Professor Dirac’s Cookbook

or
How to Construct a Dirac Operator in Infinite Dimensions

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Trondheim

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In Today’s Program

Recipe of the Day
Cooking up a Dirac operator

Ingredients Under the Microscope
Orthogonal structures

Grow Your Own
co-Riemannian structure
<table>
<thead>
<tr>
<th>Ingredients</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth manifold, $M$</td>
<td>[\text{Add } g \text{ to } M \text{ and leave until a connection appears.}]</td>
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<td>Riemannian structure, $g$</td>
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<td>Spin representations, $S$</td>
<td>[\text{Apply to } S+M \text{ to produce a covariant differential operator.}]</td>
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<table>
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<td>[\text{Combine } g, c \text{ to produce the Dirac operator.}]</td>
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## The Basic Recipe

### Ingredients

1. Smooth manifold, $M$

### Method

### Remarks

- oriented, spin, even dimensional, ...
## The Basic Recipe

### Ingredients
1. Smooth manifold, $M$
2. Riemannian structure, $g$

### Method

### Remarks
\[ g_p \quad T_pM \quad T_pM \]
# The Basic Recipe

## Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

## Method

1. Add $g$ to $M$ and leave until a connection appears.

## Remarks


# The Basic Recipe

## Ingredients

1. Smooth manifold, $M$
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3. Spin structure, $Q$

## Method

1. Add $g$ to $M$ and leave until a connection appears.

## Remarks

2. $\text{Spin}_n$
2. $\text{SO}_n$
2. $Q$
2. $P$
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1. Add $g$ to $M$ and leave until a connection appears.
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## Remarks

Has Clifford Multiplication $c^n S^+ S$ (bilinear)
The Basic Recipe

Ingredients

1. Smooth manifold, $M$
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1. Add $g$ to $M$ and leave until a connection appears.
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3. Combine $S$ with $Q$ to produce bundles $S_M$.

Remarks

Clifford Multiplication becomes

$$c_\ast TM S^+_M S_M$$

(fibrewise bilinear)
The Basic Recipe

Ingredients
- Smooth manifold, $M$
- Riemannian structure, $g$
- Spin structure, $Q$
- Spin representations, $S$

Method
- Add $g$ to $M$ and leave until a connection appears.
- Place $Q$ over the mixture and allow to infuse upwards.
- Combine $S$ with $Q$ to produce bundles $S_M$.
- Apply to $S^+_M$ to produce a covariant differential operator.
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3. Combine $S$ with $Q$ to produce bundles $S_M$.
4. Apply to $S_M^+$ to produce a covariant differential operator.
5. Combine $g$, $c$, and $S_M$ to produce the Dirac operator $/$.

### Remarks

$$/(S_M^+)((TM,S_M^+))\quad (TM,S_M^+)\quad g^1(TM,S_M^+)\quad c(S_M)$$
## First Variation

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### Remarks

- **1** Add $g$ to $X$ and leave until a connection appears.
- **2** Place $Q$ over the mixture and allow to infuse upwards.
- **3** Combine $S$ with $Q$ to produce bundles $S \times X$.
- **4** Apply to $S \times X$ to produce a covariant differential operator.
- **5** Combine $g$, $c$, and $\bar{c}$ to produce the Dirac operator $\mathcal{D}$. 

**Result**: Total collapse of Dirac soufflé.
## First Variation

### Ingredients

1. Smooth manifold, \( X \)

### Remarks

- infinite dimensional, polarised, oriented, spin …

### Method
## First Variation

### Ingredients

1. Smooth manifold, $X$
2. Riemannian structure, $g$

### Method

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$g_p, T_pX, T_pX, g_p, T_pX, T_pX$
# First Variation

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1. Add $g$ to $X$ and leave until a connection appears.

## Remarks

First Variation

Ingredients
1. Smooth manifold, $X$
2. Riemannian structure, $g$
3. Spin structure, $Q$

Method
1. Add $g$ to $X$ and leave until a connection appears.

Remarks

\[
\begin{array}{cccc}
S^1 & \text{Spin}_J & \text{SO}_J \\
S^1 & Q & P \\
\end{array}
\]
## First Variation

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1. Smooth manifold, $X$
2. Riemannian structure, $g$
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1. Add $g$ to $X$ and leave until a connection appears.
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- No longer canonical
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Has Clifford Multiplication

$$c \cdot V S^+ S \text{ (bilinear)}$$
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Clifford Multiplication becomes

$$ c \ TX \ S^+_X \ S_X $$

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4. Apply to $S_X^+$ to produce a covariant differential operator.
5. Combine $g$, and $c$ to produce the Dirac operator $/\left( S_X^+ \right)$.

**Remarks**

\[
\begin{align*}
/ \left( S_X^+ \right) \quad &\left( (TX, S_X^+) \right) \\
&\left( TX \ S_X^+ \right) \\
g^1 \quad &\left( TX \ S_X^+ \right) \\
c \quad &\left( S_X \right)
\end{align*}
\]
# First Variation

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3. Combine $S$ with $Q$ to produce bundles $S_X$.
4. Apply to $S^+_X$ to produce a covariant differential operator.
5. Combine $g$, $c$, and $c$ to produce the Dirac operator $/$. 

## Remarks

\[
/ (S^+_X) ~ ((TX, S^+_X)) \\
~ (TX, S^+_X) \\
g^1 (TX, S^+_X) \\
c (S_X)
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**Remarks**

\[
/ (S_X^+) \quad ((TX, S_X^+)) \\
(\ TX \ S_X^+) \\
g^1 (\ TX \ S_X^+) \\
c (S_X)
\]

Result

Total collapse of Dirac soufflé.
# First Variation

## Ingredients
1. Smooth manifold, $X$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

## Method
1. Add $g$ to $X$ and leave until a connection appears.
2. Place $Q$ over the mixture and allow to infuse upwards.
3. Combine $S$ with $Q$ to produce bundles $S_X^+$. 
4. Apply to $S_X^+$ to produce a covariant differential operator.
5. Combine, $g$, and $c$ to produce the Dirac operator $\slashed{\partial}$. 

## Remarks
\[
/ \ (S_X^+) \quad ((TX, S_X^+)) \\
\ (TX \ S_X^+) \\
g^1 \ (TX \ S_X^+) \\
c \ (S_X)
\]
First Variation

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4. Apply to $S_X^+$ to produce a covariant differential operator.
5. Combine $g$, $c$, and $S_X$ to produce the Dirac operator $/$. 

**Remarks**

/ $(S_X^+) \quad ((TX, S_X^+)) \\
(TX \ S_X^+) \\
g \quad (TX \ S_X^+) \\
c \quad (S_X)$
First Variation

**Ingredients**
1. Smooth manifold, $X$
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2. Place $Q$ over the mixture and allow to infuse upwards.
3. Combine $S$ with $Q$ to produce bundles $S_X$.
4. Apply to $S_X^+$ to produce a covariant differential operator.
5. Combine $g$, $S_X^+$, and $c$ to produce the Dirac operator $\mathcal{D}$.

**Remarks**
$\mathcal{D} = (S_X^+) \left( TX, S_X^+ \right)$

**Result**
Total collapse of Dirac soufflé.
The Collapse

\[(S_X^+) \rightarrow ((TX, S_X^+))\]

\[(TX, S_X^+)\]

\[(TX, S_X^+), c\]

\[(S_X)\]
The Collapse

\[(S^+_x) \longrightarrow ((TX, S^+_x))\]

\[\rightarrow \quad \text{fv}(uf(u)v)\]

\[\rightarrow \quad \text{vg}(v, )\]

\[\text{needs completion}\]

\[c\]

\[(S_x)\]
The Collapse

\[(S^+_X) \rightarrow ((TX, S^+_X))\]

\[(TX S^+_X)\]

\[(TX S^+_X)\]

\[(S^+_X)\]

fv(uf(u)v)

needs completion

vg(v,)

trace-like

c

needs completion
The Collapse

\[(S_X^+) \rightarrow ((TX, S_X^+))\]

iso $\leftrightarrow$ X nuclear

\[(TX, S_X^+)\] needs completion

\[(TX, S_X^+)\] trace-like

\[c\]

\[(S_X)\]
The Collapse

$$(S^+_X) \rightarrow ((TX, S^+_X))$$

iso $\leftrightarrow$ X nuclear

$$(TX S^+_X)$$

fv(uf(u)v)

needs completion

$$(TX S^+_X)$$

vg(v,)

c

iso $\leftrightarrow$ X Hilbert

$$(S_X)$$

trace-like
The Collapse

\[(S_X^+) \rightarrow ((TX, S_X^+))\]

iso $\leftrightarrow$ X nuclear

\[\text{exists } \leftrightarrow \dim X \downarrow\]

iso $\leftrightarrow$ X Hilbert

\[f(v, uf(u)\nu)\]

needs completion

\[\text{trace-like}\]

\[\text{exists}\]

\[\text{dim}\]

\[\text{iso}\]

\[\text{X}\]

\[\text{nuclear}\]
The Improved Basic Recipe

Ingredients

Method

Remarks

1 Smooth manifold, $M$
2 Riemannian structure, $g$
3 Spin structure, $Q$
4 Spin representations, $S$

Remarks

1 Add $g$ to $M$ and leave until a connection appears.
2 Place $Q$ over the mixture and allow to infuse upwards.
3 Combine $S$ with $Q$ to produce bundles $S^M$.
4 Apply to $S^M$ to produce a covariant differential operator.
5 Combine and $c$ to produce the Dirac operator $\gamma$. 
## The Improved Basic Recipe

### Ingredients
1. Smooth manifold, $M$

### Method

### Remarks
- oriented, spin, even dimensional, ...
## The Improved Basic Recipe

### Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

### Method

### Remarks

$$ g_p \quad T_p M \quad T_p M \\
 g_p \quad T_p M \quad T_p M $$
### The Improved Basic Recipe

#### Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

#### Method

1. Add $g$ to $M$ and leave until a connection appears.

#### Remarks
### The Improved Basic Recipe

#### Ingredients
1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$

#### Method
1. Add $g$ to $M$ and leave until a connection appears.

#### Remarks

| 2 | Spin$_n$ | SO$_n$ |
| 2 | $Q$      | $P$    |
The Improved Basic Recipe

**Ingredients**
1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$

**Method**
1. Add $g$ to $M$ and leave until a connection appears.
2. Place $Q$ over the mixture and allow to infuse upwards.

**Remarks**
The Improved Basic Recipe

Ingredients

1 Smooth manifold, $M$
2 Riemannian structure, $g$
3 Spin structure, $Q$
4 Spin representations, $S$

Method

1 Add $g$ to $M$ and leave until a connection appears.
2 Place $Q$ over the mixture and allow to infuse upwards.

Remarks

Build from $n$
Has Clifford Multiplication
$c^n S^+ S$ (bilinear)
The Improved Basic Recipe

Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

Method

1. Add $g$ to $M$ and leave until a connection appears.
2. Place $Q$ over the mixture and allow to infuse upwards.
3. Combine $S$ with $Q$ to produce bundles $S_M$.

Remarks

Clifford Multiplication becomes
\[ c \ TM \ S^+_M \ S_M \]
(fibrewise bilinear)
# The Improved Basic Recipe

## Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

## Method

1. Add $g$ to $M$ and leave until a connection appears.
2. Place $Q$ over the mixture and allow to infuse upwards.
3. Combine $S$ with $Q$ to produce bundles $S_M^+$.
4. Apply to $S_M^+$ to produce a covariant differential operator.

## Remarks
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2. Place $Q$ over the mixture and allow to infuse upwards.
3. Combine $S$ with $Q$ to produce bundles $S_M$.
4. Apply to $S_M^+$ to produce a covariant differential operator.
5. Combine and $c$ to produce the Dirac operator $\gamma$. 

**Remarks**
\[
\gamma = (S_M^+) \left( ((TM, S_M^+)) \right)
(\quad)
_{\quad}(TM S_M^+)
_{\quad}c \quad (S_M)
\]}
## The Improved Basic Recipe

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### Remarks

Isomorphism

$g: TM \rightarrow TM$

gives equivalence

Apply to $S_M^+$ to produce the Dirac operator $/$. 
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**Remarks**

Does it collapse?
Second Variation

**Ingredients**

1. Smooth manifold, $X$

**Remarks**

- infinite dimensional, polarised, oriented, spin ...

**Method**

1. Add $g$ to $X$ and leave until a connection appears.
2. Place $Q$ over the mixture and allow to infuse upwards.
3. Combine $S$ with $Q$ to produce bundles $S_X$.
4. Apply to $S + X$ to produce a covariant differential operator.
5. Combine and $c$ to produce the Dirac operator $/$. 

**Question**

Does it collapse?
### Second Variation

#### Ingredients

1. Smooth manifold, $X$
2. Riemannian structure, $g$

#### Remarks

\[
g_p \quad T_pX \quad T_pX \quad g_p \quad T_pX \quad T_pX
\]

#### Method

---

---
## Second Variation

### Ingredients
1. Smooth manifold, $X$
2. Riemannian structure, $g$

### Method
1. Add $g$ to $X$ and leave until a connection appears.

### Remarks
Second Variation

Ingredients

1. Smooth manifold, $X$
2. Riemannian structure, $g$
3. Spin structure, $Q$

Method

1. Add $g$ to $X$ and leave until a connection appears.

Remarks

$S^1$  Spin$_J$  SO$_J$

$S^1$  $Q$  $P$

Question

Does it collapse?
# Second Variation

## Ingredients
1. Smooth manifold, $X$
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1. Add $g$ to $X$ and leave until a connection appears.
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## Remarks
No longer canonical
Second Variation

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4. Spin representations, $S$

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#### Remarks

Clifford Multiplication becomes

$$c \ TX \ S_X^+ \ S_X$$

(fibrewise bilinear)
Second Variation

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5. Combine and $c$ to produce the Dirac operator $/$. 

**Remarks**

\[
/ (S_X^+) \quad ((TX, S_X^+)) \\
(TX \ S_X^+) \\
c (S_X)
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5 Combine and $c$ to produce the Dirac operator $/.$

Remarks

$/$ $(S_X^+)$ $((TX,S_X^+))$
$(TX,S_X^+)$
$c$ $(S_X)$

Question
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# Second Variation

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<td>5</td>
<td>Combine and $c$ to produce the Dirac operator $/$.</td>
</tr>
</tbody>
</table>

## Remarks

\[
\frac{1}{(S_X^+)} ((TX, S_X^+)) \\
(TX S_X^+) \\
c (S_X)
\]

## Question

Does it collapse?
Collapse?

\((S_X^+) \rightarrow ((TX, S_X^+))\)

iso \(\iff\) X nuclear

\((TX S_X^+)\)

fv(uf(u)v)

needs completion

c

trace-like

\((S_X)\)
Collapse? Not if we’re nuclear

\[(S_X^+) \rightarrow ((TX, S_X^+))\]

\[\text{iso} \leftrightarrow X \text{ nuclear}\]

\[\text{fv}(uf(u)v)\]

\[\text{needs completion}\]

\[c\]

\[(TX S_X^+)\]

\[(S_X)\]

\[\text{trace-like}\]
Part II

Ingredients Under the Microscope
# A Close Examination of the Ingredients

<table>
<thead>
<tr>
<th>Ingredients</th>
<th>Under the Microscope</th>
</tr>
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</table>

**Remarks**

1. Smooth manifold, \(X\)
2. Riemannian structure, \(g\)
3. Spin structure, \(Q\)
4. Spin representations, \(S\)

**Remarks**

Build from \((V, g)\)

Has Clifford Multiplication

\(c \cdot V \cdot S + V \cdot S \cdot X + X \cdot S\)

Construction of \(S\) starts from \(Cl(W)\)

\(B_T(w) / w \cdot w \cdot g(w, w)\)

1. So need \(g\) on \(V\) not \(V\)

Ratatouille

Gusteau: "Anyone can cook"

Anton Ego: "No, I don't think anyone can"
## A Close Examination of the Ingredients

### Ingredients

1. Smooth manifold, $X$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

### Under the Microscope

### Remarks
## A Close Examination of the Ingredients

### Ingredients

1. Smooth manifold, \( X \)
2. Riemannian structure, \( g \)
3. Spin structure, \( Q \)
4. Spin representations, \( S \)

### Under the Microscope

### Remarks

Build from \( V \)

Has Clifford Multiplication

\[
\begin{align*}
& c \ V \ S^+ \ S \\
& c \ TX \ S^+_X \ S_X 
\end{align*}
\]
# A Close Examination of the Ingredients

## Ingredients

1. Smooth manifold, $\mathcal{X}$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

## Under the Microscope

Construction of $S$ starts from

$$\text{Cl}(\mathcal{W}) := T(w) /_{w} w \ g(w, w)$$

## Remarks

Build from $V$

Has Clifford Multiplication

$$c \ V S^+ \ S$$
$$c \ TX S^+_\mathcal{X} \ S_\mathcal{X}$$
A Close Examination of the Ingredients

<table>
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<td>1 Smooth manifold, $X$</td>
<td>Construction of $S$ starts from</td>
</tr>
<tr>
<td>2 Riemannian structure, $g$</td>
<td>$\text{Cl}(\mathcal{W}) := T(w) / w \circ g(w, w) 1$</td>
</tr>
<tr>
<td>3 Spin structure, $Q$</td>
<td>So need $g$ on $V$ not $V$</td>
</tr>
<tr>
<td>4 Spin representations, $S$</td>
<td></td>
</tr>
</tbody>
</table>

Remarks

Build from $V$

Has Clifford Multiplication

$$ c \ V \ S^+ S $$

$$ c \ TX \ S^{+}_{X} S_{X} $$
# A Close Examination of the Ingredients

## Ingredients
1. Smooth manifold, $X$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

## Under the Microscope
Construction of $S$ starts from

$$\text{Cl}(\mathcal{W}) := T(w) \bigg/ w \ g(w, w)1$$

So need $g$ on $V$ not $V$

## Remarks
Build from $(V, g)$
Has Clifford Multiplication

$$c \ V \ S^+ \ S$$

$$c \ TX \ S_X^+ \ S_X$$
A Close Examination of the Ingredients

**Ingredients**

1. Smooth manifold, $X$
2. co-Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

**Under the Microscope**

Construction of $S$ starts from

$$\text{Cl}(\mathcal{W}) := T(w) w \ g(w, w)$$

So need $g$ on $V$ not $V$

**Remarks**

Build from $(V, g)$

Has Clifford Multiplication

\[
\begin{align*}
  c & \ V \ S^+ \ S \\
  c & \ TX \ S^+_X \ S_X
\end{align*}
\]
# A Close Examination of the Ingredients

## Ingredients

1. Smooth manifold, $X$
2. co-Riemannian structure, $g$
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## Under the Microscope

Construction of $S$ starts from

$$\text{Cl}(W) := T(w) /_w w g(w, w) 1$$

So need $g$ on $V$ not $V$

## Remarks

Build from $(V, g)$

Has Clifford Multiplication

$$c \ V \ S^+ \ S$$

$$c \ TX \ S^+_X \ S_X$$

## Ratatouille

Gusteau:

“Anyone can cook”

or

Anton Ego:

“No, I don’t think anyone can”
Then the guard looks in politely and will ask you very brightly “Do you like your morning tea weak or strong?” But Skimble’s just behind him and was ready to remind him, For Skimble won’t let anything go wrong.  

T. S. Eliot
Then the guard looks in politely and will ask you very brightly “Do you like your morning tea weak or strong?” But Skimble’s just behind him and was ready to remind him, For Skimble won’t let anything go wrong.  

T. S. Eliot

**Tea Through the Ages**

**1911**: Weak or Strong
Then the guard looks in politely and will ask you very brightly “Do you like your morning tea weak or strong?”
But Skimble’s just behind him and was ready to remind him,
For Skimble won’t let anything go wrong.  

T. S. Eliot

Tea Through the Ages

1911: Weak or Strong

2011: Black, green, chamomile, strawberry, jasmine, Earl Grey, chai, ...
... (but not Lipton)
## Classifying Orthogonal Structures

<table>
<thead>
<tr>
<th>Weak</th>
<th>$g_p E_p E_p$</th>
<th>inner product space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>$g_p E_p E_p$</td>
<td>Hilbert space</td>
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## Classifying Orthogonal Structures

<table>
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<tr>
<th>Weak</th>
<th>$g_p$</th>
<th>$E_p$</th>
<th>$\bar{E}_p$</th>
<th>$E_p$</th>
<th>fibrewise Hilbert completion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>$g_p$</td>
<td>$E_p$</td>
<td>$E_p$</td>
<td></td>
<td>Hilbert space</td>
</tr>
</tbody>
</table>
Classifying Orthogonal Structures

| Weak  | \( g_p E_p \overline{E}_p E_p \) | fibrewise Hilbert completion 
| Strong| \( g_p E_p \overline{E}_p E_p \) | Hilbert space

- global structure
- Hilbert completion
- map to completion
- cts
- ideal isometry
- class of completion
- equivalent isometry
- fibrewise bundle
- group

\( E_p \overline{E}_p \)
Question

1. Do loop spaces have orthogonal structures on their cotangent bundles?
2. If so, how good?
Question

1. Do loop spaces have orthogonal structures on their cotangent bundles?
2. If so, how good?

Answer

1. Yes
2. Good, but not quite as good as on the tangent bundle.
Grow Your Own Orthogonal Structure

In algebraic topological soil
Grow Your Own Orthogonal Structure

\[
\begin{align*}
L_{\text{pol}} U_n \\
\downarrow \\
LU_n \\
\downarrow \\
L^n \\
\downarrow \\
L^{2n} (L^n)
\end{align*}
\]

In algebraic topological soil
Grow Your Own Orthogonal Structure

In algebraic topological soil
Grow Your Own Orthogonal Structure

In algebraic topological soil

\[
\begin{align*}
L_{\text{pol}} U_n & \quad \downarrow \quad LU_n \\
& \quad \downarrow \\
L_{\text{pol}}^n & \quad L^2_n & \quad L^n & (L^n) & (L^2_n) & (L_{\text{pol}}^n)
\end{align*}
\]
Grow Your Own Orthogonal Structure

Any two choices differ by a polynomial.

In differential soil
Grow Your Own Orthogonal Structure

Parallel Transport

$T_{(0)} M$

In differential soil
Grow Your Own Orthogonal Structure

Parallel Transport

In differential soil
Grow Your Own Orthogonal Structure

Parallel Transport

In differential soil
Grow Your Own Orthogonal Structure

\[ T_{(0)} M \]

Parallel Transport

In differential soil
Grow Your Own Orthogonal Structure

\[ T_{(0)} M \]

Parallel Transport

\[ \text{Holonomy} \]

In differential soil
Any two choices differ by a polynomial.

In differential soil
Grow Your Own Orthogonal Structure

\[ \text{Parallel Transport} \]

\[ \text{Holonomy} \]

\[ Hv = v \]

\[ t \mathbb{II}(v) \]

\[ T_{(0)} M \]

\[ T_{(1)} M \]

In differential soil
Grow Your Own Orthogonal Structure

Any two choices differ by a polynomial

In differential soil
The Best Recipe

<table>
<thead>
<tr>
<th>Ingredients</th>
<th>Method</th>
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<tbody>
<tr>
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<td></td>
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</table>

<table>
<thead>
<tr>
<th>Remarks</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## The Best Recipe

### Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

### Method

1. Add $g$ to $M$ and leave until a connection appears.

### Remarks

- oriented, spin, string, even dimensional, ...
- Riemannian structure on $M$
## The Best Recipe

### Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

### Method

1. Add $g$ to $M$ and leave until a connection appears.
2. Apply to $TM$ to produce parallel transport $\mathbb{II}$.

### Remarks

$$g_p \quad T_p M \quad T_p M$$
$$g_p \quad T_p M \quad T_p M$$
# The Best Recipe

## Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

## Method

1. Add $g$ to $M$ and leave until a connection appears.
2. Apply to $TM$ to produce parallel transport $\mathbb{I}$.
3. Use $\mathbb{I}$ to extract $L_{pol} TM$ from $LTM$.

## Remarks
# The Best Recipe

## Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$

## Method

1. Add $g$ to $M$ and leave until a connection appears.
2. Apply $\parallel$ to $TM$ to produce parallel transport $\parallel$.
3. Use $\parallel$ to extract $L_{pol}TM$ from $LTM$.
4. Add to $LTM$ to get $(L_{r}^{2}TM)$.

## Remarks
The Best Recipe

Ingredients
1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$

Remarks
$S^1$, $LSO_n$, $LSO_n$
$S^1$, $Q$, $P$

Method
1. Add $g$ to $M$ and leave until a connection appears.
2. Apply to $TM$ to produce parallel transport $\mathbb{II}$.
3. Use $\mathbb{II}$ to extract $L_{pol}TM$ from $LTM$.
4. Add to $LTM$ to get $(L^2_{r}TM)$.
## The Best Recipe

### Ingredients
1. Smooth manifold, \( M \)
2. Riemannian structure, \( g \)
3. Spin structure, \( Q \)

### Method
1. Add \( g \) to \( M \) and leave until a connection appears.
2. Apply \( \nabla \) to \( TM \) to produce parallel transport \( \mathbb{P} \).
3. Use \( \mathbb{P} \) to extract \( L_{\text{pol}} TM \) from \( LTM \).
4. Add to \( LTM \) to get \( (L_{\gamma}^2 TM) \).
5. Place \( Q \) over the mixture and allow \( L \) to infuse upwards.

### Remarks
# The Best Recipe

## Ingredients
1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

## Method
1. Add $g$ to $M$ and leave until a connection appears.
2. Apply $\nabla$ to $TM$ to produce parallel transport $\nabla$.
3. Use $\nabla$ to extract $L_{\text{pol}} TM$ from $LTM$.
4. Add to $LTM$ to get $(L^2 TM)$.
5. Place $Q$ over the mixture and allow $L$ to infuse upwards.

## Remarks
Has Clifford Multiplication $c L^n S^+ S$ (bilinear)
# The Best Recipe

## Ingredients
1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

## Method
1. Add $g$ to $M$ and leave until a connection appears.
2. Apply to $TM$ to produce parallel transport $\nabla$.
3. Use $\nabla$ to extract $L_{pol}TM$ from $LTM$.
4. Add to $LTM$ to get $(L^2_{\gamma}TM)$.
5. Place $Q$ over the mixture and allow $L$ to infuse upwards.
6. Combine $S$ with $Q$ to produce bundles $S_{LM}$.

## Remarks
Clifford Multiplication becomes
\[ c \cdot TLM \cdot S_{LM}^+ \cdot S_{LM} \]
(fibrewise bilinear)
The Best Recipe

Ingredients

1. Smooth manifold, \( M \)
2. Riemannian structure, \( g \)
3. Spin structure, \( Q \)
4. Spin representations, \( S \)

Remarks


Method

1. Add \( g \) to \( M \) and leave until a connection appears.
2. Apply to \( TM \) to produce parallel transport \( \Pi \).
3. Use \( \Pi \) to extract \( L_{\text{pol}}TM \) from \( LTM \).
4. Add to \( LTM \) to get \( (L^2 TM) \).
5. Place \( Q \) over the mixture and allow \( L \) to infuse upwards.
6. Combine \( S \) with \( Q \) to produce bundles \( S_{LM} \).
7. Apply \( L \) to \( S^+_{LM} \) to produce a covariant differential operator.
The Best Recipe

Ingredients

1. Smooth manifold, $M$
2. Riemannian structure, $g$
3. Spin structure, $Q$
4. Spin representations, $S$

Method

1. Add $g$ to $M$ and leave until a connection appears.
2. Apply to $TM$ to produce parallel transport $-II$.
3. Use $II$ to extract $L_{\text{pol}}TM$ from $LTM$.
4. Add to $LTM$ to get $(L^2_rTM)$.
5. Place $Q$ over the mixture and allow $L$ to infuse upwards.
6. Combine $S$ with $Q$ to produce bundles $S_{LM}^-$.
7. Apply $L$ to $S_{LM}^+$ to produce a covariant differential operator $c$.
8. Combine and $c$ to produce the Dirac operator $/$. 

Remarks

\[ / (S_{LM}^+) \left( (TLM,S_{LM}^+) \right) \]
\[ (TLMS_{LM}^+) \]
\[ c \left( S_{LM}^- \right) \]
A perfect Dirac soufflé.