \rightarrow M$ via $\pi_\xi : E \rightarrow M$. The resulting bundle is isomorphic to TE and the map $\mu\pi_{\xi_*} : \pi_\xi^*(TM \oplus E) \rightarrow E$ is a local addition. This can then be used to show that the smooth structure coming from the charts constructed above is the same as the natural smooth structure on LE .

We return to the charts defined by $\mu : TM \times_M E \rightarrow E$. The domain of a chart is $\Gamma_{S^1}(\alpha^*(TM \oplus E))$. The total space of $\alpha^*(TM \oplus E)$ is the subset of $S^1 \times (TM \times_M E)$ such that $\alpha(t) = \pi(u, v)$. From the definition of $TM \times_M E$ we can see that this is the subset of $S^1 \times TM \times E$ such that $\alpha(t) = \pi(u) = \pi_\xi(v)$. Therefore a section of $\alpha^*(TM \oplus E)$ can be written in the form $t \rightarrow (t, \beta(t), \gamma(t))$ where $\beta : S^1 \rightarrow TM$ and $\gamma : S^1 \rightarrow E$ are such that $\alpha = \pi^L\beta = \pi_\xi^L\gamma$ for all $t \in S^1$. Thus the map:

$$(t \rightarrow (t, \beta(t), \gamma(t))) \rightarrow \left((t \rightarrow (t, \beta(t))), (t \rightarrow (t, \gamma(t))) \right)$$

identifies $\Gamma_{S^1}(\alpha^*(TM \oplus E))$ with $\Gamma_{S^1}(\alpha^*TM) \times \Gamma_{S^1}(\alpha^*E)$. (It is simple to show that the identification $L\mathbb{R}^{n+m} \cong L\mathbb{R}^n \times L\mathbb{R}^m$ preserves all the structure including the smooth structure, so the identification of the spaces of sections given above also preserves the smooth structure.)

We have:

$$\Theta_\alpha : \Gamma_{S^1}(\alpha^*TM) \times \Gamma_{S^1}(\alpha^*E) \rightarrow LE_{U_\alpha}$$

satisfying:

$$\Theta_\alpha(\beta, \gamma)(t) = \mu(\tilde{\beta}(t), \tilde{\gamma}(t)) = P_{\tilde{\beta}(t)}(\tilde{\gamma}(t)),$$

where $\beta(t) = (t, \tilde{\beta}(t))$ and $\gamma(t) = (t, \tilde{\gamma}(t))$.

Let $\gamma_1, \gamma_2 \in \Gamma_{S^1}(\alpha^*E)$. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in LE$ be the corresponding loops. Let $v \in L\mathbb{R}$. The loop in LE corresponding to the section $\gamma_1 + v\gamma_2$ is $\tilde{\gamma}_1 + v\tilde{\gamma}_2$. Therefore, since $v \rightarrow P_u(v)$ is linear:

$$\begin{aligned} \Theta_\alpha(\beta, \gamma_1 + v\gamma_2)(t) &= P_{\tilde{\beta}(t)}(\tilde{\gamma}_1(t) + v(t)\tilde{\gamma}_2(t)) \\ &= P_{\tilde{\beta}(t)}(\tilde{\gamma}_1(t)) + v(t)P_{\tilde{\beta}(t)}(\tilde{\gamma}_2(t)) \\ &= (\Theta_\alpha(\beta, \gamma_1) + v\Theta_\alpha(\beta, \gamma_2))(t). \end{aligned}$$

Hence Θ_α is $L\mathbb{R}$ -linear in its second argument.

Thus $LE \rightarrow LM$ is a locally trivial $L\mathbb{R}$ -module bundle. The fibres are modelled on the spaces $\Gamma_{S^1}(\alpha^*E)$. Since we assumed that E is orientable, these are all isomorphic to $L\mathbb{R}^n$ (as $L\mathbb{R}$ -modules) via a smooth trivialisation of $\alpha^*E \rightarrow S^1$.

4.3 Principal and Gauge Bundles

The key in the analysis of the vector bundle was the existence of the parallel transport operator and the fact that it is linear. There are other bundles with such operators, the most common being principal bundles.

Let G be a finite dimensional Lie group, $Q \rightarrow M$ a principal G -bundle. As always, we shall assume that it is *orientable* in that $\alpha^*Q \rightarrow S^1$ admits a section for any loop $\alpha : S^1 \rightarrow M$. Now LQ inherits an action of the loop group LG by $(\alpha \cdot \gamma)(t) = \alpha(t) \cdot \gamma(t)$. The space of sections of α^*Q also has such an action. A trivialisation, $\alpha^*Q \cong S^1 \times G$, identifies $\Gamma_{S^1}(\alpha^*Q)$ with LG and this is an isomorphism of LG -spaces.

A connection on Q defines a parallel transport operator along any path and this preserves the fibrewise G -action. Therefore the same analysis as for the case of the vector bundle leads to maps:

$$\Theta_\alpha : \Gamma_{S^1}(\alpha^*TM) \times \Gamma_{S^1}(\alpha^*Q) \rightarrow LQ_{U_\alpha}$$

which are LG -equivariant. This shows that $LQ \rightarrow LM$ is a locally trivial principal LG -bundle.

Suppose that S is a (finite dimensional) smooth manifold with a G -action. We can form the locally trivial bundle $R := Q \times_G S \rightarrow M$. It is clear that LR can be described as $LQ \times_{LG} LS \rightarrow LM$ and that this is a locally trivial bundle modelled on LS . Structure on S that is preserved by G defines structure on R so since LG preserves the “looped structure” on LS , there is a corresponding structure on LR . We have already seen this in the vector bundle situation in that the \mathbb{R} -module (i.e. vector space) structure of the fibres of E looped to give an $L\mathbb{R}$ -module structure on the fibres of LE .

Another important situation where this arises is when G acts on itself via the adjoint action. This defines the *gauge group* corresponding to Q , $Q^{\text{ad}} := Q \times_{\text{ad}} G$ which is a bundle of groups over M modelled on G . The corresponding loop space, LQ^{ad} , is a bundle of groups over LM modelled on LG . We have the identity $L(Q^{\text{ad}}) = (LQ)^{\text{ad}} := LQ \times_{\text{ad}} LG$.

Returning to the vector bundle case, if $E = Q \times_G \mathbb{R}^n$ then $LE = LQ \times_{LG} L\mathbb{R}^n$, as $L\mathbb{R}$ -modules. Now as Q is the frame bundle of E , it would be nice to be able to say that LQ is the frame bundle of LE . This is not true as a statement about vector bundles, since $\text{Gl}(L\mathbb{R}^n) \neq L\text{Gl}(\mathbb{R}^n)$, but is true as a statement about $L\mathbb{R}$ -modules as a consequence of:

Proposition 4.5 *Let $g : L\mathbb{R}^n \rightarrow L\mathbb{R}^n$ be an isomorphism of $L\mathbb{R}$ -modules. Then $g \in L\text{Gl}(\mathbb{R}^n)$.*

Proof. For a ring R , module M , and subset $S \subseteq M$, let $\langle S \rangle_R$ denote the R -linear span of S in M . We add the subscript R to the usual notation as we will have a space being a module over two different rings.

It is mildly obvious that the space $L\mathbb{R}^n$ is a free $L\mathbb{R}$ -module of rank n . Let \mathcal{B} be an n -element generating set (basis).

Let $t \in S^1$. The evaluation map $e_t : L\mathbb{R} \rightarrow \mathbb{R}$ is a ring map and so converts any \mathbb{R} -module (i.e. vector space) into an $L\mathbb{R}$ -module. The corresponding evaluation map $e_t : L\mathbb{R}^n \rightarrow \mathbb{R}^n$ is then a map of $L\mathbb{R}$ -modules. Then:

$$\mathbb{R}^n = e_t(L\mathbb{R}^n) = \langle e_t(\mathcal{B}) \rangle_{L\mathbb{R}} = \langle e_t(\mathcal{B}) \rangle_{\mathbb{R}}.$$

The first equality is because $e_t : L\mathbb{R}^n \rightarrow \mathbb{R}^n$ is surjective and the last because the action of $L\mathbb{R}$ on \mathbb{R}^n factors through \mathbb{R} . As $e_t(\mathcal{B})$ is an n -element set, it is therefore a basis for \mathbb{R}^n .

Let $g : L\mathbb{R}^n \rightarrow L\mathbb{R}^n$ be an $L\mathbb{R}$ -isomorphism. For $j = 1, \dots, n$ let $\alpha_j \in L\mathbb{R}^n$ be the constant loop at the j th element of the standard basis of \mathbb{R}^n . Define a loop

γ in the space of $n \times n$ matrices, $M_n(\mathbb{R})$, by arranging the loops $g\alpha_j$ in columns:

$$\gamma(t) = ((g\alpha_1)(t), \dots, (g\alpha_n)(t)).$$

Now $\{\alpha_1, \dots, \alpha_n\}$ is a generating set for $L\mathbb{R}^n$ as an $L\mathbb{R}$ -module. As g is an $L\mathbb{R}$ -isomorphism, $\{g\alpha_1, \dots, g\alpha_n\}$ is also a generating set for $L\mathbb{R}^n$. Therefore at each $t \in S^1$, $\{(g\alpha_1)(t), \dots, (g\alpha_n)(t)\}$ is a basis for \mathbb{R}^n . Hence γ takes values in $\text{Gl}(\mathbb{R}^n)$.

Now the action of $L\text{Gl}(\mathbb{R}^n)$ on $L\mathbb{R}^n$ is $L\mathbb{R}$ -linear as it is the loop of the \mathbb{R} -linear action of $\text{Gl}(\mathbb{R}^n)$ on \mathbb{R}^n . As α_j is the constant loop at the j th standard basis element, $\gamma\alpha_j$ selects the j th column of γ , which is $g\alpha_j$. Hence g and γ are both $L\mathbb{R}^n$ -linear maps which agree on a generating set, thus are the same map. \square

The proof of this result shows that the correct interpretation of an $L\mathbb{R}$ -frame of $LE \rightarrow LM$ at a loop $\alpha \in LM$ is a smooth choice of frame of each $E_{\alpha(t)}$. Thus an $L\mathbb{R}$ -frame of LE is a trivialisation of α^*E . To get back the isomorphism $L\mathbb{R}^n \rightarrow L_\alpha E$, we use the trivialisation to identify $L\mathbb{R}^n$ with $\Gamma_{S^1}(\alpha^*E)$ which is naturally identified with the fibre $L_\alpha E$.

4.4 Connections

In this section we consider connections on vector bundles. As with the vector bundles themselves, we are interested in structure on the loop space that arises from structure on the original space. Our main theorem in this section is:

Theorem 4.6 *A principal connection in finite dimensions loops to a principal connection on the loop space.*

There are several ways of thinking of connections in finite dimensions and these carry over to infinite dimensions. The general theory is contained in [KM97, §37] of which we recall the basics here.

Let G be a Lie group, $\pi : P \rightarrow M$ a principal G -bundle. We are considering both the finite and infinite dimensional cases here. The two main ways that we think of a connection are:

1. A projection $\Phi : TP \rightarrow VP$, where $VP = \ker d\pi$ is the vertical tangent bundle, which is G -equivariant for the G action on P . That is, for each $g \in G$ and $u \in P$ the following diagram commutes:

$$\begin{array}{ccc} T_u P & \xrightarrow{dr^g} & T_{ug} P \\ \Phi \downarrow & & \Phi \downarrow \\ V_u P & \xrightarrow{dr^g} & V_{ug} P, \end{array}$$

where $r^g : P \rightarrow P$, $r^g(u) = ug$, is the action of $g \in G$.

2. The connection one-form $\omega : TP \rightarrow \mathfrak{g}$, the Lie algebra of G , satisfying:
 - (a) $w(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$, where $\zeta_X \in \mathfrak{X}(P)$ is the vector field defined by $\zeta_X(u) := dr_{(u,e)}(0, X)$ (note that $dr_{(u,e)}$ is a linear map $T_u P \times T_e G \rightarrow T_u P$);
 - (b) w is G -equivariant: $((r^g)^*\omega)(X) = \text{Ad}_{g^{-1}} \omega(X)$ for all $g \in G$ and $X \in T_u P$.

The equivalence is given by $\Phi(X) = \zeta_{\omega(X)}(u)$ for $X \in T_uP$.

We can loop both maps to get $\Phi^L : LTP \rightarrow LVP$ and $\omega^L : LTP \rightarrow Lg$. Using theorem 4.2 we can identify LTP with TLP in both cases. To proceed further, we need some technical results:

Lemma 4.7 *Let G be a finite dimensional Lie group, $\pi : P \rightarrow M$ a principal G -bundle over a finite dimensional manifold. Then:*

1. *The Lie algebra of LG is canonically isomorphic to Lg . Although we shall not need it here, it seems an appropriate place to mention that the exponential map $Lg \rightarrow LG$ is the loop of the exponential map $g \rightarrow G$.*
2. *Under the isomorphism $TLP \cong LTP$, the vertical tangent bundle VLP is mapped to LVP .*
3. *Under the isomorphism $TLP \cong LTP$, the vector field mapping $\zeta : Lg \rightarrow \mathfrak{X}(LP)$ is the loop of $\zeta : g \rightarrow \mathfrak{X}(P)$ followed by the canonical inclusion $L\mathfrak{X}(P) \rightarrow \Gamma_{LP}(LTP) \cong \mathfrak{X}(LP)$ which is given by: $\chi_\alpha(t) = \chi(t)_{\alpha(t)}$.*
4. *The adjoint action of LG on Lg is the loop of the adjoint action of G on g .*

Proof. 1. The Lie algebra of any Lie group is the (kinematic) tangent space at the identity. Since the identity of LG is the constant loop at the identity, e , of G , the Lie algebra of LG is, by corollary 4.3, naturally identified with $\Gamma_{S^1}(\gamma_e^*TG)$. As γ_e is constant at e , $\gamma_e^*TG = S^1 \times T_eG$ and so $T_{\gamma_e}LG$ is naturally identified with loops in T_eG , i.e. with Lg .

The statement about the exponential maps follows because the evaluation maps, $e_t : LG \rightarrow G$, are group homomorphisms. The induced Lie algebra map is also the evaluation map, $e_t : Lg \rightarrow g$. Thus for $t \in S^1$, $s \in \mathbb{R}$, and $\chi \in Lg$, $\exp(s\chi)(t) = \exp(s\chi(t))$.

2. We need to identify $\ker d(\pi^L) : TLP \rightarrow TLM$ under the equivalence $TLP \cong LTP$ and $TLM \cong LTM$. From proposition 4.4, $d(\pi^L)$ is $(d\pi)^L$. Thus $(d\pi)^L\beta = 0$ if and only if $d\pi\beta(t) = 0$ for all t and so $\ker(d\pi)^L \subseteq LTP$ is the loop of $\ker d\pi \subseteq TP$, i.e. VLP corresponds to LVP .
3. Let $\chi \in Lg$ and $\alpha \in LP$. By definition:

$$\zeta_\chi(\alpha) = dr_{(\alpha, \gamma_e)}(0, \chi) : T_\alpha LP \times T_{\gamma_e} LG \rightarrow T_\alpha LP.$$

As the action of LG on LP is the loop of the action of G on P , the various isomorphisms mean that: $dr_{(\alpha, \gamma_e)}(0, \chi)(t) = dr_{(\alpha(t), e)}(0, \chi(t))$. Hence:

$$\zeta_\chi(\alpha)(t) = \zeta_{\chi(t)}(\alpha(t))$$

as required.

4. The adjoint action of LG on Lg is the derivative of the conjugation action at the identity. The conjugation action is the loop of the conjugation action of G on g . Hence its derivative is the loop of the derivative. \square

We can now prove theorem 4.6.

Proof of theorem 4.6. Let G be a finite dimensional Lie group and let $P \rightarrow M$ be a principal G -bundle over a finite dimensional manifold. Let $\Phi : TP \rightarrow VP$ be a principal connection on P . Using the equivalences $TLP \cong LTP$ and $VLP \cong LVP$, we loop Φ to a map $\Phi^L : TLP \rightarrow VLP$ which we claim is a principal connection on LP . It is certainly a fibre projection as at a loop $\alpha \in LP$ it is the projection $\Gamma_{S^1}(\alpha^*TP) \rightarrow \Gamma_{S^1}(\alpha^*VP)$. It is also LG -equivariant as $\Phi : TP \rightarrow VP$ is G -equivariant and the action is defined pointwise.

For the other view of a connection, let $\omega : TP \rightarrow \mathfrak{g}$ be the connection form associated to Φ . Looping this, using the equivalences, yields $\omega^L : TLP \rightarrow L\mathfrak{g}$. For $\chi \in L\mathfrak{g}$ and $\alpha \in LP$ we have, for all $t \in S^1$:

$$\omega^L(\zeta_\chi(\alpha))(t) = \omega(\zeta_\chi(\alpha)(t)) = \omega(\zeta_{\chi(t)}(\alpha(t))) = \chi(t)$$

and hence $\omega^L(\zeta_\chi(\alpha)) = \chi$. Secondly, ω^L is LG -equivariant as all the actions are loops of the actions on the original spaces.

Finally, the equation $\Phi(X) = \zeta_{\omega(X)}(u)$ for $X \in T_uP$ shows that for $\chi \in T_\alpha LP$:

$$\Phi(\chi)(t) = \Phi(\chi(t)) = \zeta_{\omega(\chi(t))}(\alpha(t))$$

and hence $\Phi^L(\chi) = \zeta_{\omega^L(\chi)}(\alpha)$. \square

Associated to a connection is a parallel transport operation that lifts curves in the base to curves in the fibre such that the derivative is horizontal (i.e. in $\ker \Phi$). The parallel transport operations on M and on LM are related as follows:

Proposition 4.8 *The parallel transport operator exists in LP and corresponds to the parallel transport operator in P under the evaluation maps, $e_t : LP \rightarrow P$. That is, if $\tilde{c} : \mathbb{R} \rightarrow LP$ is a horizontal lift of $c : \mathbb{R} \rightarrow LM$ then $e_t \tilde{c} : \mathbb{R} \rightarrow P$ is a horizontal lift of $e_t c : \mathbb{R} \rightarrow M$.*

Proof. As a loop is completely determined by the values it takes, this characterisation completely specifies the parallel transport operator in LP . As parallel transports are unique providing they exist, [KM97, 37.6], we just need to show that this characterisation *defines* the parallel transport.

Let $c : \mathbb{R} \rightarrow LM$ be a curve and $v \in LP$ a lift of $c(0)$. For $t \in S^1$ let $c_t := e_t c : \mathbb{R} \rightarrow M$ and $v_t := e_t v \in P$, so that v_t is a lift of $c_t(0)$. Let $\tilde{c}_t : \mathbb{R} \rightarrow P$ be the parallel transport of v_t along c_t . We therefore have a map $S^1 \times \mathbb{R} \rightarrow P$, $(t, s) \rightarrow \tilde{c}_t(s)$. Let $\tilde{c} : \mathbb{R} \rightarrow LP$ be the adjoint of this.

As parallel transport in P is smooth in all initial conditions, the map $(t, s) \rightarrow \tilde{c}_t(s)$ is smooth. Hence \tilde{c} is smooth. Clearly, $\pi^L \tilde{c} = c$ and $\tilde{c}(0) = v$. Finally, $e_t \Phi^L \tilde{c}' = \Phi \tilde{c}'_t = 0$ so $\Phi^L \tilde{c}' = 0$ and hence \tilde{c} is horizontal. Thus it is the parallel transport of v along c . \square

Given an action of G on a (finite dimensional) manifold S we get an associated fibre bundle $R := P \times_G S \rightarrow M$. The connection on P induces a connection on R . Similarly, the connection on LP induces a connection on LR . That these correspond is straightforward to deduce.

In particular, for a representation of G on \mathbb{R}^n and $E := P \times_G \mathbb{R}^n$ we get linear connections on E and LE . From these we construct covariant differentiation operators. We start with the canonical isomorphism $\text{vl}_E : E \times_M E \rightarrow VE$, where $\text{vl}_E(u_x, v_x)$ is represented by the short curve $s \rightarrow u_x + sv_x$. Using this, we define the *connector* K_E of the connection Φ_E by:

$$K_E := \text{pr}_2 \circ (\text{vl}_E)^{-1} \circ \Phi_E : TE \rightarrow VE \rightarrow E \times_M E \rightarrow E.$$

The covariant derivative on vector bundles is defined, following [KM97, 37.28], as follows: for any manifold N , smooth mapping $s : N \rightarrow E$, and (kinematic) vector field $X \in \mathfrak{X}(N)$ we define the *covariant derivative* of s along X by:

$$\nabla_X^E s := K_E \circ ds \circ X : N \rightarrow TN \rightarrow TE \rightarrow E.$$

If s is a lift of some fixed map $f : N \rightarrow M$ then $\nabla_X^E s$ is also a lift of the same map and so we get an induced operation on sections of f^*E . In particular, taking $f : M \rightarrow M$ to be the identity we get the usual operator:

$$\nabla^E : \mathfrak{X}(M) \times \Gamma_M(E) \rightarrow \Gamma_M(E).$$

The reason for taking the more general approach is that it is the better setting for expressing the relationship between ∇^E and ∇^{LE} , which is defined analogously, because by theorem 3.28 there is an equivalence $C^\infty(N, LE) \cong C^\infty(S^1 \times N, E)$. Therefore given a map $s : N \rightarrow LE$ we take its adjoint to get a map $s^\vee : S^1 \times N \rightarrow E$. A vector field X on N defines one on $S^1 \times N$ in the obvious way which we also denote by X .

Theorem 4.9 $(\nabla_X^{LE} s)^\vee = \nabla_X^E (s^\vee)$.

Proof. It is obvious that K_{LE} is the loop of K_E since each map in the definition of K_{LE} is the loop of the corresponding map in the definition of K_E . It is straightforward to show that the identification $f \rightarrow f^\vee$ of theorem 3.28 together with the identification of TLM with LTM means that:

$$(df)^\vee = d(f^\vee) \circ (\zeta \times 1) : S^1 \times TN \rightarrow TS^1 \times TN \rightarrow TM,$$

where $\zeta : S^1 \rightarrow TS^1$ is the zero section. Now the vector field on $S^1 \times N$ corresponding to X can be thought of as $(\zeta \times 1)(1 \times X)$. Thus $\nabla_X^E (s^\vee)$ is:

$$S^1 \times N \xrightarrow{1 \times X} S^1 \times TN \xrightarrow{\zeta \times 1} TS^1 \times TN \xrightarrow{d(s^\vee)} TE \xrightarrow{K_E} E.$$

the central two terms contract to $(ds)^\vee$ so we have:

$$S^1 \times N \xrightarrow{1 \times X} S^1 \times TN \xrightarrow{(ds)^\vee} TE \xrightarrow{K_E} E.$$

This is the adjoint of:

$$N \xrightarrow{X} TN \xrightarrow{ds} LTE \xrightarrow{K_E} LE,$$

which is $\nabla_X^{LE} s$ as required. \square

In particular, the covariant differential operator:

$$\nabla^{LE} : \mathfrak{X}(LM) \times \Gamma_{LM}(LE) \rightarrow \Gamma_{LM}(LE)$$

is adjoint to the covariant differential operator on $e^*E \rightarrow S^1 \times LM$ where $e : S^1 \times LM \rightarrow M$ is the evaluation map.

Lemma 4.10 Fix $f : N \rightarrow LM$ and consider only those maps that are lifts of f . For such sections, the connection ∇^{LE} is $\mathbb{L}\mathbb{R}$ -linear.

Proof. We need to restrict our attention to lifts of a fixed map in order to have an addition on the maps $s : N \rightarrow LE$ since we need to know that $s_1(x)$ and $s_2(x)$ end up in the same fibre of $LE \rightarrow LM$.

Standard properties of connections imply \mathbb{R} -linearity so all that we need to prove is the correct behaviour under multiplication by an element of $L\mathbb{R}$. This follows from the pleasant properties of the differential of a map under the adjoint mapping. Namely, that for $s : N \rightarrow LE$ with adjoint $s^\vee : S^1 \times N \rightarrow E$, the adjoint of ds is the N -directional derivative of s^\vee . That is,

$$(ds)^\vee = d(s^\vee) \circ (\zeta \times 1) : S^1 \times TN \rightarrow TS^1 \times TN \rightarrow TE.$$

Now for $\beta \in L\mathbb{R}$, $(\beta s)^\vee$ is the map $(t, x) \rightarrow \beta(t)s^\vee(t, x)$. When taking the N -directional derivative, only the s^\vee -factor contributes and we find that $d(\beta s)^\vee = \beta ds^\vee$. Hence, undoing the adjoints, $d(\beta s) = \beta ds$.

Since K_{LE} is the loop of K_E , it is $L\mathbb{R}$ -linear and hence $K_{LE}(\beta ds) = \beta K_{LE}ds$. Thus $\nabla_X^{LE}s$ is $L\mathbb{R}$ -linear in s . \square

Even when N is a loop space, such as LM itself, the connection will not be $L\mathbb{R}$ -linear in the other variable. This is because for $\gamma \in L\mathbb{R}^n$ and $\beta \in L\mathbb{R}$, the vectors γ and $\beta\gamma$ represent completely different directions. It is tempting to view $\beta\gamma$ as an elaborate stretch of γ but really they are unrelated vectors.

A connection on TM loops to one on LTM , whence to one on TLM . Thus we can consider its torsion:

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

As in finite dimensions this is a tensor and so is given by a fibrewise bilinear, skew-symmetric map $TLM \times TLM \rightarrow TLM$.

Proposition 4.11 *The torsion on LM is the loop of the torsion on M .*

Proof. We start with a couple of facts about $L\mathbb{R}^n$:

1. Any loop in \mathbb{R}^n is the sum of two never-zero loops.

Let $\beta : S^1 \rightarrow \mathbb{R}^n$. As S^1 is compact the image of β is bounded and so there is some non-zero $v \in \mathbb{R}^n$ such that $\beta(t) \neq v$ for all t . Thus $\beta - \gamma_v$ is never zero, where γ_v is the constant loop at v . Hence β is the sum $(\beta - \gamma_v) + \gamma_v$ of two never-zero loops.

2. If $n \geq 2$, given a pair of never-zero loops β and δ , there are never-zero loops $\beta_1, \beta_2, \delta_1, \delta_2$ such that: $\beta = \beta_1 + \beta_2$ and $\delta = \delta_1 + \delta_2$ and, for all t and for $i, j \in \{1, 2\}$, $\{\beta_i(t), \delta_j(t)\}$ is a linearly independent set.

If $n \geq 3$ this is simple: choose a never-zero loop γ such that $\gamma(t)$ is not in the linear span of $\{\beta(t), \delta(t)\}$. Let $\beta_1 = \frac{1}{2}(\beta + \gamma)$, $\beta_2 = \frac{1}{2}(\beta - \gamma)$, $\delta_1 = \delta_2 = \frac{1}{2}\delta$. Then β_i and δ_i satisfy the required properties.

If $n = 2$ this is slightly more complicated. The idea relies on the fact that if the images of two loops lie on the same side of a line through the origin then at no time can they be colinear. Thus we arrange matters so that the four loops lie in the four quadrants with the β_i in opposite quadrants (hence also the δ_i in opposite quadrants). Then each pair $\{\beta_i, \delta_j\}$ lie on the same side of one of the axes and so can never be colinear. Specifically, let $R > 0$ be such that the images of β and δ lie within the open square

with vertices at $(\pm R, \pm R)$. Let γ_1 and γ_2 be constant loops at two adjacent vertices of this square. Let $\beta_1 = \beta - \gamma_1, \beta_2 = \gamma_1, \delta_1 = \delta - \gamma_2, \delta_2 = \gamma_2$. Then these satisfy the requirements.

3. Hence if $\rho_i : L\mathbb{R}^n \times L\mathbb{R}^n \rightarrow L\mathbb{R}^n, i = 1, 2$, are bilinear and such that $\rho_1(\beta, \gamma) = \rho_2(\beta, \gamma)$ whenever β, γ are never-zero and never-colinear then $\rho_1 = \rho_2$.

These extend to sections of an orientable vector bundle over S^1 via a trivialisation since the concepts of “never-zero” and “never-colinear” are preserved under bundle maps. Hence they transfer to the fibres of $TLM \rightarrow LM$. Note that we really need orientable here rather than it being a convenient simplification as there are no never-zero sections of the Möbius line.

The other information that we need to know about is how the covariant derivative on LE – on any vector bundle – transforms under maps of the source. As in finite dimensions, the covariant derivative of $s : N \rightarrow LE$ along $X \in \mathfrak{X}(N)$ at $x \in N$ depends on X only up to $X(x)$. That is, the map $\nabla_{X(x)} : C^\infty(N, LE) \rightarrow LE$ makes sense and satisfies:

$$\begin{aligned}\pi(\nabla_{X(x)}s) &= s(x), \\ (\nabla_X s)(x) &= \nabla_{X(x)}s.\end{aligned}$$

Also given a map $g : Q \rightarrow N$ and $Y_y \in T_yQ$ we have:

$$\nabla_{d_g Y_y} s = \nabla_{Y_y} (s \circ g).$$

Moreover, if $Y \in \mathfrak{X}(Q)$ and $X \in \mathfrak{X}(N)$ are g -related in that $d_g(Y)(y) = X(g(y))$ for all $y \in Q$ then:

$$\nabla_Y (s \circ g) = (\nabla_X s) \circ g.$$

We now turn to the covariant derivative on TLM .

Let $\alpha \in LM$ and let $\beta, \gamma \in T_\alpha LM$ be never-zero and never-colinear. Let $t \in S^1$ and put $x := \alpha(t) \in M$. Let $\eta : TM \rightarrow M$ be a local addition on M . It is straightforward to show that the restriction of η to $T_x M$ is a diffeomorphism onto an open neighbourhood of x of M . Write this as $\eta_x : T_x M \rightarrow V_x$.

As $\alpha^* TM$ is orientable, we can choose a trivialisation $\alpha^* TM \cong S^1 \times T_x M$. From our assumptions on β and γ we can arrange matters so that under the induced isomorphism $\Gamma_{S^1}(\alpha^* TM) \cong LT_x M$, they are taken to constant loops. Define $\psi : V \rightarrow LM$ as:

$$V \xrightarrow{\eta_x^{-1}} T_x M \subseteq LT_x M \cong \Gamma_{S^1}(\alpha^* TM) \xrightarrow{\Psi_\alpha} U_\alpha \subseteq LM.$$

Here, $T_x M \subseteq LT_x M$ is identified with the subspace of constant loops.

To determine the image of x , we see that $\eta_x^{-1}(x) = 0 \in T_x M$ which gets mapped to the zero loop and hence to the zero section, which is then mapped to α . Hence $\psi(x) = \alpha$.

Apart from the map $T_x M \rightarrow LT_x M$, all the maps are diffeomorphisms. Thus to identify the image of $d\psi_x$ it is sufficient to identify the image of the derivative of $T_x M \rightarrow LT_x M$ at the origin. As this map is linear, its derivative is itself and so we get the constant loops. Thus, by construction, the image of $d\psi_x$ contains β and γ . Let $u, v \in T_x M$ be their respective pre-images.

Using the charts $\eta_x : T_x M \rightarrow V$ and $\Psi_\alpha : \Gamma_{S^1}(\alpha^* TM) \rightarrow U_\alpha$ we can choose vector fields X_β, X_γ on LM extending β and γ and Y_u, Y_v on M extending u and v that are ψ -related and such that $[X_\beta, X_\gamma] = 0, [Y_u, Y_v] = 0$. Thus, writing ∇^L for ∇^{TLM} :

$$\begin{aligned}\tau(\beta, \gamma) &= (\nabla_{X_\beta}^L X_\gamma)(\alpha) - (\nabla_{X_\gamma}^L X_\beta)(\alpha) \\ &= (\nabla_{X_\beta}^L X_\gamma)(\psi(x)) - (\nabla_{X_\gamma}^L X_\beta)(\psi(x)) \\ &= (\nabla_{Y_u}^L (X_\gamma \circ \psi))(x) - (\nabla_{Y_v}^L (X_\beta \circ \psi))(x).\end{aligned}$$

As τ is a map into $T_\alpha LM \subseteq TLM \cong LTM$, we can take its adjoint, τ^\vee . Taking adjoints on the right, we use theorem 4.9 to simplify as follows:

$$\left(\nabla_{Y_u}^L (X_\gamma \circ \psi)\right)^\vee = \nabla_{Y_u} (X_\gamma \circ \psi)^\vee.$$

Now $X_\gamma \circ \psi : M \rightarrow LM \rightarrow TLM$ adjoints to:

$$S^1 \times M \xrightarrow{1 \times \psi} S^1 \times LM \xrightarrow{1 \times X_\gamma} S^1 \times LTM \xrightarrow{e} TM.$$

Hence $(X_\gamma \circ \psi)^\vee(t, x) = e(t, X_\gamma(\alpha)) = \gamma(t) = v$. Thus we can consider $(X_\gamma \circ \psi)^\vee$ as a vector field on $S^1 \times M$ extending v at (t, x) . We extend the covariant derivative to $S^1 \times M$ using the product of the connection on M with the standard connection on S^1 . As the S^1 -part is torsion free, the torsion of this extension is the torsion of M . Thus:

$$\nabla_u (X_\gamma \circ \psi)^\vee - \nabla_v (X_\beta \circ \psi)^\vee = \tau(u, v).$$

Hence $\tau(\beta, \gamma)(t) = \tau(\beta(t), \gamma(t))$ and thus the torsion of the looped connection is the loop of the torsion of the original connection. \square

Corollary 4.12 *A torsion-free connection on M loops to a torsion-free connection on LM .*

4.5 The Enemy of my Enemy is not my Friend

Section 4.2 showed that the loop space of a vector bundle has a pleasant $L\mathbb{R}$ -module structure. We extend this further by observing that:

$$\text{hom}_{L\mathbb{R}}(LE, L\mathbb{R}) = L \text{hom}_{\mathbb{R}}(E, \mathbb{R}),$$

and more generally:

$$\begin{aligned}\text{hom}_{L\mathbb{R}}(LE, LF) &= L \text{hom}_{\mathbb{R}}(E, F), \\ \text{ISO}_{L\mathbb{R}}(LE, LF) &= L \text{ISO}_{\mathbb{R}}(E, F).\end{aligned}$$

This last is a generalisation of the fact that if Q is the frame bundle of E then LQ is the $L\mathbb{R}$ -frame bundle of LE .

However, when viewing LE as a mere vector bundle this pleasant functorality is not preserved. The most distressing case of this is that the cotangent bundle of the loop space is not the loop space of the cotangent bundle. The simple explanation for this is that $L(E^*)$ is modelled on the space $L\mathbb{R}^n$, as is LE ,

but $(LE)^*$ is modelled on its dual which is the space of \mathbb{R}^n -valued distributions on the circle¹.

Thus although $TLM = LTM$ it is not true that $T^*LM = LT^*M$. This rules out any possibility of an isomorphism between the tangent and cotangent bundles, either via an inner product or via a symplectic structure. One can (easily) define *weak* such structures where the induced map $TLM \rightarrow T^*LM$ is injective but it can never be an isomorphism.

A choice of linear map $f : L\mathbb{R} \rightarrow \mathbb{R}$ defines a map $L(E^*) \rightarrow (LE)^*$ via:

$$L(E^*) = \text{hom}_{L\mathbb{R}}(LE, L\mathbb{R}) \subseteq \text{hom}_{\mathbb{R}}(LE, L\mathbb{R}) \xrightarrow{f} \text{hom}_{\mathbb{R}}(LE, \mathbb{R}) = (LE)^*.$$

The most usual choice is the map $\alpha \rightarrow \int_{S^1} \alpha$. This has various properties that bode well for further construction, not least being its equivariance under the natural action of the circle. With this choice, the induced map $L(E^*) \rightarrow (LE)^*$ is injective. Using this, any isomorphism $E \rightarrow E^*$ loops to give an injection $LE \rightarrow (LE)^*$, but never an isomorphism.

Thus over a loop space we have a schizophrenia as to whether bundles should be vector spaces or $L\mathbb{R}$ -modules. On the one hand, most of the constructions that one wishes to generalise from finite dimensional topology definitely use vector spaces and cannot be modified to use $L\mathbb{R}$ -modules – for the simple reason that the differential of an arbitrary function $f : LM \rightarrow \mathbb{R}$ is an \mathbb{R} -linear map $TLM \rightarrow \mathbb{R}$ and not $L\mathbb{R}$ -linear. On the other hand, the theory of bundles as $L\mathbb{R}$ -modules is very nice. In addition to the properties outlined above there is also the fact that one never wants to consider *any old* vector bundle with infinite dimensional fibres as the corresponding full general linear group is usually either not a Lie group or is contractible, neither of which is much help. Therefore we usually work with a subgroup, such as $LGL(\mathbb{R}^n)$, which is a Lie group, does have some interesting topology, and coincidentally preserves the $L\mathbb{R}$ -module structure.

¹We are fortunate in this situation that dualising the dual – which results in the *bidual* – ends us up where we started. This is *not* always true in infinite dimensions.

5 Submanifolds and Tubular Neighbourhoods

There are two important sources of submanifolds of loop spaces: those that arise from coincidences and those that arise from the obvious circle action. The “coincidental” submanifolds are ones where some constraint is imposed on the value of a loop at some specific times; the most obvious being the based loop space. The submanifolds arising from the circle action are the fixed point sets of the various subgroups of the circle. The main result of this section is that all of these submanifolds have tubular neighbourhoods. We conclude this section by exhibiting a submanifold without a tubular neighbourhood.

5.1 The Fundamental Fibration

The space of *based* loops is one of the key concepts in algebraic topology. All of the above analysis works equally well for the based loop space as for the free loop space. The model spaces for ΩM are sections of α^*TM which are zero at time 0. The tangent space of ΩM is thus ΩTM where the base point in TM is the zero above the base point in M . The inclusion of based loops in free loops is smooth and fits into the sequence:

$$\Omega M \rightarrow LM \xrightarrow{e_0} M$$

where e_0 is the map which evaluates a loop at time 0. This is a fibration sequence and is split since M includes in LM as the subspace of constant loops.

This is great as far as algebraic topology is concerned. However for differential topology we would like to know that this is a locally trivial fibration.

Theorem 5.1 *Let M be a connected smooth manifold with base point x_0 . Then $\Omega M \rightarrow LM \rightarrow M$ is a locally trivial fibration.*

In the non-connected case we simply use this result together with the fact that the loop spaces – based or free – of a disjoint union are a disjoint union of the loop spaces of the components. Thus $LM \rightarrow M$ remains a locally trivial fibration although the fibre may differ on components.

Proof. We need to show local triviality. This will imply that the diffeomorphism type of the fibres is locally constant and thus will allow us to identify every fibre with ΩM .

We start with a technical result on diffeomorphisms of \mathbb{R}^n . What we are looking for is a family $\{\psi_v\}$ of compactly supported diffeomorphisms of \mathbb{R}^n indexed by points of \mathbb{R}^n such that $\psi_v(0) = v$. In other words, we are looking for a splitting of the map $\text{Diff}_c(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ which evaluates a diffeomorphism at the origin.

We construct this using the exponential map for compactly supported vector fields. It is a standard corollary of ODE theory that the exponential map $\exp : \mathcal{X}_c(\mathbb{R}^n) \rightarrow \text{Diff}_c(\mathbb{R}^n)$ is well-defined, where $\mathcal{X}_c(\mathbb{R}^n)$ is the space of compactly supported vector fields on \mathbb{R}^n .

We define a map $\mathbb{R}^n \rightarrow \mathcal{X}_c(\mathbb{R}^n)$ as follows: let $\rho : \mathbb{R} \rightarrow [0, 1]$ be a bump function such that:

$$\rho(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 2 \leq t. \end{cases}$$

Define $\mathbb{R}^n \rightarrow \mathcal{X}_c(\mathbb{R}^n)$, $v \rightarrow X_v$ by:

$$X_v(u) = \rho(\|u\|^2)v.$$

The exponential map $\exp : \mathcal{X}_c(\mathbb{R}^n) \rightarrow \text{Diff}_c(\mathbb{R}^n)$ is such that the path $t \rightarrow \exp(tX)y_0$ is the solution to the ODE $y' = X(y)$ with initial condition y_0 . Thus $\exp(X_v)0$ is the value at time 1 of the solution to the ODE $y' = v$ with initial condition $y_0 = 0$. Hence $\exp(X_v)0 = v$. Thus the map $v \rightarrow \exp(X_v)$ is the required splitting. Note that X_0 is the zero vector field and so $\exp(X_0)$ is the identity.

Let $x \in M$ and let $\Omega_x M$ be the fibre of $LM \rightarrow M$ at x . Let $\phi : \mathbb{R}^n \rightarrow U$ be a chart at x with $\phi(0) = x$. Now ϕ takes any compactly supported diffeomorphism of \mathbb{R}^n to one of U . As such a diffeomorphism is compactly supported, it is the identity near the boundary of U . It therefore extends to a diffeomorphism of M by defining it to be the identity outside U .

Using the above family of diffeomorphisms of \mathbb{R}^n we thus have a smooth family of diffeomorphisms of M indexed by the points of U with the property that $\phi_u(x) = u$.

Let $L_U M := \{\alpha \in LM : \alpha(0) \in U\}$. This is an open submanifold of LM . Define $\Omega_x M \times U \rightarrow L_U M$ by $(\alpha, u) \rightarrow \phi_u(\alpha)$. Since $\phi_u(x) = u$, this is a fibrewise map of spaces over U . Its inverse is $\beta \rightarrow (\phi_{\beta(0)}^{-1}(\beta), \beta(0))$. This is the required local trivialisation. \square

The key part of this proof is showing that the submanifold ΩM of LM has a tubular neighbourhood. The length and intricacy of this proof should be contrasted with the corresponding statement about the submanifold M of LM , where M is identified with the space of constant loops in LM .

Proposition 5.2 *The inclusion $M \rightarrow LM$ admits a tubular neighbourhood. The normal bundle is $\Omega^v TM$, the space of fibrewise loops in TM which are based at the zero section; that is, the fibre of $\Omega^v TM$ at a point p is $\Omega T_p M$. This can be identified with $TM \otimes \Omega \mathbb{R}$.*

Proof. Let $\eta : TM \rightarrow M$ be a local addition on M and let $V \subseteq M \times M$ be the corresponding neighbourhood of the diagonal. Since $\Omega^v TM$ is a subset of the set of smooth loops in TM , we can define $\eta : \Omega^v TM \rightarrow LM$ by composition. This is clearly a smooth map.

Let $p \in M$. The domain of the chart map defined by η at the constant map at p can be naturally identified with $LT_p M$. The map $\eta : \Omega^v TM \rightarrow LM$ restricted to the fibre above p is the restriction of the chart map to the subspace $\Omega T_p M$. Therefore $\eta : \Omega^v TM \rightarrow LM$ is injective when restricted to any fibre. The images of the fibres can be distinguished in LM since for α in the fibre of $\Omega^v TM$ above $p \in M$, $\eta\alpha(0) = \eta(O_p) = p$.

The image of this map is the set of $\alpha \in LM$ such that $(\alpha(0), \alpha(t)) \in V$ for all $t \in S^1$. This is open in LM as it is the preimage of LV under the continuous map $LM \rightarrow M \times LM \rightarrow LM \times LM$ given by sending α to $(\alpha(0), \alpha)$.

The inverse of this map is thus $\alpha \rightarrow (\pi \times \eta)^{-1}(\alpha(0), \alpha)$. It is therefore a diffeomorphism onto its image. \square

This is not the tubular neighbourhood of M that is usually wanted as it is not S^1 -equivariant. We postpone the construction of that neighbourhood to section 5.3.

5.2 Tubular Neighbourhoods

The vector fields that we defined in section 5.1 on \mathbb{R}^n did not use any structure of \mathbb{R}^n beyond its being an inner product space. Therefore we can define similar vector fields on a vector bundle over a manifold. We can use this to prove a generalisation of this result involving tubular neighbourhoods. One important application of this generalisation is the following result:

Proposition 5.3 *Let $LM \times_M LM$ be the family of pairs of loops which coincide at time 0. Then $LM \times_M LM \rightarrow LM \times LM$ is an embedded submanifold with a tubular neighbourhood.*

This result is used in [CJ02] in the construction of the loop product in the cohomology of the loop space. A generalisation of it is used in [CG04] to define the other operations of string topology in the cohomological setting.

We shall prove a little more than that a tubular neighbourhood exists. We shall prove that the *obvious* neighbourhood is a tubular neighbourhood. To explain this remark, observe that there is a pull-back diagram:

$$\begin{array}{ccc} LM \times_M LM & \longrightarrow & LM \times LM \\ e_0 \downarrow & & \downarrow e_0 \times e_0 \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

where Δ is the inclusion of the diagonal. The lower line is an embedded submanifold with a tubular neighbourhood, say V , so define:

$$LM \times_V LM : \{(\alpha, \beta) \in LM \times LM : (\alpha(0), \beta(0)) \in V\}.$$

This fits in to the above diagram very neatly:

$$\begin{array}{ccc} LM \times_M LM & \longrightarrow & LM \times_V LM \subseteq LM \times LM \\ e_0 \downarrow & & \downarrow e_0 \times e_0 \\ M & \xrightarrow{\Delta} & V \subseteq M \times M \end{array}$$

It would be nice if not only did $LM \times_M LM$ have a tubular neighbourhood in $LM \times LM$ but that $LM \times_V LM$ were an example of such.

Before proving that this is so, let us examine what we get for free and thus what extra is needed to be shown. As the normal bundle to the diagonal embedding is isomorphic to TM , the existence of the lower tubular neighbourhood means that there is a diffeomorphism $TM \rightarrow V \subseteq M \times M$ where V is an open neighbourhood of the diagonal such that the composition of this with the zero section is the embedding of the diagonal. For convenience, we shall assume that this diffeomorphism comes from a local addition on M , see definition 3.13. Thus we have an identification between tangent vectors and pairs of suitably close points. Our assumption means that the first of those points is the anchor for the tangent vector.

What we do get for free is that the normal bundle on the upper level is the pull-back of the normal bundle on the lower level. Thus on the upper level we seek an identification between the spaces:

$$\begin{aligned} e_0^* TM &= \{(\alpha, \beta, v) : \alpha(0) = \beta(0), v \in T_{\alpha(0)} M\}, \\ LM \times_V LM &= \{(\alpha, \beta) : (\alpha(0), \beta(0)) \in V\}. \end{aligned}$$

To make everything fit nicely into the pull-back diagram, we want the diffeomorphism between these two to project down to the diffeomorphism that we already have. Since $\alpha(0)$ therefore should not change, we may as well – for simplicity – assume that α does not move. Thus we want $(\alpha, \beta, v) \rightarrow (\alpha, \tilde{\beta})$ such that on evaluation at 0 we get the lower diffeomorphism.

The difficulty is that v only tells us what to do with $\beta(0)$. As β is smooth, we need to know what to do with the rest of it. This involves some choices and some careful analysis. Fortunately, we have already laid the necessary foundations.

Proposition 5.4 *Let M be a smooth finite dimensional manifold, $P \subseteq M$ an embedded submanifold with normal bundle E and tubular neighbourhood $V \subseteq M$ with diffeomorphism $v : E \rightarrow V$. Let $L_P M := \{\alpha \in LM : \alpha(0) \in P\}$ and $L_V M := \{\alpha \in LM : \alpha(0) \in V\}$.*

*The inclusion $L_P M \rightarrow LM$ is a smooth embedding with normal bundle e_0^*E and tubular neighbourhood $L_V M$. Moreover, there is a diffeomorphism $e_0^*E \rightarrow L_V M$ covering $v : E \rightarrow V$.*

We view P as an actual subset of M rather than taking $j : P \rightarrow M$ as an embedding to reduce the number of maps that we need to make explicit.

Proof. We omit the full proof that $L_P M$ is a submanifold of LM . The proof that it is a manifold is a repetition of the proof that LM is a manifold. The embedding follows from the fact that the isomorphism: $L\mathbb{R}^k \cong \Omega\mathbb{R}^k \oplus \mathbb{R}^k$ together with the fact that P is embedded in M . The case of $L_V M$ is simpler as it is an open subset of LM .

The bundle $e_0^*E \rightarrow L_P M$ is the pull-back bundle via the evaluation map $\alpha \rightarrow \alpha(0)$. As a space,

$$e_0^*E = \{(\alpha, v) \in L_P M \times E : \alpha(0) = \pi(v)\}.$$

We equip E with inner products on the fibres, varying smoothly over P . The pull-back, e_0^*E inherits these inner products. Let $\|\cdot\|$ be the corresponding fibrewise norm.

Using the local triviality of E and paracompactness of P we wish to choose an open cover of P over which E trivialises together with a variation on the theme of a subordinate partition of unity. The variation that we want is that the *squares* of our functions should be a partition of unity. This presents no technical difficulties: recall that the final step in constructing a partition of unity is to renormalise a family of bump functions with respect to their sum; if one instead renormalised with respect to the square-root of the sum of their squares, the resulting family would have the required property. This square-root results in a smooth function as it is the square-root of a strictly positive function.

Thus we choose, for an indexing set Λ :

1. an open cover $\{U_\lambda : \lambda \in \Lambda\}$,
2. trivialisations $\phi_\lambda : E_\lambda \rightarrow U_\lambda \times \mathbb{R}^k$,
3. smooth functions $\rho_\lambda : P \rightarrow \mathbb{R}$ with compact support such that $\{\rho_\lambda^2\}$ is a partition of unity subordinate to $\{U_\lambda\}$ with the support of ρ_λ contained in U_λ .

Let $\tilde{\phi}_\lambda : E_\lambda \rightarrow \mathbb{R}^k$ be the composition of ϕ_λ with the projection onto \mathbb{R}^k . Define $s : E \rightarrow \Gamma(E)$ by:

$$s(v)(x) = \sum_{\lambda \in \Lambda} \rho_\lambda(\pi(v)) \rho_\lambda(x) \phi_\lambda^{-1}(x, \tilde{\phi}_\lambda(v)).$$

Note that the λ -summand is zero unless both $\rho_\lambda(x)$ and $\rho_\lambda(\pi(v))$ are non-zero. Therefore the support of the section $s(v)$ is contained in the union of the supports of the ρ_λ for which $\rho_\lambda(\pi(v)) \neq 0$. As the supports of the ρ_λ form a locally finite family of compact sets, the support of $s(v)$ is compact and hence s takes values in $\Gamma_c(E)$, sections with compact support.

This function is smooth and has the following properties:

1. the restriction to a fibre is linear, and
2. $s(v)(\pi(v)) = v$.

The first of these follows from the fact that the ϕ_λ are linear on fibres. The second uses the fact that the ρ_λ square to a partition of unity. Note that as a consequence we have that if v is a zero vector then $s(v)$ is the zero section.

From this family of sections of E , we define a family of compactly supported vector fields on E which, on fibres, look like the vector fields that we used in the proof that $LM \rightarrow M$ was locally trivial. We note that the tangent bundle of E contains a canonical copy of π^*E as the vertical tangent bundle. Thus a section σ of E defines a vector field on E by $v \rightarrow \sigma(\pi(v))$. Let $\tau : \mathbb{R} \rightarrow [0, 1]$ be a bump function with $\tau(t) = 1$ for $0 \leq t \leq 1$ and $\tau(t) = 0$ for $t \geq 2$. Define $X : \Gamma(E) \rightarrow \mathcal{X}(E)$ by:

$$X_\sigma(v) = \tau(\|v\|^2) \sigma(\pi(v)).$$

By construction, X_σ has fibrewise compact support and its horizontal support agrees with that of σ . Therefore this restricts to a map $X : \Gamma_c(E) \rightarrow \mathcal{X}_c(E)$. Combining this with the above map $E \rightarrow \Gamma_c(E)$ and the exponential map $\Gamma_c(E) \rightarrow \text{Diff}_c(E)$ we obtain a map $\psi : E \rightarrow \text{Diff}_c(E)$.

Now the vector field corresponding to a point $v \in E$ takes values in the vertical tangent space of E . Therefore the diffeomorphism is a fibre-preserving diffeomorphism covering the identity on the base. On fibres, it looks like the diffeomorphisms we had in the previous proof. Thus as $s_v(\pi(v)) = v$, $\psi(v)(0_{\pi(v)}) = v$, where 0_x is the zero vector in E_x .

The diffeomorphism $\nu : E \rightarrow V$ defines $\text{Diff}_c(E) \rightarrow \text{Diff}_c(V)$ and thence to $\text{Diff}_c(M)$ since a compactly supported diffeomorphism in V extends by the identity to the whole of M . Hence we have $\theta : V \rightarrow \text{Diff}_c(M)$ such that $\theta(v)(\pi(\nu^{-1}v)) = v$.

We now define our tubular neighbourhood diffeomorphism as:

$$(\alpha, v) \rightarrow \theta(\nu(v))(\alpha)$$

with inverse:

$$\beta \rightarrow (\theta(\beta(0))^{-1}(\beta), \nu^{-1}(\beta(0))).$$

Evaluating at zero, on the left we get $v \in E_{\alpha(0)}$ whilst on the right we get $\theta(\nu(v))(\alpha(0))$. Now as $\pi(v) = \alpha(0)$, $\theta(\nu(v))(\alpha(0)) = \theta(\nu(v))(\pi(v)) = \nu(v)$. Hence the diffeomorphism on the loop spaces projects down to $\nu : E \rightarrow V$ under evaluation. \square

Proposition 5.3 follows immediately using $\Delta : M \rightarrow M \times M$. Theorem 5.1 also follows from this proposition using the embedding of a single point.

5.3 Equivariant Tubular Neighbourhoods

The previous section deals with submanifolds arising from “coincidences”: loops that happen to coincide with each other or with some submanifold of the target manifold. Another source of submanifolds comes via the natural circle action on the loop space.

Definition 5.5 Define the circle action $\rho : S^1 \times LM \rightarrow LM$ by $\rho(t, \alpha)(s) = \alpha(t + s)$.

The adjoint of ρ is the composition:

$$S^1 \times S^1 \times LM \xrightarrow{(s,t,\gamma) \rightarrow (s+t,\gamma)} S^1 \times LM \xrightarrow{e} M$$

which is obviously smooth.

This action induces an action by any subgroup of S^1 . We shall be concerned with the *compact* subgroups which are the finite cyclic groups and S^1 itself. We wish to consider the fixed point subsets of these actions. It is straightforward to show that for $G = S^1$ the fixed points are the constant loops and so the fixed point set is diffeomorphic to M while for $G \neq S^1$ the fixed point set is the set of loops of period $1/|G|$ and this is diffeomorphic to LM . What is more intricate is showing that these all have S^1 -equivariant tubular neighbourhoods.

Theorem 5.6 Let $G \subseteq S^1$ be a compact subgroup (including the case $G = S^1$). The fixed point set of the induced action of G on LM is an S^1 -invariant embedded submanifold with an S^1 -equivariant tubular neighbourhood. The normal bundle is $T(LM^G) \otimes C^\infty(G, \mathbb{R})_0$ where $C^\infty(G, \mathbb{R})_0$ is the G -invariant complement of the constant maps in $C^\infty(G, \mathbb{R})$.

Strangely, for $G \neq S^1$ the results about the G -fixed points in LM depend on the structure of $\text{Map}(G, M)$. For more on the links between loop spaces and the spaces $\text{Map}(G, M)$ see [Jon87] and [Sta05b]. Note that as G is finite, $\text{Map}(G, M)$ is a product of copies of M .

To prove this theorem we need more structure on M . In studying the whole loop space, LM , we used a *local addition* to enable us to use the tangent spaces. Now we need a *local averaging function*. The point is that we need to be able to find an S^1 -equivariant map from sufficiently small loops in M to M . When considering the non-equivariant tubular neighbourhood of the constant loops in LM we could just take evaluation at a point. This is not S^1 -equivariant so is not adequate for our purposes. In \mathbb{R}^n we would average the values taken by the loop (equivalently, take the constant Fourier component). A *local averaging function* is precisely what we need in order to extend this to an arbitrary manifold. It is somewhat more complicated than a local addition and so we shall give an explicit construction rather than a definition.

The starting point for this construction is an embedding of M in some Euclidean space \mathbb{R}^k . We shall identify M with its image to avoid excess maps. This also identifies TM with its image in $T\mathbb{R}^k$. We consider $T_x\mathbb{R}^k$ to be an affine space anchored at x and isomorphic to \mathbb{R}^k . Thus the addition in $T_x\mathbb{R}^k$ is $(u, v) \rightarrow (u - x) + (v - x) + x = u + v - x$.

This identifies T_pM with an affine subspace of \mathbb{R}^k anchored at $p \in M$. Let $\pi : N \rightarrow M$ be the vector bundle defined by setting N_p to be the affine orthogonal complement to T_pM (also anchored at p). Thus as an affine space, $N_p = p + (T_pM - p)^\perp$.

For $p \in M$ we have an orthogonal projection map $\mathbb{R}^k \rightarrow T_pM$ which we restrict to M to define $\lambda_p : M \rightarrow T_pM$. This map varies smoothly in p so we

define $\lambda : M \times M \rightarrow TM$ by:

$$\lambda(p, q) = \lambda_p(q) \in T_pM.$$

This map has the following properties:

1. Consider $M \times M$ as a bundle over M via projection onto the first factor. Then λ is a bundle map.
2. The composition of λ with the diagonal map $M \rightarrow M \times M$ is the zero section of TM .
3. The derivative at $(p, p) \in M \times M$ is an isomorphism. This is because $d\lambda_p$ at p is the identity.

Standard techniques of differential topology involving judicious use of the inverse function theorem thus allow us to find a neighbourhood of the diagonal in $M \times M$ on which λ restricts to a diffeomorphism. We therefore have $\lambda : M \times M \supseteq V \xrightarrow{\cong} U \subseteq TM$. Let $\eta : TM \supseteq U \rightarrow M$ be the composition of λ^{-1} with the projection onto the second factor. As λ is a bundle map, the projection onto the first factor is just $\pi : TM \rightarrow M$ so $\pi \circ \eta = \lambda^{-1}$. By item (2) above, η composed with the zero section is the identity on M . Thus η is almost a local addition, the only variation is that its domain is not the whole of TM .

There is a natural map $\iota : N \rightarrow \mathbb{R}^k$ given by the natural inclusions $N_p \rightarrow \mathbb{R}^k$. The derivative of ι at a point in the image of the zero section is an isomorphism as it corresponds to the isomorphism $T_pM \oplus N_p = T_p\mathbb{R}^k$. The composition of ι with the zero section $M \rightarrow N$ is just the embedding $M \rightarrow \mathbb{R}^k$. Thus by similar techniques of differential topology, there is a neighbourhood W of the zero section in N and a neighbourhood X of $M \subseteq \mathbb{R}^k$ such that $\iota : W \rightarrow X$ is a diffeomorphism. Thus we have a map $\pi\iota^{-1} : X \rightarrow M$ with the property that for $x \in X$, $x - \pi\iota^{-1}(x)$ is orthogonal to $T_{\pi\iota^{-1}(x)}M$.

The two maps η and ι have been defined without reference to each other. We shall use them together so we need to modify their domains and codomains so that they interact nicely. The modification that we need to make is to shrink U so that the closure of the convex hull of $\eta(U \cap T_pM)$ – taken in \mathbb{R}^k – is contained in X , the codomain of ι . This ensures that if $C \subseteq U \cap T_pM$ is any set then the closure of the convex hull of $\eta(C)$ is contained in the codomain of ι .

We shall now explain how we are going to use these two maps to construct the required tubular neighbourhoods. Let G be a compact subgroup of S^1 . Let $C_0^\infty(G, U)$ be the space of smooth maps $\alpha : G \rightarrow U$ with the following properties:

1. There is some $p \in M$ depending on α such that $\alpha(G) \in U \cap T_pM$.
2. Considering α as a map into \mathbb{R}^k via the inclusion $T_pM \rightarrow \mathbb{R}^k$, the following holds:

$$\int_G \alpha = p.$$

In this last property, if G is finite then the integral is simply the average value of α on G , if $G = S^1$ then we take the standard S^1 -invariant measure of total volume 1. Note that in our view p is the zero in T_pM .

Composition with η defines $C_0^\infty(G, U) \rightarrow C^\infty(G, M)$.

Lemma 5.7 *The map $C_0^\infty(G, U) \rightarrow C^\infty(G, M)$ is a diffeomorphism onto its image which is open in $C^\infty(G, M)$.*

Proof. Let $\text{Cvx}(G, M)$ denote the family of smooth maps $\beta : G \rightarrow M$ such that when considered as a map into X the closed convex hull of $\beta(G)$ lies inside X .

As G is compact and β continuous, the closed convex hull of $\beta(G)$ is compact. Therefore $\text{Cvx}(G, M)$ is open within $C^\infty(G, M)$. For $\beta \in \text{Cvx}(G, M)$, the value of $\int_G \beta$ lies within the closed convex hull of $\beta(G)$ and therefore in X . Hence we have a well-defined smooth map $\tau : \text{Cvx}(G, M) \rightarrow M$ given by $\tau(\beta) = \iota^{-1} \int_G \beta$.

Consider the set:

$$Y_G := \{\beta \in \text{Cvx}(G, M) : (\tau\beta, \beta(t)) \in V \text{ for all } t \in S^1\}.$$

We assume that $V \subseteq M \times M$ was modified at the same time as U so that $\lambda : U \rightarrow V$ remains a diffeomorphism. The set Y_G is open in $\text{Cvx}(G, M)$, whence in LM , as it is the preimage of $LV \subseteq LM \times LM$ under the map $\text{Cvx}(G, M) \rightarrow M \times LM \rightarrow LM \times LM, \beta \rightarrow (\tau\beta, \beta)$.

Now for $\beta \in Y_G$, $\lambda(\tau\beta, \beta)$ is a map $G \rightarrow U \subseteq TM$. Since $\pi\lambda(p, q) = p$, it takes values in the fibre $U \cap T_{\tau\beta}M$. As $\lambda_p : M \rightarrow T_pM$ is the orthogonal projection, the difference $\beta - \lambda(\tau\beta, \beta)$ is orthogonal to $T_{\tau\beta}M$. Hence $\int_G \lambda(\tau\beta, \beta) \in N_{\tau\beta}$. Since $T_{\tau\beta}M$ is convex, this integral also lies in $T_{\tau\beta}M$. It is therefore $\tau\beta$, the zero point in $T_{\tau\beta}M$. Hence $\lambda(\tau\beta, \beta) \in C_0^\infty(G, U)$.

Now $\eta\lambda(p, q) = q$ so $\eta\lambda(\tau\beta, \beta) = \beta$. Hence the map $Y \rightarrow C_0^\infty(G, U)$ is inverse to the map $\alpha \rightarrow \eta\alpha$. As these maps are both smooth, they are diffeomorphisms. \square

Lemma 5.8 *There is a smooth G -equivariant map $C_0^\infty(G, TM) \rightarrow C_0^\infty(G, U)$ which is a diffeomorphism onto an open subset.*

Proof. We build this map in two stages: fibrewise and then extend over M .

For the fibrewise situation, let $I : C^\infty(G, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ be the integration map. As this is G -equivariant with respect to the trivial G action on \mathbb{R}^n , $\ker I$ is a G -invariant subspace. We wish to define a G -equivariant diffeomorphism $\ker I \rightarrow \ker I \cap C^\infty(G, D^n)$ where D^n is the open unit disc in \mathbb{R}^n .

We do this by noting that $C^\infty(G, D^n)$ is also the intersection of $C^\infty(G, \mathbb{R}^n)$ with the unit ball in $C(G, \mathbb{R}^n)$, the space of continuous maps with the standard sup-norm. A symmetric diffeomorphism $\phi : \mathbb{R} \rightarrow (-1, 1)$ thus defines a diffeomorphism $\hat{\phi} : C^\infty(G, \mathbb{R}^n) \rightarrow C^\infty(G, D^n)$ via (for $\alpha \neq 0$):

$$\alpha \rightarrow \frac{\phi(\|\alpha\|_\infty)}{\|\alpha\|_\infty} \alpha.$$

Therefore $I\hat{\phi}(\alpha) = 0$ if and only if $I\alpha = 0$ so $\hat{\phi}$ preserves $\ker I$. It therefore defines a diffeomorphism $\ker I \rightarrow \ker I \cap C^\infty(G, D^n)$.

We extend this over the manifold by choosing a smooth function $\epsilon : M \rightarrow (0, \infty)$ such that the ϵ -ball in T_pM is contained in U . The map $\alpha \rightarrow \epsilon(p)\hat{\phi}(\alpha)$ defines the required diffeomorphism. \square

The part of theorem 5.6 for $G = S^1$ follows immediately by putting $G = S^1$ in the above. The rest of theorem 5.6 is equally simple but requires a word or two of explanation.

The required neighbourhood of LM^G in LM consists of those loops which, when evaluated on the cosets of G in S^1 , take values in the neighbourhood Y_G of the constant maps in $C^\infty(G, M)$. We use the contraction of Y_G onto M to define the corresponding contraction of the neighbourhood onto LM^G . Thus by restricting a loop to each coset of G in turn we move from the infinite case to the finite case. The G -invariance of everything in finite dimensions means that when evaluating on a coset you get the same answer no matter which point you choose as the initial point, thus the result is well-defined.

5.4 A Not-So-Nice Submanifold

We conclude this section with an example of a submanifold that does not have a tubular neighbourhood. As with so many counterexamples or counterintuitive results in infinite dimensions, the failure is due to a linear problem.

Let $L_b M$ denote the space of loops in M that are infinitely flat at the basepoint of S^1 . We allow the value of this basepoint to vary, the flatness condition is concerned with the derivatives. This is a smooth manifold modelled on $L_b \mathbb{R}^n$ and is a submanifold of LM . However, it does not possess a tubular neighbourhood.

This is for the simple reason that the exact sequence:

$$0 \rightarrow L_b \mathbb{R} \rightarrow LR \rightarrow \mathbb{R}^N \rightarrow 0$$

does not split. The map $LR \rightarrow \mathbb{R}^N$ sends a map to its derivatives at 0. That this sequence is exact and does not split is a corollary of [KM97, Lemma 21.5]. Therefore the inclusion $L_b M \rightarrow LM$ does not have a normal bundle.

6 A Miscellany

We conclude this document with two topics designed to lead the interested reader out of the basic differential topology of the loop space and into more interesting areas. The first topic is the differential geometry of the loop space. This is generally more complicated than finite dimensional geometry, but still a certain amount can be said without too much difficulty. The second topic is the semi-infinite structure of a loop space, which has no true analogy in finite dimensions and is an important area of current interest.

6.1 Weak Riemannian Manifolds

Fairly early on in any text on Riemannian geometry is the statement that any finite dimensional manifold admits a Riemannian structure. We used this fact in demonstrating that any such manifold admits a local addition. In finite dimensions the definition of a Riemannian structure is straightforward: it consists of a smooth choice of inner product on each fibre of the tangent bundle. In infinite dimensions things are more complicated. The following discussion clearly generalises to inner products on arbitrary vector bundles.

The issues that one needs to deal with are:

1. Fibrewise questions:

- (a) Do the fibres admit (smooth) inner products?

For example, the space of all \mathbb{R} -valued sequences with its inverse limit topology does not.

For the model space of the loop space, $L\mathbb{R}^n$, the answer is “yes”.

- (b) Up to equivalence, how many inner products are there?

Equivalence means that there is a topological isomorphism taking one inner product to the other. The answer is likely to be that standard mathematical answer: none, one, or infinity.

For “none”, the previous example works. For “one” we take its dual: the space of all \mathbb{R} -valued sequences that are eventually zero. For “infinity” we can take the Hilbert space of square-integrable \mathbb{R} -valued sequences.

For the model space of the loop space, the answer is “infinity”.

- (c) Does a particular inner product induce an isomorphism to the dual space?

The global version of this – the induced isomorphism of the tangent and cotangent bundles – is one of the mainstays of finite dimensional geometry since it allows free movement between vector fields and one-forms. In infinite dimensions a positive answer to this question means that one is dealing with a Hilbert space (with the correct choice of inner product). Therefore if one wishes to work with more than just Hilbert manifolds one must be prepared for a negative answer.

What one *always* has is a linear injection into the dual space.

For the model space of the loop space, the answer is “no”.

Once one has answered these questions, and has a positive answer to the first, then the partition-of-unity argument applies and one can define a smooth global choice of inner product on the fibres of the tangent bundle. However, the questions don't stop there:

2. Global questions:

- (a) Is this Riemannian structure *weak* or *strong*?

The difference is whether or not the inner products identify the tangent and cotangent bundles, with “strong” meaning that they do. A strong Riemannian structure is only possible when one has a Hilbert bundle and the inner product on each fibre is equivalent to the standard one.

Thus with a weak structure all vector fields are one-forms but the converse only holds for a strong structure.

For a loop space, the answer is always “weak”.

- (b) Is the equivalence class of the inner product (locally) constant?

The problem here is that the partition of unity construction paid no attention to the question of equivalence. As an example, consider a space E with two inequivalent inner products g_0 and g_1 . Let $\rho : [0, 1] \rightarrow [0, 1]$ be the identity map, then $\{\rho, 1 - \rho\}$ is a partition of unity on $[0, 1]$. Define a fibrewise inner product on $[0, 1] \times E$ by $g(t) := \rho(t)g_1 + (1 - \rho(t))g_0$. By construction, the equivalence class of this inner product is not locally constant.

This is closely related to the next question:

- (c) Is there a bundle (i.e. locally trivial) of Hilbert spaces which can be considered as the fibrewise completions of the fibres of the tangent bundle?

The connection with the previous question comes about because an inner product defines a Hilbert completion. An equivalence between two inner products extends to an isometric isomorphism of the corresponding completions. Therefore if the equivalence class is locally constant the Hilbert completions fit together to define a locally trivial Hilbert bundle.

Note that one can define this bundle without reference to an actual inner product but only to an equivalence class. Essentially, one breaks down the choice of inner product to an initial choice of equivalence class – which defines the bundle of completions – and then to a choice of inner product within that class.

- (d) Is the tangent bundle with its family of inner products *isometrically* locally trivial?

By this we mean that there is one fixed inner product on the model space and the tangent bundle can be locally trivialised in such a way that the fibrewise inner products are all carried to this reference one. In finite dimensions this follows from the Gram-Schmidt algorithm.

A positive answer to this question implies a positive answer to the previous one since there is a fixed Hilbert completion corresponding to the fixed inner product.

- (e) Assuming the existence of a smaller group than the full general linear group, can the construction be done in such a way that the transition functions lie in this group?

If the answer to this question is yes (or is suspected to be), this provides a simpler route to the construction: first fix the reference inner product and, consequently, Hilbert completion. Then prove that the given group preserves this inner product, and hence the Hilbert completion. Finally, use this group action to transfer the whole structure to the manifold.

We illustrate this with the loop space, LM , of a finite dimensional Riemannian manifold M . There is a canonical weak Riemannian structure on LM coming from the Riemannian structure on M . There are two ways to define these inner products.

The direct way is to use the strategy of section 4.5. The inner product on the tangent space of M is a symmetric fibrewise bilinear map $TM \times_M TM \rightarrow \mathbb{R}$ with the property that the induced map $g : TM \rightarrow T^*M$ satisfies $g(v)(v) > 0$ for $v \neq 0$ (i.e. not in the image of the zero section). This loops to a symmetric bi-LR-linear map $LTM \times_{LM} LTM \rightarrow LR$ such that the induced map $g : LTM \rightarrow LT^*M$ satisfies $g(\alpha)(\alpha) > 0$ for $\alpha \neq 0$. The inequality now holds in LR and is defined by $\beta > \gamma$ if $\beta(t) \geq \gamma(t)$ for all t and $\beta \neq \gamma$ (equivalently, there is some t such that the inequality is strict). We then apply the integration map $\int_{S^1} : LR \rightarrow \mathbb{R}$. This is an order-preserving linear map and so the symmetric bilinear map $LTM \times_{LM} LTM \rightarrow \mathbb{R}$ has the property that the induced map $\int g : LTM \rightarrow LT^*M \rightarrow T^*LM$ satisfies $\int g(\alpha)(\alpha) > 0$ for $\alpha \neq 0$. Untangling all of that yields the formula:

$$(\beta, \gamma)_\alpha = \int_{S^1} \langle \beta(t), \gamma(t) \rangle_{\alpha(t)} dt,$$

for $\beta, \gamma \in \Gamma_{S^1}(\alpha^*TM) = T_\alpha LM = L_\alpha TM$.

The indirect way is to observe that the structure group of M is, because of the choice of Riemannian structure, O_n . Therefore the structure group of LM is LO_n . Now the action of LO_n on $L\mathbb{R}^n$ preserves the standard inner product coming from the inclusion $L\mathbb{R}^n \rightarrow L^2\mathbb{R}^n$ (in fact, it is precisely the subgroup of $LGL_n(\mathbb{R})$ which does so). Therefore we can define a locally trivial inner product on the fibres TLM and a corresponding bundle of Hilbert completions.

The equivalence of the two approaches comes from the fact that the principal LO_n -bundle of TLM is the loop of the principal O_n -bundle of TM . An element of the LO_n -bundle above $\alpha \in LM$ is an isometric trivialisation of the bundle $\alpha^*TM \rightarrow S^1$. This defines an isometric isomorphism $\Gamma_{S^1}(\alpha^*TM) \rightarrow L\mathbb{R}^n$ (assuming orientability to avoid twisting) and hence identifies the inner product given by the above formula with the standard one.

We note that both approaches have their advantages. In the first there is an explicit formula for the inner product that one can work with. In the second, the local triviality and the existence of the bundle of Hilbert completions are straightforward.

For this weak Riemannian structure on LM , it is straightforward to prove that certain geometric objects on M loop to the corresponding objects on LM .

Proposition 6.1 *The Levi-Civita connection on M loops to the Levi-Civita connection on LM .*

Proof. From corollary 4.12, the loop of the Levi-Civita connection is torsion-free. To see that it respects the inner product, we use the fact that the orthogonal structure group of $TLM \cong LTM$ is the loop of the orthogonal structure group of TM . Hence as the Levi-Civita connection is an orthogonal connection, its loop is also orthogonal. \square

The Koszul formula that is often used to prove existence and uniqueness of the Levi-Civita connection can only, in infinite dimensions, be used to prove uniqueness. This is because the existence part of the proof uses the isomorphism of vector fields and one-forms at a crucial stage but the uniqueness only uses the injectivity of the map from vector fields to one-forms. Hence a corollary of the above result is that the Levi-Civita connection on LM does exist.

Proposition 6.2 *For X either M or LM and $v \in TX$, let $\gamma_v : I_v \rightarrow X$ denote the geodesic corresponding to v with maximal domain I_v . Then for $v \in TLM$,*

$$e_t \gamma_v = \gamma_{e_t v},$$

and $I_v = \cap I_{e_t v}$.

Proof. Let $\gamma : I \rightarrow LM$ be a path. For $t \in S^1$, let $\gamma_t : I \rightarrow M$ be the adjoint of γ restricted to $\{t\} \times I$. Differentiating, we get $\gamma' : I \rightarrow TLM$ and its adjoint when restricted to $\{t\} \times S^1$ is γ_t' .

We have the covariant derivative of γ' along the canonical vector field ∂x of I : $\gamma'' := \nabla_{\partial x}^L \gamma'$. This is again a map $I \rightarrow TLM$ above γ . By theorem 4.9, this has adjoint:

$$\left(\nabla_{\partial x}^L \gamma'\right)^\vee = \nabla_{\partial x}(\gamma'^\vee).$$

Hence the adjoint of γ'' restricted to $\{t\} \times S^1$ is γ_t'' . Thus γ'' vanishes if and only if each γ_t'' vanishes and so γ is a geodesic if and only if $e_t \gamma$ is a geodesic for each $t \in S^1$. The rest of the proposition then follows directly. \square

Corollary 6.3 *If M is geodesically complete then LM is geodesically complete.*

In fact, this is an “if and only if” as M is a Riemannian submanifold of LM . However, although in finite dimensions geodesic completeness is equivalent to a lot of things, that no longer holds for loop spaces. In particular, although $\exp : T_x M \rightarrow M$ is surjective, it need not be the case that $\exp : T_x LM \rightarrow LM$ is surjective. For example, take the sphere, S^2 , and the exponential map based at the constant loop at the south pole. Consider a loop which is a great circle through the south pole. When we try to lift this to the tangent space at the south pole, we find that it must lift to a segment of a straight line between preimages of the south pole, as it is a geodesic segment. This cannot be made into a loop and so there is no lift. Thus there is no geodesic between the constant loop at the south pole and any great circle through this point.

6.2 More Fun with Based Loops

We conclude with a brief introduction to the topic of *polarisations*. For simplicity, we shall work with complex vector bundles.

The space LC has a very simple description using Fourier analysis. It is an appropriate completion of the space of Laurent polynomials, $\mathbb{C}[z, z^{-1}]$. In particular it decomposes as two pieces: $L_- \mathbb{C} \oplus L_+ \mathbb{C}$ according to the powers of

z. We assign the constant loops, corresponding to z^0 , to $L_+\mathbb{C}$. These have more stylish descriptions as the space of loops that extend holomorphically over an inner or outer disc (modulo the assignment of the constant loops).

One might ask whether this structure is preserved on a loop space. That is, given a complex vector bundle $E \rightarrow M$, is there a similar fibrewise splitting of LE ? If one restricts LE to the constant loops then this does exist, but over the whole of LM then it does not except in very special circumstances (see [CS04] and [Sta05a]).

However, it almost works. When moving from one chart to another one finds that the projections $L_\pm\mathbb{C}^n \rightarrow \widetilde{L}_\pm\mathbb{C}^n$ are Fredholm and $L_\pm\mathbb{C}^n \rightarrow \widetilde{L}_\mp\mathbb{C}^n$ are compact. This means that, morally, one is only shifting a finite dimensional amount from one side to the other.

Such a structure is called a *polarisation*. The definitive reference is [PS86]. The theory of polarisations is intimately connected with that of representations of loop groups which is why it is of particular interest to students of loop spaces. We shall just mention a few highlights here.

The reason for the title of this section is that the based loop space, ΩU_n is closely linked to polarisations. Given a complex vector bundle $E \rightarrow M$, one can consider the bundle over LM the points of which are the splittings of LE which are equivalent to the canonical polarisation. These are only fibrewise splittings so always exist. A global section of this bundle would define a global splitting of LE which, by work of [CS04] and [Sta05a], would mean that $LE \rightarrow LM$ was “almost” trivial. If one imposes a little extra structure on the splittings, namely that they are orthogonal and behave well under the natural $L\mathbb{C}$ -action, then this bundle is very easy to identify: let $Q \rightarrow M$ be the principal U_n -bundle associated to E . The group LU_n acts on ΩU_n via $\gamma \cdot \beta = \gamma\beta\gamma(0)^{-1}$. The bundle of “nice” polarisations is:

$$LQ \times_{LU_n} \Omega U_n.$$

The bundle of *all* polarisations is homotopy equivalent to $LQ \times_{LU_n} \Omega U$.

One can give an alternative interpretation of this bundle. A point of LQ consists of a trivialisation of α^*E for some $\alpha \in LM$. That is, it is a fibrewise isomorphism $\alpha^*E \cong S^1 \times \mathbb{C}^k$. A point of $LQ \times_{LU_n} \Omega U_n$ is also a trivialisation of α^*E but not to a “standard” reference space such as \mathbb{C}^k . Rather, it is a fibrewise isomorphism $\alpha^*E \cong S^1 \times E_{\alpha(0)}$. Thus a global section of this bundle defines an isomorphism $LE \cong e_0^*E \otimes L\mathbb{C}$ which is what is meant by a bundle being “almost” trivial.

Another highlight of the topic of polarisations is that although the subspaces L_+E and L_-E are not well-defined, the exterior algebra $\Lambda^\bullet(L_+E)^* \widetilde{\otimes} \Lambda^\bullet L_-E$ is well-defined². The grading here is slightly odd in that the L_-E part is negatively graded. Thus the degree of $\Lambda^p(L_+E)^* \widetilde{\otimes} \Lambda^q L_-E$ is $p - q$. This is known as the *semi-infinite exterior power* of LE . This is because in finite dimensions a choice of isomorphism $\Lambda^{\dim W} W \cong \mathbb{C}$ defines an isomorphism:

$$\Lambda^{\bullet+\dim W} V^* \cong \Lambda^\bullet W^0 \otimes \Lambda^\bullet W$$

for $W \subseteq V$ where W^0 is the annihilator of W in W^* and $\Lambda^\bullet W$ is negatively graded. Hence as LE is infinite dimensional and L_-E is of half the dimension of LE , we get:

$$\Lambda^{\bullet+\infty/2}(LE)^* \cong \Lambda^\bullet(L_+E)^* \otimes \Lambda^\bullet L_-E.$$

²The tilde on the tensor product is to denote an appropriate completion.

One simple reason for wanting to take account of such structure can be seen from the idea of the signature of a manifold. This turns out to depend on the middle dimension cohomology. If one wanted to generalise this to loop spaces, one would need a notion of “middle dimension” cohomology, i.e. of *semi-infinite* cohomology. A deeper reason is that this is part of the link between polarisations and representations of loop groups, about which I shan’t go into more detail except to say that it is closely linked to the theory of spin and spin bundles on loop spaces. A good place to start reading is the book [PS86] and a good place to end reading is the recent result of Freed, Hopkins, and Teleman [FHT]. En route, I would take in [PR94] and my preprint on the construction of the Dirac operator in [Sta].

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