The Geometry of the Loop Space and a Construction of a Dirac Operator

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Abstract

We describe a construction of fibrewise inner products on the cotangent bundle of the smooth free loop space of a Riemannian manifold. Using this inner product, we construct an operator over the loop space of a string manifold which is directly analogous to the Dirac operator of a spin manifold.

Contents

1 Introduction

The problem addressed in this paper is that of constructing an inner product on the cotangent bundle of the manifold of smooth unbased loops in a smooth finite dimensional Riemannian manifold. The main motivation for this is the problem of constructing for the loop space an analogue of the Dirac operator of a finite dimensional spin manifold.

In this introduction we start with an overview of the construction of the Dirac operator in finite dimensions, explain what can and cannot be generalised to infinite dimensions, and show how an inner product on the cotangent bundle solves the problems that occur. We follow this with a short discussion on the connection between inner products and Hilbert completions and explain what exactly we aim to construct in the paper. The main part of this introduction finishes with an outline of the method of construction.

1.1 The Dirac Operator in Finite Dimensions

The construction of the Dirac operator is the main motivation for the construction of the inner product on the cotangent bundle of the loop space so we explain this first. We start with the construction in finite dimensions. As this is laid out in detail elsewhere we shall focus on the pieces that lead to the difficulties in infinite dimensions. For more on the details of the construction in finite dimensions see [LM89]. For details of the spin representation in all dimensions see [PR94].

There are two methods of constructing the Dirac operator in finite dimensions. Both follow the same general outline and have the same initial data, namely a spin manifold M. Part of what is meant by the statement that "M is spin" is that M is a Riemannian manifold and so there is an inner product on the tangent bundle. Since the Riemannian structure defines an isomorphism of the tangent and cotangent bundles we can transfer this inner product to the cotangent bundle.

The first step is to construct two finite dimensional unitary vector bundles over M called the *spinor bundles* of M. What is relevant for our purposes is that the construction starts from a vector bundle with an inner product. This is where the two methods diverge: one starts with the tangent bundle, the other with the cotangent bundle. We shall write S_T^+ and S_T^- for the bundles constructed from the tangent bundle and $S_{T^*}^+$, $S_{T^*}^-$ for those from the cotangent bundle. When we wish to refer to something that holds for both methods, we shall use the notation S^+ and S^- for the spinor bundles and $T^?M$ for the correct choice of tangent or cotangent bundle.

The key properties of these spinor bundles are the following: first, there is an operation called *Clifford multiplication* which is a vector bundle map:

$$c : TM \otimes S_T^+ \to S_T^-, \qquad c : T^*M \otimes S_{T^*}^+ \to S_{T^*}^-.$$

Second, there is a natural covariant differential operator ∇ on S^+ arising from the Levi-Civita connection on M.

The Clifford multiplication map extends in the natural way to a linear map on sections. Together with the differential operator, we therefore have maps:

$$
\begin{tikzcd}
\Gamma(S^+) \arrow[r, "\nabla"] & \Gamma(\mathcal{L}(TM, S^+)) & \\
& & \\
& \Gamma(T^?M \otimes S^+) \arrow[r, "c"] & \Gamma(S^-).
\end{tikzcd}
\tag{1.1}
$$

Here $\mathcal{L}(TM, S^+)$ means the bundle with fibres the linear maps between the corresponding fibres of TM and S^+. Note that the TM that appears here is definitely TM and not $T^?M$.

We wish to compose these maps to define the Dirac operator. In order to do so we must find a vertical map to fill the gap. This is not difficult. First, we observe that for finite dimensional spaces V and W the space of linear maps from V to W, $\mathcal{L}(V, W)$, is naturally isomorphic to $V^* \otimes W$ where V^* is the linear dual of V. Therefore $\mathcal{L}(TM, S^+) \cong T^*M \otimes S^+$. If we are using the cotangent method, we can stop here as this is the domain of the Clifford multiplication map. If we are using the tangent method we must use the inner product on the tangent bundle to identify TM with T^*M and thus $TM \otimes S^+$ with $T^*M \otimes S^+$. Thus we obtain the Dirac operator $\partial\!\!\!/ : \Gamma(S^+) \to \Gamma(S^-)$ as one of the compositions:

$$
\begin{tikzcd}
\text{(cotangent:)}\ \Gamma(S^+_{T^*}) \arrow[r, "\nabla"] & \Gamma(\mathcal{L}(TM, S^+_{T^*})) \arrow[d, "\cong"'] & \\
& \Gamma(T^*M \otimes S^+_{T^*}) \arrow[r, "c"] & \Gamma(S^-_{T^*}).
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{(tangent:)}\ \Gamma(S^+_{T}) \arrow[r, "\nabla"] & \Gamma(\mathcal{L}(TM, S^+_{T})) \arrow[d, "\cong"'] & \\
& \Gamma(T^*M \otimes S^+_{T}) \arrow[d, "\cong"'] & \\
& \Gamma(TM \otimes S^+_{T}) \arrow[r, "c"] & \Gamma(S^-_{T}).
\end{tikzcd}
$$

The identification of the tangent and cotangent bundles via the inner product on the tangent bundle defines an isomorphism of the spinor bundles and thus the two methods lead to isomorphic operators.

At first sight it appears that the tangent method uses only the inner product on the tangent bundle whilst the cotangent method uses only the inner product on the cotangent bundle (ignoring, for the moment, the fact that the inner product on the cotangent bundle was defined using the one on the tangent bundle). The first part of that statement is not strictly true. The inner product on the cotangent bundle appears surreptitiously in the identification of the tangent and cotangent bundles. The inner product on the tangent bundle defines an injective map $TM \to T^*M$ which, for dimension reasons, is an isomorphism. In the construction of the Dirac operator it is not this map which is used but rather its inverse, $T^*M \to TM$. Whilst we can think of this as merely the inverse to the above map, it is useful to think of it as the natural map coming from the inner product on the cotangent bundle, $T^*M \to T^{**}M$.

Therefore once the inner product has been transferred to the cotangent bundle, the cotangent method only uses that inner product while the tangent method uses both.

1.2 Generalising to Loop Spaces

We now consider what is known to generalise – prior to this paper – from the finite dimensional construction of the Dirac operator to the case of loop spaces. Essentially, everything generalises for the tangent method up to and including diagram (1.1). Thus for a loop space which is spin there are bundles $S_T^+, S_T^- \to LM$ – now unitary Hilbert bundles – together with a covariant differential operator and a Clifford multiplication map as before (although the differential operator is not as natural as the finite dimensional one). These bundles are constructed from the tangent bundle with its natural inner product.

The cotangent method is dead in the water as it requires an inner product as part of its initial data and – prior to this paper – such has not been defined.

The next step for the tangent method is to fill in the gap in the analogous diagram to (1.1). This gap in finite dimensions was filled in by two maps. The first of these came from the natural isomorphism, in finite dimensions, of $\mathcal{L}(V, W)$ with $V^* \otimes W$. The natural map is $V^* \otimes W \to \mathcal{L}(V, W)$, $f \otimes w \to (v \to f(v)w)$, and this map exists for any vector spaces. It is not generally an isomorphism in infinite dimensions.

In the case that we are dealing with, V is a *complete, nuclear, reflexive* space and W is a Hilbert space. It is a remarkable fact that for such spaces the completion of $V^* \otimes W$ with respect to the projective tensor product topology *is* isomorphic to $\mathcal{L}(V, W)$ under the natural map above. Essentially, this is because the completion of $V^* \otimes W$ is the space of all *compact* maps from V to W and under the assumptions on V and W, all continuous maps are compact. We prove this remarkable isomorphism in proposition 5.10.

The Clifford multiplication map extends over the corresponding completion so we can complete all tensor products in this diagram with respect to the projective topology. Thus we can fill in half the gap. Here, however, the method stalls. The inner product on the tangent bundle defines, as before, an injective map $TLM \to T^*LM$ but this is not – and cannot be made to be – an isomorphism. This is purely a linear question and is due to the fact that the model spaces for the fibres are not isomorphic. Since we want the inverse of this map, our construction of the Dirac operator by the tangent method falls here.

Both methods fail due to the same problem: a lack of an inner product on the cotangent bundle. If we had such an object, we could resurrect the cotangent method – providing some technical conditions are satisfied. Whereupon the gap in (1.1) for the cotangent method can be filled by the remarkable isomorphism; ergo: the cotangent method will yield a Dirac operator. We could also restart the stalled tangent method since the inner product on the cotangent bundle would define the injective map $T^*LM \to TLM$ (the model space is reflexive so $T^{**}LM \cong TLM$ just as in finite dimensions) which would fill in the last bit of the gap. It wouldn't be an isomorphism, but we have said that we can't have an isomorphism so this is the next best thing.

Thus both methods would lead to a Dirac operator. This raises two questions: are the Dirac operators isomorphic? and which is the best method? The answers are: "yes" and "the cotangent method". The first one comes from the fact that the spinor bundles are constructed using *completions* of the tangent and cotangent bundles with respect to the inner product topology, rather than the bundles themselves, and these completions are isomorphic.

Since the operators are equivalent, it may seem surprising that we therefore claim that one method is superior to the other. The reasoning is simple: the tangent method uses the inner products on both the tangent and cotangent bundles (the former in the construction, the latter in filling the gap) whereas the cotangent method only uses the inner product on the cotangent bundle (which is no longer induced by that on the tangent bundle). The two inner products are now independent and therefore given a choice between using both or using only one, we lean towards the simpler option.

Before proceeding, we note that one option when encountering problems of this nature in loop spaces is to alter the type of loop used. Certainly, using something like H^1-Sobolev loops would define a Hilbert manifold of loops and thus the inner product on the tangent bundle would identify the tangent and cotangent bundles as in finite dimensions. However, the remarkable isomorphism would then fail and so we would be looking for a way to construct a map which on fibres looks like: $\mathcal{L}(H_1, H_2) \to H_1^* \widetilde{\otimes} H_2$. It may not seem so, but this is exactly the same type of problem as we have above: the space $\mathcal{L}(H_1, H_2)$ is isomorphic to the dual of $H_1^* \widetilde{\otimes} H_2$ and so we are looking for a map from a space to its dual when we already have a map the other way around. Therefore we gain nothing by altering the type of loop.

1.3 Inner Products and Hilbert Completions

In this section we pick up on a remark made above. In the previous section it was stated that the two constructions of the Dirac operator in infinite dimensions produce equivalent operators because the construction depends on the completions of the bundles with respect to the inner product topology, rather than the bundles themselves. It is this property that gives a little more substance to the study of inner products on infinite dimensional vector bundles.

The question of *existence* of an inner product on an infinite dimensional vector bundle is solved in a similar manner to that in finite dimensions. We need two conditions to be satisfied, one on the base space and one on the typical fibre:

1. The base manifold is *smoothly paracompact*, in that it admits smooth partitions of unity.

2. The typical fibre admits inner products[1].

Providing these two are satisfied we can define an inner product exactly as in finite dimensions by picking local inner products and summing them using a partition of unity.

Thus mere existence is not a problem and one might feel that 50 pages is a little long for a discussion as to why one particular inner product is better than any other. The truth of the matter is that one wants more than just an inner product. What is needed is that the fibrewise completions of the cotangent bundle with respect to the inner product fit together to yield a bundle of Hilbert spaces. If one starts with an arbitrary inner product there is no guarantee that this will happen.

To make this specific, we recall the definition of equivalent inner products:

[1] This is not a trivial condition. The direct product of a countable number of copies of $\mathbb{R}$ does not admit any inner products.

Definition 1.1 *Let V be a locally convex topological vector space. Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be two inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. We say that these inner products are* equivalent *if the norms are equivalent. That is, there are constants $a, b > 0$ such that $a\|v\|_1 \leq \|v\|_2 \leq b\|v\|_1$ for all $v \in V$.*

The following results are standard from Banach space theory:

Lemma 1.2 *Let H_1 and H_2 be the completions of V with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Then $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are equivalent if and only if the identity map on V extends to an isomorphism $H_1 \cong H_2$.*

Let $\langle\cdot,\cdot\rangle$ be an inner product on V with Hilbert completion H. Let $g \in \mathrm{Gl}(V)$. Define $\langle\cdot,\cdot\rangle_g$ by $\langle u, v\rangle_g = \langle gu, gv\rangle$. Then $\langle\cdot,\cdot\rangle$ is equivalent to $\langle\cdot,\cdot\rangle_g$ if and only if g extends to an operator in $\mathrm{Gl}(H)$.

From this it is clear that for the fibrewise Hilbert completions to form a bundle then the equivalence class of the inner product must be constant. This leads to four types of inner product which we define in terms of an associated principal bundle. We think of a point in this principal bundle as being an isomorphism from the corresponding fibre to the model space. In infinite dimensions it is rare to use the full general linear group as this is either not a Lie group or is contractible.

Definition 1.3 *We classify the inner products on a vector bundle according to how many of the following statements are satisfied.*

1. *The* basic *inner product: the vector bundle admits a smooth choice of inner product on its fibres.*
2. *The* completable *inner product: the vector bundle admits a smooth choice of inner product on its fibres which map to a fixed equivalence class under the action of the principal bundle.*
3. *The* weakly locally trivial *inner product: the vector bundle admits a completable inner product and the principal bundle can be altered by a homotopy so that the inner product is mapped to a fixed inner product under its action.*
4. *The* locally trivial *inner product: the vector bundle admits a completable inner product which maps to a fixed inner product under the action of the principal bundle.*

Now that we have explicitly introduced a principal bundle, we can rephrase these definitions in terms of the action of the group on the vector space. This will make clearer what is meant by a "weakly locally trivial" inner product. Using the fact that an inner product on a space is the same thing as an inclusion with dense image into a Hilbert space with an inner product[2], we can also phrase the statements using Hilbert spaces rather than inner products.

Proposition 1.4 *Let G be a Lie group acting on a vector space V. Let $P \to X$ be a principal G-bundle over a manifold X (which we assume to be smoothly paracompact). Let $E := P \times_G V$ be the associated vector bundle. The following four conditions are equivalent, in order, to the different types of inner product given above:*

[2]Unless otherwise stated, when equipping a Hilbert space with an inner product we shall assume that it generates the given topology.

1. *V admits an inner product; equivalently, there is an inclusion $V \to H$ with dense image of V into a Hilbert space, H.*

2. *The action of G on V preserves an equivalence class of an inner product on V; equivalently, G acts on the diagram $V \to H$ but not necessarily by isometries.*

3. *The action of G on V preserves an equivalence class of an inner product on V and there is a subgroup K of G homotopic to G with an action on V by isometries such that this action is homotopic to the action which factors through G; equivalently, the induced action of K on the diagram $V \to H$ can be altered by homotopy so that it acts by isometries.*

4. *The action of G on V is by isometries with respect to a fixed inner product; equivalently, the action of G on the diagram $V \to H$ is by isometries.*

We can now state the main theorem of this paper:

Theorem 1.5 *The cotangent bundle of the loop space of a smooth manifold considered as a bundle with structure group $L\,\mathrm{Gl}_n(\mathbb{R})$ admits a weakly locally trivial inner product.*

The cotangent bundle of the loop space of a Riemannian manifold considered as a bundle with structure group LO_n does not admit a locally trivial inner product.

Compare this with the well-known analogous theorem for the tangent bundle:

Theorem 1.6 *The tangent bundle of the loop space of a smooth manifold considered as a bundle with structure group $L\,\mathrm{Gl}_n(\mathbb{R})$ admits a weakly locally trivial inner product.*

The tangent bundle of the loop space of a Riemannian manifold considered as a bundle with structure group LO_n admits a locally trivial inner product.

This inner product being given by the formula:

$$\langle \alpha, \beta \rangle_\gamma = \int_{S^1} \langle \alpha(t), \beta(t) \rangle_{\gamma(t)} dt,$$

where we identify the tangent space of the loop space with the loop space of the tangent space.

The use of principal bundles has a significant advantage over just writing down formulæ such as the one above. When writing down a formula one then has to go to considerable lengths to prove any local triviality statements that one may wish to use whereas the local triviality follows naturally if everything is done using principal bundles.

This seems an appropriate point to mention one aspect of the theory that influences the flavour of the discussion without introducing any change in the mathematics. As part of our quest we shall have to pick a Hilbert completion of $(L\mathbb{R})^*$. Because all the spaces involved are reflexive, we can equally choose a dense Hilbert subspace of $L\mathbb{R}$. Since it is conceptually easier to visualise subspaces of $L\mathbb{R}$ than superspaces of $(L\mathbb{R})^*$, we tend to work in this dual picture. Reflexivity ensures that we introduce no complications by doing so.

1.4 The Inner Product on the Cotangent Bundle - an Overview

We shall now give an overview of the construction of the inner product on the cotangent bundle. The method that we employ is to look at the structure group.

From the point of view of algebraic topology the construction is very simple. There is no equivalence class of inner product on the model space of the cotangent bundle that is preserved by the action of $L\,\mathrm{Gl}_n(\mathbb{R})$, or even by LO_n, but the polynomial loop group, $L_{\mathrm{pol}}O_n$ preserves many equivalence classes. Since the polynomial loop group is homotopic to the smooth loop group, see for example [PS86], we choose a reduction of the structure group from the smooth loop group to the polynomial loop group. There is a little work to show that the action of the polynomial loop group can be altered through homotopies to one by isometries with respect to some fixed inner product, but this is not hard. Thus we have a weakly locally trivial inner product.

The purpose of the rest of the 50 pages of this paper is to reduce the number of choices in that last paragraph to a minimum and to make them as global as possible. Ultimately, we end up with one global choice which is essentially the reference inner product on the model space of the cotangent bundle. Our input to the machinery is a Riemannian manifold – which is the same input needed to define the inner product on the tangent bundle. However, our method of construction is somewhat more complicated.

We shall explain the method for the space of based loops, ΩM. This will enable us to get to the central idea without too many details. We assume that M is simply-connected so that its loop spaces are connected. This implies that M is orientable.

The Riemannian structure on M defines the Levi-Civita connection. This in turn defines the holonomy operator, $h : \Omega M \to SO_n$. Now SO_n is the classifying space of the group ΩSO_n and the holonomy operator is a classifying map for the principal ΩSO_n-bundle associated to the tangent bundle of ΩM – and thus also to the cotangent bundle. The parallel transport operator defines an explicit isomorphism from the (co)tangent bundle to the corresponding pull-back bundle.

Since the polynomial loop group, $\Omega_{\mathrm{pol}}SO_n$, is homotopic to the smooth loop group, ΩSO_n, the classifying spaces are the same. Thus there is a principal $\Omega_{\mathrm{pol}}SO_n$-bundle over SO_n which includes into the natural ΩSO_n-bundle. Therefore, using the holonomy and parallel transport maps, we get a principal $\Omega_{\mathrm{pol}}SO_n$-bundle over ΩM which is a natural subbundle of the principal ΩSO_n-bundle associated to the tangent and cotangent bundles.

As stated above, the action of $\Omega_{\mathrm{pol}}SO_n$ on the model space of the cotangent preserves an equivalence class of an inner product, though does not act by isometries. The choice of this equivalence class – for there are many – is part of our one choice. We can therefore define the Hilbert completion of the cotangent bundle. The action of $\Omega_{\mathrm{pol}}SO_n$ can be gently altered to one of isometries whereupon we get an inner product. The choice of this inner product is the other part of our one choice (we count these as one choice since the inner product determines its equivalence class).

When considering the full loop space we use the fact that there is a locally trivial fibration $\Omega M \to LM \to M$ and essentially repeat the above construction fibre-by-fibre on $LM \to M$.

We give an explicit construction of the $\Omega_{\mathrm{pol}}SO_n$-bundle over SO_n (actually, we construct an $L_{\mathrm{pol}}SO_n$-bundle) and show that it is locally trivial. Thus although the homotopy equivalence $\Omega_{\mathrm{pol}}SO_n \to \Omega SO_n$ is part of the background of the construction, we never actually use it. In fact, the bundle that we construct shows that the homotopy groups of ΩSO_n are a direct summand of those of $\Omega_{\mathrm{pol}}SO_n$: the existence of this bundle means that there is a map $B\Omega SO_n \to B\Omega_{\mathrm{pol}}SO_n$. Since the $\Omega_{\mathrm{pol}}SO_n$-bundle includes naturally in the ΩSO_n-bundle, the composition of the above map on classifying spaces with the natural map $B\Omega_{\mathrm{pol}}SO_n \to B\Omega SO_n$ is homotopic to the identity. Ralph Cohen has suggested that further study of this bundle might yield an alternative proof of the homotopy equivalence of $\Omega_{\mathrm{pol}}SO_n$ with ΩSO_n, however that is beyond the scope of this paper.

1.5 Acknowledgements and History

The central idea of this paper – the construction of the polynomial loop bundle – places this paper as the latest in a loosely defined series: [Mor01], [CS04], and [Sta]. In the first of these, Morava attempted to construct an isomorphism for an almost complex manifold M between the tangent bundle of the loop space, LTM, and a bundle of the form $e_1^* TM \otimes_{\mathbb{C}} L\mathbb{C}$. Here, $e_1 : LM \to M$ is the map which evaluates a loop at time 1 and every bundle is considered to be complex. The argument broke down at one crucial step and the papers [CS04] and [Sta] grew out of considering the question as to when that crucial step could be made to work. This was found to be highly restrictive and implied, for example, that the tangent bundle of the based loop space of M was trivial.

One consequence which would follow from the existence of an isomorphism $LTM \cong e_1^* TM \otimes_{\mathbb{C}} L\mathbb{C}$ would be the existence of a sub-bundle modelled on the polynomial loop space. In fact, for any class of loops there would be a bundle with the appropriate fibre constructed as $e_1^* TM \otimes_{\mathbb{C}} L^{\alpha}\mathbb{C}$. Close examination of [Mor01] reveals that Morava's method was essentially to construct the polynomial loop bundle fibrewise. His mistake was to assume that from this one could globally pick-out a finite dimensional sub-bundle.

The point of view of this paper is that the polynomial loop bundle is as far as one needs to go – as well as being as far as one can go. We proffer the construction of the Dirac operator as evidence for this.

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1.6 Structure of the Paper

This paper is structured as follows:

Section 2: In this section we gather in one place all the non-standard or unusual notation that we shall use in this paper. This section is more than a reference section in that notation defined here will not be formally defined elsewhere.

Section 3: In this section we study the polynomial loop groups and construct the universal polynomial loop bundles over the classifying spaces. For technical reasons – which are given – we concentrate on the cases U_n, SU_n, and SO_n.

Section 4: In this section we use the work of section 3 to construct an inner product on the dual of a loop bundle and the associated Hilbert bundle. We also consider the properties of this construction and the relation of loop bundles to twisted K-theory.

Section 5: In this section we construct a Dirac operator over the loop space of a string manifold using the inner product on the cotangent bundle. This section also contains a brief summary of the main results on infinite dimensional spin.

Appendix: This contains some results that may be of interest about the general problem of inner products on the space of distributions.

2 Notation

This paper is somewhat heavy on notation. Therefore, we have included this section here as a reference point for the bemused reader. Here we have collected together the notation for all the reasonably standard objects that we use. The following definitions have not been included here as they are the main subject of study in various sections of this paper:

1. The polynomial loop groups: $\Omega_{\text{pol}}G$ and $L_{\text{pol}}G$. Section 3.1.
2. The periodic and polynomial path spaces: $P_{\text{per}}G$ and $P_{\text{pol}}G$. Section 3.2.
3. The periodic and polynomial vector bundles: $P_{\text{per}}V$ and $P_{\text{pol}}V$. Section 3.3.
4. The polynomial loop bundles: $L_{\text{pol}}E$, $L_{\text{pol}}Q$, and $L_{\text{pol}}Q^{\text{ad}}$. Section 4.1.

   Also defined in section 4.1 are various spaces used in the construction of the polynomial loop bundles. As these are not used elsewhere we shall not list them here.

We have tried to choose notation that is as clear as possible by choosing notation that is relatively bracket free. The issue is further complicated by the fact that this topic mixes geometry and functional analysis. Notation that is clear when geometrically viewed may not be so from the point of view of a functional analysts. As the intended audience consists primarily of geometers, we have gone for clarity in the geometrical viewpoint, with apologies to any functional analysts that may be present.

2.1 The Circle

In this paper we have two views of the circle. One is as the domain of loops, the other as a Lie group. We regard loops as periodic paths from $\mathbb{R}$ and thus wish to identify the domain of loops with $\mathbb{R}/\mathbb{Z}$. When thinking of the circle as a Lie group, we think of it as U_1 sitting inside $M_1(\mathbb{C}) = \mathbb{C}$. We shall use the notation S^1 for $\mathbb{R}/\mathbb{Z}$ and $\mathbb{T}$ for U_1. We shall write t for the parameter in S^1 and z in $\mathbb{T}$, with relationship $z = e^{2\pi i t}$.

2.2 Loop and Path Spaces of Fibre Bundles

Let M be a finite dimensional smooth manifold. We shall write LM for the manifold of smooth maps $S^1 \to M$ and PM for the manifold of smooth maps $\mathbb{R} \to M$. Since we are viewing a loop as a periodic path with period 1, LM is a submanifold of PM. By regarding M as the space of constant paths, we view M as a submanifold of LM, whence also of PM.

Let $F \to X \to M$ be a locally trivial fibre bundle which is either a vector bundle, a principal bundle, or a bundle of Lie groups. The loop space of X is a locally trivial fibre bundle over LM. If X is *orientable* – that is, trivialisable over any loop – then the fibre is LF; otherwise it will vary on the components of LM. In the situations encountered in this paper the bundles will always be orientable. In all cases, $PX \to PM$ is a locally trivial fibre bundle with fibre PF.

We shall define various pull-backs of the bundles $LX \to LM$ and $PX \to PM$. The guide to our notation is that we shall label the pull-backs by adjoining appropriate superscripts to the L or P. The convention will be to read from left to right: that is, the leftmost label happened first. Thus $L^{a,b}X$ denotes the bundle LX pulled back via first a and then b.

We shall label the fibre of a bundle over a particular point by adjoining the label of point as a subscript to the L or P. When the L or P is decorated by an additional subscript, say *abc*, the fibre label will be to the right of this. Thus $L_{\mathrm{abc},\gamma}X$ is the fibre of $L_{\mathrm{abc}}X$ over $\gamma \in LM$ (the additional subscripts are *pol* and *per* which will be defined in section 3).

We shall now describe the various pull-back bundles that we shall use:

1. $P^L X$ is the pull-back (or restriction) of PX to LM; thus for $\gamma \in LM$, $P^L_\gamma X = P_\gamma X$. Note that LX is a sub-bundle of $P^L X$.

2. $P^M X$ and $L^M X$ are the pull-backs of, respectively, PX and LX to M. Again, $P^M_p X = P_p X$.

   Note that as p is here regarded as a constant path, the paths (resp. loops) in $P_p X$ (resp. $L_p X$) lie above a single point in M. Thus they lie in a single fibre of X. Hence $P_p X = P(X_p)$ (resp. $L_p X = L(X_p)$).

3. For $t \in \mathbb{R}$, let $e_t : PM \to M$ be the map which evaluates a path at time t. Let $X^t \to PM$ be the pull-back of X via e_t. We shall use the same notation for X^t restricted to LM to avoid too many superscripts. Likewise, let $P^{M,t}X$ and $L^{M,t}X$ be the pull-backs of $P^M X$ and $L^M X$, respectively, via e_t. Thus $X^t_\gamma = X_{\gamma(t)}$, $P^{M,t}_\gamma X = P(X_{\gamma(t)})$, and $L^{M,t}_\gamma X = L(X_{\gamma(t)})$.

   For each $t \in \mathbb{R}$, there are evaluation maps $PX \to X^t$ which we denote again by e_t. Over LM, we have the identity: $X^{t+1} = X^t$.

2.3 Function Spaces and Function Bundles

The function spaces that we shall use in this paper will be spaces of maps from S^1 to some finite dimensional real or complex vector space. We shall base our notation on that from differential geometry rather than functional analysis and use a similar convention to that above. Thus a space of maps from S^1 will be denoted by an L decorated in some fashion.

Since we are using L to denote maps from the circle into some vector space, we shall use $\mathcal{L}(X, Y)$ for continuous linear maps from one topological vector space to another. Where the target space is the same as the source, we shall abbreviate this to $\mathcal{L}(X)$.

We shall now describe the various function spaces that we shall use in terms of maps from the circle into $\mathbb{C}$. For maps into $\mathbb{C}^n$, we tensor with $\mathbb{C}^n$: thus $L\mathbb{C}^n = L\mathbb{C} \otimes \mathbb{C}^n$; for maps into $\mathbb{R}^n$, we take the underlying real space of the maps into $\mathbb{C}^n$.

1. $L\mathbb{C}$: smooth maps.

2. $L^2\mathbb{C}$: square-integrable maps.

3. $L_{\mathrm{pol}}\mathbb{C} = \mathbb{C}[z^{-1}, z]$: Laurent polynomials in $\mathbb{C}$.

4. $L^*\mathbb{C}$: distributions – the dual of $L\mathbb{C}$.

5. $L^{2,*}\mathbb{C}$: the dual of $L^2\mathbb{C}$.

6. $L_r\mathbb{C}$, $r > 1$: smooth maps which extend holomorphically over an annulus of outer radius r and inner radius r^{-1} and are smooth on the boundary.

7. $L_r^2\mathbb{C}$, $r > 1$: smooth maps which extend holomorphically over an annulus of outer radius r and inner radius r^{-1} and are square-integrable on the boundary.

8. $L_r^*\mathbb{C}$, $r > 1$: smooth maps which extend holomorphically over an annulus of outer radius r and inner radius r^{-1} and are distributions on the boundary.

9. $L_r^{2,*}\mathbb{C}$, $r > 1$: the dual of $L_r^2\mathbb{C}$.

10. $L_r^{(2,*)}\mathbb{C}$, $r > 1$: smooth maps which extend holomorphically over an annulus of outer radius r and inner radius r^{-1} and are dual to square-integrable on the boundary.

The last space is, of course, just $L_r^2\mathbb{C}$. However, we are viewing it as the image in $L^{2,*}\mathbb{C}$ of $L_r^2\mathbb{C}$ under the conjugate linear isomorphism $L^2\mathbb{C} \to L^{2,*}\mathbb{C}$.

The penultimate space in the above list has an interesting interpretation. Within the space of formal power series, $\mathbb{C}[\![z^{-1}, z]\!]$, one can consider those power series that converge on a *formal* annulus of *outer* radius r^{-1} and *inner* radius r for some $r > 1$ and satisfy some condition on the boundary. It is not hard to show that this space is (conjugate) dual to some space of the form $L_r^a\mathbb{C}$ for some appropriate boundary condition. Thus $L_r^{2,*}\mathbb{C}$ is conjugate dual to $L_{r^{-1}}^2\mathbb{C}$. One consequence of this interpretation is the following identity:

$$(L_r^{2,*})_r\mathbb{C} = (L_{r^{-1}}^2)_r\overline{\mathbb{C}} = L^2\overline{\mathbb{C}} = L^{2,*}\mathbb{C}.$$

The crucial step here is the observation that the annuli of radii (r, r^{-1}) and of radii (r^{-1}, r) cancel out.

Let $E \to M$ be a finite dimensional orientable vector bundle over a finite dimensional smooth manifold. The loop space, LE, is an infinite dimensional vector bundle over LM modelled on $L\mathbb{F}^n$ for some n, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. We shall consider various related bundles where we modify the fibre of LE from $L\mathbb{F}^n$ to some other function space. We shall label these by decorating the L as for the function spaces above.

The standard ones are L^2E and L^*E having fibre $L^2\mathbb{F}^n$ and $L^*\mathbb{F}^n$ respectively. The bundle L^*E is the dual of LE: a fibre, L_γ^*E, is the space of continuous linear maps $L_\gamma E \to \mathbb{F}$. The simplest way to define L^2E is to observe that the action on $L\mathbb{F}^n$ of the structure group of LE, namely $L\,\mathrm{Gl}_n(\mathbb{F})$, extends to an action on $L^2\mathbb{F}^n$. Thus L^2E is constructed from the principal bundle of LE in the usual way. Fibrewise, it can be viewed as the Hilbert completion of LE with respect to the inner product:

$$\langle \alpha, \beta \rangle_\gamma := \int_{S^1} (\alpha(t), \beta(t))_{\gamma(t)}\, dt$$

where $(\cdot, \cdot)$ is some smooth choice of inner product on the fibres of E. With this approach, one needs to show that this fibrewise completion does result in a locally trivial Hilbert bundle.

The definitions of the bundles $L^2_r E$, $L^*_r E$, and the others is the core of this paper. They will turn out to be locally trivial bundles modelled on the corresponding function spaces.

Finally, it is a standard fact from the differential topology of loop spaces that $TLM = LTM$ but that $T^*LM \neq LT^*M$. For the second, observe that in the case of $\mathbb{R}^n$, $T^*L\mathbb{R}^n = L^*\mathbb{R}^n \times L\mathbb{R}^n$ but $LT^*\mathbb{R}^n = L\mathbb{R}^n \times L\mathbb{R}^n$. Thus T and L commute whilst T^* and L do not. The notation we have introduced above provides another way of writing the cotangent bundle, namely L^*TM. With this notation, T and L continue to behave well since $T^*LM = L^*TM$.

To continue into absurdity, note that L^* and T do not commute even when L^*M makes sense (i.e. when M is $\mathbb{R}^n$) since $TL^*\mathbb{R}^n = L^*\mathbb{R}^n \times L^*\mathbb{R}^n$ and $L^*T\mathbb{R}^n = L^*\mathbb{R}^n \times L\mathbb{R}^n$. To break the bounds of absurdity and enter in to the ridiculous, observe that $T^*L^*\mathbb{R}^n = L\mathbb{R}^n \times L^*\mathbb{R}^n = T^*L\mathbb{R}^n = L^*T\mathbb{R}^n$. Thus our identities are: $TL = LT$, $T^*L = L^*T = T^*L^*$, and $TL^* = (TL)^*$.

3 The Polynomial Loop Group

In this section we consider the group of polynomial loops in a compact, connected Lie group. This was studied extensively in [PS86] with some further work appearing in [Seg89] in the case of U_n. We start with some general results on polynomial loops before constructing the $L_{\mathrm{pol}}G$-bundle over G for G each of U_n, SU_n, and SO_n. We conclude by constructing the corresponding vector bundles.

3.1 Polynomial Loops

The definition of the polynomial loop group appears in [PS86, §3.5]. We repeat that definition here.

Definition 3.1 *Let G be a compact, connected Lie group. Fix an embedding of G as a subgroup of U_n for some n. This exhibits G as a submanifold of $M_n(\mathbb{C})$. The* polynomial loop group *of G, $L_{pol}G$, is defined as the space of those loops in G which when expanded as a Fourier series in $M_n(\mathbb{C})$ are finite Laurent polynomials. The group of based loops, $\Omega_{pol}G$, is the subgroup of $L_{pol}G$ of loops γ with $\gamma(0) = 1_G$.*

Remark 3.2 The following comments appear in [PS86, §3.5]:

1. The choice of the embedding of G in U_n is immaterial.

2. The space $L_{\mathrm{pol}}G$ is the union of the subspaces $L_{\mathrm{pol},N}G$ consisting of those loops with Fourier series of the form:

$$\sum_{k=-N}^{N} \gamma_k z^k.$$

   These spaces are naturally compact. The topology on $L_{\mathrm{pol}}G$ is the direct limit topology of this union.

3. The free polynomial loop group is the semi-direct product of the based polynomial loop group and the constant loops.

4. The group $L_{\mathrm{pol}}G$ does not have an associated Lie algebra, although the Lie algebra $L_{\mathrm{pol}}\mathfrak{g}$ is often linked to it.

5. If G is semi-simple then $L_{\mathrm{pol}}G$ is dense in LG.

6. In the case of the circle, $\Omega_{\mathrm{pol}}S^1 = \mathbb{Z}$ and so $L_{\mathrm{pol}}S^1 = S^1 \times \mathbb{Z}$.

The following is [PS86, proposition 8.6.6]:

Proposition 3.3 *The inclusion $\Omega_{pol}G \to \Omega G$ is a homotopy equivalence.*

Since $L_{\mathrm{pol}}G \cong \Omega_{\mathrm{pol}}G \times G$ and $LG \cong \Omega G \times G$ as spaces (although not generally as groups), this holds for the unbased loops as well.

Although the definition of $L_{\mathrm{pol}}G$ does not depend on the embedding of G in U_n, it is useful to have such an embedding to investigate the structure of $L_{\mathrm{pol}}G$ in a little more detail. We consider loops of the form $t \to \exp(t\xi)$ for suitable $\xi \in \mathfrak{g}$. The main result is the following:

Proposition 3.4 *Let G be a compact, connected Lie group, $\mathfrak{g}$ its Lie algebra. For $\xi \in \mathfrak{g}$, let $\eta_\xi : \mathbb{R} \to G$ denote the path $\eta_\xi(t) = \exp(t\xi)$.*

Let $\xi_1, \xi_2 \in \mathfrak{g}$ be such that $\exp(\xi_1) = \exp(\xi_2)$. Then $\eta_{-\xi_1}\eta_{\xi_2}$ is a polynomial loop in G.

As part of the proof of this, we shall prove the following useful result for the unitary group:

Lemma 3.5 *Let $g \in U_n$. There exists $\zeta \in \exp^{-1}(g) \subseteq \mathfrak{u}_n$ such that $[\zeta, \xi] = 0$ for all $\xi \in \exp^{-1}(g)$.*

The proofs of these rely on the simple structure in U_n of the centraliser of any particular element. For $g \in U_n$, define $C(g)$ and $Z(g)$ to be the centraliser of g and its centre. That is, $C(g) := \{h \in G : h^{-1}gh = g\}$ and $Z(g) = Z(C(g))$. Clearly, $g \in Z(g)$.

Lemma 3.6 *For any $g \in U_n$, $Z(g)$ is a torus.*

Proof. The group $C(g)$ is a closed subgroup of U_n, hence its centre is a closed abelian subgroup of U_n. In particular, it is compact. Therefore, it is a torus if and only if it is connected.

Recall that two diagonalisable matrices commute if and only if they are simultaneously diagonalisable. This condition does not rely on the eigenvalues of either matrix but only on the eigenspaces.

Let $h \in Z(g)$. As h is unitary, it is orthogonally diagonalisable. Let $\lambda_1, \dots, \lambda_l$ be the distinct eigenvalues of h with associated eigenspaces $E_1, \dots, E_l$. For each j, let $s_j \in [-i\pi, i\pi)$ be such that $e^{s_j} = \lambda_j$.

Define $\alpha : [0,1] \to U_n$ to be the path such that $\alpha(t)$ has eigenvalues e^{ts_j} and corresponding eigenspaces E_j. Then $\alpha(0) = 1_n$ and $\alpha(1) = h$ so α is a path from 1_n to h. By construction, $\alpha(t)$ for $t \neq 0$ has the same eigenspaces as h and therefore $\alpha(t)$ commutes with exactly the same elements of U_n that h commutes with. Hence as $h \in Z(g)$, $\alpha(t) \in Z(g)$. □

Proof of lemma 3.5. As $Z(g)$ is a torus, it is a connected compact Lie group. Therefore, the exponential map is surjective and so there is some $\zeta \in \mathfrak{z}(g) \subseteq \mathfrak{u}_n$ with $\exp(\zeta) = g$. As $\zeta \in \mathfrak{z}(g)$, $\exp(t\zeta) \in Z(g)$ for all $t \in \mathbb{R}$.

Let $\xi \in \mathfrak{u}_n$ be such that $\exp(\xi) = g$. Then for all $t \in \mathbb{R}$, $\exp(t\xi)$ commutes with g. Hence $\exp(t\xi) \in C(g)$ for all t. Thus $\exp(t\zeta)$ and $\exp(t'\xi)$ commute for all $t, t' \in \mathbb{R}$. Hence $[\zeta, \xi] = 0$. □

Using this we can prove proposition 3.4.

Proof of proposition 3.4. Firstly, note that it is sufficient to prove this in the case of the unitary group. For if $\eta_{-\xi_1}\eta_{\xi_2}$ is a loop in G which is a polynomial loop when G is considered as a subgroup of U_n then, by definition, $\eta_{-\xi_1}\eta_{\xi_2}$ is a polynomial loop in G.

Secondly, note that it is sufficient to consider the case where $\xi_2 = 0$. This forces $\exp(\xi_1) = 1_n$. To deduce the general case from this simpler one, note that by lemma 3.5 that there is some $\zeta \in \mathfrak{u}_n$ with $\exp(\zeta) = \exp(\xi_1)$ (whence also $\exp(\xi_2)$) such that $[\zeta, \xi_j] = 0$. Then $\exp(\xi_j - \zeta) = 1_n$ so, by assumption, $\eta_{(\xi_j - \zeta)}$ is a polynomial loop. The identity:

$$\eta_{-\xi_1}\eta_{\xi_2} = \eta_{-\xi_1}\eta_\zeta\eta_{-\zeta}\eta_{\xi_2} = \eta_{(-\xi_1+\zeta)}\eta_{(\zeta-\xi_2)}.$$

demonstrates that this is a polynomial loop.

Thus we need to show that η_ξ is a polynomial loop if $\exp(\xi) = 1$. To show this, we diagonalise ξ. If s is an eigenvalue of ξ then e^s is an eigenvalue of $\exp(\xi) = 1$. The eigenvalues of ξ therefore lie in $2\pi i\mathbb{Z}$. Hence there is a basis of $\mathbb{C}^n$ with respect to which η_ξ is the path:

$$t \to \begin{bmatrix} e^{2\pi i t k_1} & 0 & \dots & 0 \\ 0 & e^{2\pi i t k_2} & \dots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \dots & e^{2\pi i t k_n} \end{bmatrix}$$

for some $k_j \in \mathbb{Z}$. Since $e^{2\pi i t k} = z^k$ for $k \in \mathbb{Z}$, this is a polynomial loop (viewed as a periodic path). □

Note that for a general group G, although the loop $\eta_{\xi_1}\eta_{-\xi_2}$ lies in $\Omega_{\text{pol}}G$, there may be no factorisation in G as $\eta_{\xi_1-\zeta}\eta_{\zeta-\xi_2}$ since in a general Lie group there may not be any $\zeta \in \mathfrak{g}$ satisfying the required properties.

3.2 The Path Spaces

In the light of the homotopy equivalence, $\Omega_{\text{pol}}G \simeq \Omega G \simeq \Omega_{\text{cts}}G$, the classifying space of $\Omega_{\text{pol}}G$ is (homotopy equivalent to) G itself. Since $\Omega_{\text{pol}}G$ acts on $L_{\text{pol}}G$, there is a natural $L_{\text{pol}}G$-principal bundle over G. In this section we shall give an explicit construction of this bundle. We shall also construct a similar bundle for the smooth loop group. These bundles will be denoted by $P_{\text{pol}}G$ and $P_{\text{per}}G$ (the "per" stands for "periodic").

To demonstrate that these are principal bundles with the appropriate fibre we have to show two things: firstly, that the bundles are locally trivial; and secondly, that the fibres have an action of the appropriate loop group which identifies the fibre with that group. The second of these is straightforward, the first is simple for the smooth case but is surprisingly difficult for the polynomial loop group. We shall only consider the cases of U_n, SU_n, and SO_n.

Definition 3.7 *Let G be a compact, connected Lie group, $\mathfrak{g}$ its Lie algebra. We define $P_{per}G$ and $P_{pol}G$ as follows:*

1. *$P_{per}G$ is the space of smooth paths $\alpha : \mathbb{R} \to G$ with the property that $\alpha(t+1)\alpha(t)^{-1}$ is constant.*

2. *$P_{pol}G \subseteq P_{per}G$ consists of those paths of the form $\eta_\xi\gamma$ for some $\xi \in \mathfrak{g}$ and $\gamma \in L_{pol}G$.*

The projection map $P_{per}G \to G$ is given by $\alpha \to \alpha(1)\alpha(0)^{-1}$. Notice that when restricted to $P_{pol}G$, this maps $\eta_\xi\gamma$ to $\exp(\xi)$.

Recall from section 3.1 that for $\xi \in \mathfrak{g}$ the path $\eta_\xi : \mathbb{R} \to G$ is defined as the path $t \to \exp(t\xi)$.

Observe that a path in $P_{\text{per}}G$ is completely determined by its values on the interval $[0, 1]$. The motivation for the given definition of $P_{\text{per}}G$ (and of $P_{\text{pol}}G$) is that of holonomy.

It will sometimes be useful to consider an element of $P_{\text{per}}G$ to be a pair $(g, \alpha) \in G \times PG$ such that $\alpha(t + 1) = g\alpha(t)$. Here PG is *all* smooth paths $\mathbb{R} \to G$. Although g is completely determined by α, this viewpoint makes it more explicit.

We shall now investigate the desired properties of these spaces. Neither is a group (unlike the analogous continuous situation), but the group G acts in two ways:

Lemma 3.8 *The group G acts on $P_{per}G$ by two actions:*

$$g \cdot_m \alpha = g\alpha, \qquad g \cdot_c \alpha = g\alpha g^{-1}.$$

These actions restrict to actions on $P_{pol}G$. For both actions, the action of G on itself by conjugation makes the projection $P_{per}G \to G$ G-equivariant (hence also for $P_{pol}G \to G$).

Proof. Let $g \in G$ and $\alpha \in P_{\text{per}}G$. Both $g\alpha$ and $g\alpha g^{-1}$ are smooth paths in G so we only need to check the periodicity condition. Let $h = \alpha(t+1)\alpha(t)^{-1}$. Then:

$$\begin{aligned}(g\alpha)(t+1)(g\alpha)(t)^{-1} &= g\alpha(t+1)\alpha(t)^{-1}g^{-1} = ghg^{-1}.\\ (g\alpha g^{-1})(t+1)(g\alpha g^{-1})(t)^{-1} &= g\alpha(t+1)g^{-1}g\alpha(t)^{-1}g^{-1}\\ &= g\alpha(t+1)\alpha(t)^{-1}g^{-1} = ghg^{-1}.\end{aligned}$$

This also proves the statement about the induced action on G.

If $\alpha \in P_{\text{pol}}G$ then α is of the form $\eta_\xi\gamma$ for some $\xi \in \mathfrak{g}$ and $\gamma \in L_{\text{pol}}G$. Let h be either g^{-1} or 1_G. Then $g\eta_\xi\gamma h = \eta_{(\text{Ad}_g\,\xi)}g\gamma h$. As $L_{\text{pol}}G$ is closed under left and right multiplication by G, this lies in $P_{\text{pol}}G$ as required. □

Proposition 3.9 *Define an action of LG on $P_{per}G$ by sending $(\alpha,\gamma) \in P_{per}G \times LG$ to the path $t \to \alpha(t)\gamma(t)$. This action is well-defined and identifies the fibres of $P_{per}G \to G$ with LG. It restricts to an action of $L_{pol}G$ on $P_{pol}G$ and identifies the fibres of $P_{pol}G \to G$ with $L_{pol}G$.*

Proof. The path $t \to \alpha(t)\gamma(t)$ is a smooth path $\mathbb{R} \to G$ (considering γ as a periodic path). We need merely check the periodicity condition. Since $\gamma(t+1) = \gamma(t)$ for all $t \in \mathbb{R}$, we have:

$$\begin{aligned}(\alpha\gamma)(t+1)(\alpha\gamma)(t)^{-1} &= \alpha(t+1)\gamma(t+1)\gamma(t)^{-1}\alpha(t)^{-1}\\ &= \alpha(t+1)\alpha(t)^{-1}.\end{aligned}$$

Hence $\alpha\gamma \in P_{\text{per}}G$. This also shows that $\alpha\gamma$ lies in the same fibre as α.

For an inverse, let $\alpha, \beta \in P_{\text{per}}G$ be such that $\alpha(1)\alpha(0)^{-1} = \beta(1)\beta(0)^{-1}$. As α and β lie in $P_{\text{per}}G$, this means that $\alpha(t+1)\alpha(t)^{-1} = \beta(t+1)\beta(t)^{-1}$ for all $t \in \mathbb{R}$. Rearranging this yields $\alpha(t+1)^{-1}\beta(t+1) = \alpha(t)^{-1}\beta(t)$. Thus the path γ given by $\gamma(t) = \alpha(t)^{-1}\beta(t)$ is a loop. Moreover, it is smooth. Clearly $\alpha\gamma = \beta$ so this is the inverse map which identifies a non-empty fibre of $P_{\text{per}}G \to G$ with LG.

In the polynomial case, if $\alpha \in P_{\text{pol}}G$ and $\gamma \in L_{\text{pol}}G$ then by definition, $\alpha = \eta_\xi\beta$ for some polynomial loop β. Therefore $\alpha\gamma = \eta_\xi(\beta\gamma)$ and hence lies in $P_{\text{pol}}G$.

Conversely, suppose that $\alpha, \beta \in P_{\text{pol}}G$ lie in the same fibre. We need to show that the loop $t \to \alpha^{-1}(t)\beta(t)$ is a polynomial loop. Let $\alpha = \eta_{\xi_1}\widehat{\alpha}$ and $\beta = \eta_{\xi_2}\widehat{\beta}$ where $\widehat{\alpha}$ and $\widehat{\beta}$ are polynomial loops. Since α and β lie in the same fibre, $\exp(\xi_1) = \exp(\xi_2)$. Thus:

$$\gamma = \widehat{\alpha}^{-1}\eta_{-\xi_1}\eta_{\xi_2}\widehat{\beta}.$$

By proposition 3.4, the two terms in the centre give a polynomial loop, hence γ is a polynomial loop.

To complete the proof of the proposition, we need to show that no fibre of $P_{\mathrm{pol}}G \to G$ is empty, whence also no fibre of $P_{\mathrm{per}}G \to G$ is empty. As G is a compact, connected Lie group, for each $g \in G$ there is some $\xi \in \mathfrak{g}$ such that $\exp(\xi) = g$. The path η_ξ lies in $P_{\mathrm{pol}}G$ (and thus in $P_{\mathrm{per}}G$) and is in the fibre above g. Thus the fibres are non-empty. $\square$

Proving that $P_{\mathrm{per}}G$ is locally trivial is relatively straightforward. The case of $P_{\mathrm{pol}}G$ is harder. Therefore we deal with $P_{\mathrm{per}}G$ quickly now before passing to the – for this paper – more relevant case of the polynomial loops in the next section.

Proposition 3.10 *The space $P_{per}G$ is locally trivial over G.*

Proof. To prove this, we require local sections. Let $g \in G$. Let $\xi \in \mathfrak{g}$ be such that $\exp(\xi) = g$. Let $\rho : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ be a smooth surjection which preserves the endpoints and is constant in a neighbourhood of each endpoint. Let $\phi : V \to U$ be a chart for G with U a neighbourhood of g such that $\phi^{-1}(g) = 0$.

For $h \in U$, define a path $\alpha_h : [0, 1] \to G$ by:

$$\alpha_h(t) = \begin{cases} \exp(2\rho(t)\xi) & t \in [0, \frac{1}{2}] \\ \phi((2\rho(t - \frac{1}{2}) + 1)\phi^{-1}(h)) & t \in [\frac{1}{2}, 1] \end{cases}$$

By construction, α_h is continuous. Since α_h is constant in a neighbourhood of $\frac{1}{2}$ and is smooth either side, it is smooth. Moreover, as it is constant in neighbourhoods of 0 and 1, the concatenation $\alpha_h \sharp (\alpha_h(1)\alpha_h)$ is smooth. Hence α_h extends via the formula:

$$\alpha_h(t + n) = \alpha_h(1)^n \alpha_h(t)$$

for $t \in [0, 1)$ and $n \in \mathbb{Z}$, to a smooth path $\mathbb{R} \to G$ such that $\alpha_h(t + 1) = \alpha_h(1)\alpha_h(t)$ for all $t \in \mathbb{R}$.

Clearly, $\alpha_h(1) = h$. Also, the assignment $h \to \alpha_h$ is smooth. Therefore, $h \to \alpha_h$ is a local section of $P_{\mathrm{per}}G$ in a neighbourhood of g. $\square$

3.2.1 The Polynomial Path Space

The case of the polynomial path space is harder. Regarding local sections, it would appear from the definition that there are natural local sections, namely $g \to \eta_\xi$ where $\exp(\xi) = g$. However, except in the case of the unitary group, there is in general no way to choose ξ smoothly in g for all points $g \in G$ (it is always possible to do so for an open dense subset, but this is not good enough).

In fact, we are not able to prove that $P_{\mathrm{pol}}G \to G$ is locally trivial for all compact, connected G at this time. The methods we employ work on a case-by-case basis. This is sufficient for our needs as we are mainly interested in ordinary vector bundles with inner products and thus in the structure groups U_n and SO_n. We shall prove that $P_{\mathrm{pol}}G \to G$ is locally trivial for these groups and also for SU_n. There is no *a priori* reason why the argument for SO_n should not extend to Sp_n, using quaternionic structures in place of complex structures but we feel that this case is outside the focus of this paper.

The following result will prove useful in examining the structure of $P_{\mathrm{pol}}G$ in terms of $P_{\mathrm{pol}}U_n$.

Lemma 3.11 *Let G be a compact, connected Lie group. Consider G as a subgroup of U_n. Then $P_{pol}G = P_{per}G \cap P_{pol}U_n$.*

Proof. Clearly $P_{\mathrm{pol}}G \subseteq P_{\mathrm{per}}G \cap P_{\mathrm{pol}}U_n$. For the converse, let $\alpha \in P_{\mathrm{per}}G \cap P_{\mathrm{pol}}U_n$. Then $\alpha = \eta_\xi \gamma$ for some $\xi \in \mathfrak{u}_n$ and $\gamma \in L_{\mathrm{pol}}U_n$. Now $\exp(\xi) = \eta_\xi(1) = \alpha(1) \in G$ since $\alpha \in P_{\mathrm{per}}G$. Choose $\zeta \in \mathfrak{g}$ such that $\exp(\zeta) = \exp(\xi)$. Then:

$$\alpha = \eta_\zeta \eta_{-\zeta} \eta_\xi \gamma.$$

By proposition 3.4, $\eta_{-\zeta}\eta_\xi$ is a polynomial loop in U_n. Since α and η_ζ both take values in G, $\eta_{-\zeta}\eta_\xi\gamma$ must also take values in G. It thus lies in $LG \cap L_{\mathrm{pol}}U_n$ which is, by definition, $L_{\mathrm{pol}}G$. Therefore α is of the form $\eta_\zeta\beta$ with $\zeta \in \mathfrak{g}$ and $\beta \in L_{\mathrm{pol}}G$. Hence $\alpha \in P_{\mathrm{pol}}G$. □

3.2.2 The Unitary Group

In the case of U_n, there are local sections of the form $g \to \eta_\xi$ where $\exp(\xi) = g$. This will follow from lemma 3.5.

Proposition 3.12 *The space $P_{pol}U_n$ is locally trivial over U_n.*

Proof. Let $s \in i\mathbb{R}$. Let $V_s \subseteq U_n$ be the open subset consisting of those operators which do not have $-e^s$ as an eigenvalue. Let $\mathfrak{v}_s \subseteq \mathfrak{u}_n$ be the open subset consisting of those operators which have eigenvalue in the interval $(s - i\pi, s + i\pi)$. The exponential map restricts to a diffeomorphism $\exp : \mathfrak{v}_s \to V_s$. Let $\log_s : V_s \to \mathfrak{v}_s$ be its inverse.

For a direct construction, define the s-logarithm $\log_s : \mathbb{T} \setminus \{-e^s\} \to (s - i\pi, s + i\pi)$ as the inverse of the exponential map on this domain (note that this coincides with the above definition putting $n = 1$). Let $g \in V_s$. Let $E_1 \oplus \cdots \oplus E_l$ be the orthogonal decomposition of $\mathbb{C}^n$ into the eigenspaces of g with eigenvalues $\lambda_1, \dots, \lambda_l$. Then $\log_s g$ is the operator which acts on E_j by multiplication by $\log_s \lambda_j$.

It is a simple exercise to show that $\log_s g \in Z(g)$ for any g and s such that $\log_s g$ is defined, that $\log_s g$ is locally constant in s, and that $V_{s+2\pi i} = V_s$ and $\mathfrak{v}_{s+2\pi i} = \mathfrak{v}_s + 2\pi i 1_n$.

The local sections of $P_{\mathrm{pol}}U_n \to U_n$ are $\alpha_s : V_s \to P_{\mathrm{pol}}U_n$ given by $\alpha_s(g)(t) = \exp(t \log_s g)$. □

3.2.3 The Special Unitary Group

The method of the previous section works in U_n because every point in U_n is *exp-regular*; that is, is the image of a point in $\mathfrak{u}_n$ such that the exponential map is a diffeomorphism is a neighbourhood of that point. This is not true for a general Lie group. It is straightforward to show that the preimage of $-1 \in SU_2$ under $\exp : \mathfrak{su}_2 \to SU_2$ is a countable number of copies of $\mathbb{CP}^1$, hence $-1 \in SU_2$ is not exp-regular.

However, we can still prove that $P_{\mathrm{pol}}SU_n \to SU_n$ is locally trivial. The strategy is to use the fact that there *is* a point in $\mathfrak{u}_n$ around which the exponential map is a local diffeomorphism, and then use the fact that $SU_n \to U_n \to S^1$ is split.

Proposition 3.13 *The map $P_{pol}SU_n \to SU_n$ is locally trivial.*

Proof. Choose a unit vector $v \in \mathbb{C}^n$. Define the representation $\sigma : \mathbb{T} \to U_n$ by $\sigma(\lambda)v = \lambda v$ and $\sigma(\lambda)$ is the identity on $\langle v \rangle^\perp$.

Let $s \in i\mathbb{R}$. Let $V_s \subseteq U_n$ and $\mathfrak{v}_s \subseteq \mathfrak{u}_n$ be as in the proof of proposition 3.12. Let $\alpha_s : V_s \to P_{\text{pol}}U_n$ be the local section defined in that proposition.

Define $\beta_s : V_s \cap SU_n \to P_{\text{per}}U_n$ by:

$$\beta_s(g)(t) = \alpha_s(g)(t)\,\sigma\Big(\det\big(\alpha_s(g)(-t)\big)\Big).$$

Recall that $\det\exp(\xi) = e^{\operatorname{Tr}\xi}$. Thus for $g \in V_S \cap SU_n$:

$$\det\alpha_s(g)(-t) = e^{\operatorname{Tr}(-t\log_s(g))} = e^{-t\operatorname{Tr}\log_s(g)}.$$

As $g \in SU_n$, $e^{\operatorname{Tr}\log_s(g)} = \det g = 1$ so $\operatorname{Tr}\log_s(g) = 2\pi ik$ for some $k \in \mathbb{Z}$. Thus $t \to e^{-t\operatorname{Tr}\log_s(g)}$ is the map $t \to z^{-k}$. Hence $\sigma(\det\alpha_s(g)(-t))$ is a polynomial loop in U_n. Thus $\beta_s(g)(t) \in P_{\text{pol}}U_n$.

Then as $\det\circ\sigma : \mathbb{T} \to \mathbb{T}$ is the identity, $\det\beta_s(g)(t) = 1$ for all g, t. Hence $\beta_s(g)(t) \in SU_n$ for all g, t. Thus by lemma 3.11, $\beta_s(g) \in P_{\text{per}}SU_n \cap P_{\text{pol}}U_n = P_{\text{pol}}SU_n$. □

3.2.4 The Special Orthogonal Group

The situation for SO_n is more complicated still. The problem here is with eigenvalue -1. It can be shown that $g \in SO_n$ is *exp-regular* if and only if its -1-eigenspace has dimension at most 2. The solution comes from the theory of *unitary structures* which we now describe.

Definition 3.14 *Let E be a real vector space with an inner product. A* unitary structure *on E is an orthogonal map $J : E \to E$ such that $J^2 = -1$.*

Proposition 3.15 *Let E be a real even dimensional vector space with an inner product. The properties of unitary structures that we shall need are:*

1. *E admits a unitary structure.*
2. *The set of unitary structures on E is $O(E) \cap \mathfrak{o}(E)$.*
3. *Let J be a unitary structure on E. Then $\exp(\pi J) = -1_E$.*
4. *Let J_1, J_2 be unitary structures on E. Then: $\eta_{-\pi J_1}\eta_{\pi J_2}$ is a polynomial loop in $SO(E)$.*
5. *Let $\xi \in \mathfrak{so}(E)$ be such that ξ does not have 0 as an eigenvalue. Then there is a natural unitary structure J_ξ on E which varies smoothly in ξ. Considered as an element of $\mathfrak{so}(E)$, J_ξ satisfies $[\xi, J_\xi] = 0$. The assignment $\xi \to J_\xi$ satisfies $J_J = J$ (here J is considered as an element of $\mathfrak{so}(E)$), and $J_{\xi+cJ_\xi} = J_\xi$ for $c > 0$.*
6. *Let $g \in SO(E)$ be such that 1 is not an eigenvalue of g. Then $\log_0(-g)$ is of the form $\xi - \pi J_\xi$ for some $\xi \in \mathfrak{so}(E)$ with $\exp(\xi) = g$.*

In the last property we use the inclusion $SO(E) \to U(E \otimes \mathbb{C})$ to define $\log_0 : SO(E) \cap V_0 \to \mathfrak{u}(E)$. Since $\log_0$ commutes with complex conjugation[3], the image of $SO(E) \cap V_0$ lies in $\mathfrak{so}(E)$.

[3]It is the only one of the logarithms that we have defined with this property.

Proof. Property 1 is a standard property of complex structures whilst 2 is a simple deduction from the definition of a unitary structure. Therefore we start with property 3.

3. As an element of $\mathfrak{o}(E) = \mathfrak{so}(E)$, J is diagonalisable over $\mathbb{C}$. Since $J^2 = -1$, its eigenvalues are $\pm i$. Thus πJ has eigenvalues $\pm \pi i$. Hence $\exp(\pi J)$ has sole eigenvalue -1. As $\exp(\pi J) \in SO(E)$, it is diagonalisable over $\mathbb{C}$ and thus is -1_E.

4. This is a corollary of proposition 3.4 together with the previous property.

5. Diagonalise ξ over $\mathbb{C}$. As ξ is a real operator, its eigenvalues and corresponding eigenspaces come in conjugate pairs. As ξ is skew-adjoint, its eigenvalues lie on the imaginary axis in $\mathbb{C}$. Let $W \subseteq E \otimes \mathbb{C}$ be the sum of the eigenspaces of ξ corresponding to eigenvalues of the form is with $s > 0$. Then $\overline{W}$, resp. $W^\perp$, is the sum of the eigenspaces of ξ corresponding to eigenvalues of the form is with $s < 0$, resp. $s \leq 0$. The assumption on ξ implies that $\overline{W} = W^\perp$. Define J_ξ on $E \otimes \mathbb{C}$ to be the operator with eigenspaces W and $\overline{W}$ with respective eigenvalues i and $-i$. By construction, $J^2 = -1$ and $J^* J = 1$. As the eigenspaces and eigenvalues of J come in conjugate pairs, J is a real operator and thus is a unitary structure.

   Since J_ξ is defined from the eigenspaces of ξ, it varies smoothly in ξ. Moreover, as the eigenspaces of J_ξ decompose as eigenspaces of ξ, J_ξ and ξ are simultaneously diagonalisable over $\mathbb{C}$. Hence $[\xi, J_\xi] = 0$.

   It is clear from the construction that if ζ and ξ can be simultaneously diagonalised and the eigenvalues of ζ have the same parity on the imaginary axis as the corresponding ones of ξ then $J_\zeta = J_\xi$. In particular, $J_J = J$ and $J_{\xi + c J_\xi} = J_\xi$ for $c > 0$.

6. Let F be the -1-eigenspace of g. Then E decomposes g-invariantly as $F \oplus F^\perp$. As g does not have 1 as an eigenvalue, $\log_0(-g)$ is well-defined. Since the decomposition of E is $-g$-invariant:

$$\log_0(-g) = \log_0(-g|_F) + \log_0(-g|_{F^\perp}) = \log_0(-g|_{F^\perp}).$$

   This last step is because $-g|_F = 1_F$ so $\log_0(-g|_F) = 0_F$.

   Let $\xi_{F^\perp} = \log_0(-g\,|_{F^\perp})$. As $-g$ does not have 1 as an eigenvalue on $F^\perp$, $\xi_{F^\perp}$ does not have 0 as an eigenvalue. Let $J_{F^\perp}$ be the corresponding unitary structure. As $[\xi_{F^\perp}, J_{F^\perp}] = 0$,

$$\exp(\xi_{F^\perp} + \pi J_{F^\perp}) = \exp(\xi_{F^\perp}) \exp(\pi J_{F^\perp}) = (-g)|_{F^\perp} \, (-1_{F^\perp}) = g|_{F^\perp} \, .$$

   As $g \in SO(E)$, F must be of even dimension. Choose a unitary structure J_F on F. Then $\exp(\pi J_F) = -1_F = g|_F$. Let $\xi = \pi J_F + \xi_{F^\perp} + \pi J_{F^\perp}$. Then:

$$\exp(\xi) = \exp(\pi J_F) + \exp(\xi_{F^\perp} + \pi J_{F^\perp}) = -1_F + g|_{F^\perp} = g.$$

   Then $J_\xi = J_F + J_{F^\perp}$ so $\xi - \pi J_\xi = \xi_{F^\perp}$, whence $\xi - \pi J_\xi = \log_0(-g)$. □

Theorem 3.16 *The map $P_{pol}SO_n \to SO_n$ is locally trivial.*

Proof. We first describe a family of open sets which cover SO_n. These will be the domains of the sections of $P_{\text{pol}}SO_n$. The family is indexed by the interval $[-1,1]$ and by elements of SO_n.

Let $r \in [-1,1]$. Let W_r be the open subset of SO_n consisting of those g such that no eigenvalue of g (over $\mathbb{C}$) has real part r. For $g \in W_r$ there is a g-invariant orthogonal decomposition of $\mathbb{R}^n$ as $E^r_{-1}(g) \oplus E^1_r(g)$ where the eigenvalues (over $\mathbb{C}$) of g on $E^r_{-1}(g)$ have real part in the interval $[-1,r]$ and on $E^1_r(g)$ in the interval $[r,1]$. Note that g cannot have eigenvalue 1 on $E^r_{-1}(g)$, even if $r=1$, so as $g \in SO_n$, $E^r_{-1}(g)$ must have even dimension.

Over each W_r is a vector bundle with fibre $E^r_{-1}(g)$ at g (this will have different dimension on the different components of W_r). Over most W_r's this bundle is not trivial. Therefore we find smaller open sets over which we can trivialise it.

Let $r \in [-1,1]$ and $g \in W_r$. Define $W_r(g)$ to be the open subset of SO_n consisting of those $h \in W_r$ for which the orthogonal projection $E^r_{-1}(h) \to E^r_{-1}(g)$ is an isomorphism.

Over $W_r(g)$, therefore, the aforementioned vector bundle is trivial and of constant even dimension. Hence, we can choose a unitary structure J_h on each $E^r_{-1}(h)$ which varies smoothly in h.

Extend J_h to a skew-adjoint operator on $\mathbb{R}^n$ by defining it to be zero on $E^1_r(h)$. Let $\epsilon(h) = h\exp(-\pi J_h) \in SO_n$. Then $\epsilon(h)$ agrees with h on $E^1_r(h)$ and is $-h$ on $E^r_{-1}(h)$. Since h does not have eigenvalue -1 on $E^1_r(h)$ and does not have eigenvalue 1 on $E^r_{-1}(h)$, $\epsilon(h)$ does not have eigenvalue -1 on $\mathbb{R}^n$ and so lies in the domain of $\log_0$. Also, as J_h varies smoothly in h, $h \to \epsilon(h)$ is smooth.

Define $\beta_{r,g} : W_r(g) \to P_{\text{per}}SO_n$ by:

$$\beta_{r,g}(h)(t) = \exp\big(t\log_0(\epsilon(h))\big)\exp(t\pi J_h).$$

This is a smooth path in SO_n since both $\log_0(\epsilon(h))$ and J_h lie in $\mathfrak{so}_n$. It varies smoothly in h since both $\epsilon(h)$ and J_h are smooth in h. Since $\epsilon(h) = h\exp(-\pi J_h)$, $\beta_{r,g}(h)(1) = h$ so it is a path above h. We need to show that it lies in $P_{\text{pol}}SO_n$.

Now $\epsilon(h)$ respects the decomposition $E^r_{-1}(h) \oplus E^1_r(h)$ of $\mathbb{R}^n$, therefore so does $\log_0(\epsilon(h))$. Accordingly, write $\log_0(\epsilon(h)) = \xi^r_{-1} + \xi^1_r$.

Consider the situation on $E^r_{-1}(h)$. Since $\exp(\xi^r_{-1}) = \epsilon(h) = -h$ (all restricted to $E^r_{-1}(h)$), by property 6, $\xi^r_{-1} = \zeta - \pi J_\zeta$ for some $\zeta \in \mathfrak{so}(E^r_{-1}(h))$ with $\exp(\zeta) = h$. Extend J_ζ to $\mathbb{R}^n$ by defining it to be zero on $E^1_r(h)$. Let $\xi = \zeta + \xi^1_r$. Then $\exp(\xi) = h$, $[\xi, J_\zeta] = 0$, and $\log_0(\epsilon(h)) = \xi - \pi J_\zeta$. Therefore:

$$\beta_{r,g}(h)(t) = \exp(t\xi)\exp(-t\pi J_\zeta)\exp(t\pi J_h).$$

Since J_ζ and J_h are both extensions to $\mathbb{R}^n$ by zero of unitary structures on $E^r_{-1}(h)$, then by property 4, $\exp(-t\pi J_\zeta)\exp(t\pi J_h)$ is a polynomial loop in SO_n. Hence $\beta_{r,g}(h)$ lies in $P_{\text{pol}}SO_n$. $\square$

3.3 The Polynomial Vector Bundles

Now that we have principal bundles, given a representation we can construct vector bundles. Let V be a finite dimensional vector space with an inner product, either real or complex. Let LV be the space of smooth loops in V and $L_{\text{pol}}V$ the space of polynomial loops. If V is complex then $L_{\text{pol}}V = V[z^{-1}, z]$; if V is real then $L_{\text{pol}}V = LV \cap L_{\text{pol}}(V \otimes \mathbb{C})$.

Let G be a compact, connected Lie group which acts on V by isometries. In the polynomial case, assume that G is one of U_n, SU_n, or SO_n. Then LG acts on LV and $L_{\text{pol}}G$ acts on $L_{\text{pol}}V$. Therefore we have vector bundles over G together with a bundle inclusion:

$$P_{\text{pol}}V := P_{\text{pol}}G \times_{L_{\text{pol}}G} L_{\text{pol}}V \to P_{\text{per}}V := P_{\text{per}}G \times_{LG} LV.$$

We shall now give an alternative view of these vector bundles which will be more enlightening in terms of their structure.

Let PV be the full path space of V. Define $\tau : PV \to PV$ to be the shift operator: $(\tau\beta)(t) = \beta(t+1)$. Let D denote the differential operator: $(D\beta)(t) = \frac{d\beta}{dt}(t)$. There is a strong connection between these operators: D is the infinitesimal generator of the group of translations on PV and $\exp(D) = \tau$.

The motivation for considering these operators is that they give simple descriptions of LV and $L_{\text{pol}}V$ inside PV. The loop space, LV, is the $+1$-eigenspace of τ. The space of polynomial loops inside LV is the union of the finite dimensional D-invariant subspaces of LV.

In the complex case, we can write this as the linear span of the eigenvectors of D. This does not carry over to the real case, however, as the only eigenvectors of D are the constant maps.

Theorem 3.17 *Let g in G. The fibre of $P_{per}V$ above g is the space of $\phi \in PV$ such that $\tau\phi = g\phi$.*

The fibre of $P_{pol}V$ above g is the union of the finite dimensional D-invariant subspaces of the fibre of $P_{per}V$ above g.

Proof. An element of $P_{\text{per}}V$ in the fibre above g is represented by a pair (α, β) with $\alpha \in P_{\text{per}}G$ above g and $\beta \in LV$. Any alternative representative is of the form $(\alpha\gamma, \gamma^{-1}\beta)$ for some $\gamma \in LG$.

Thus the map $\phi : \mathbb{R} \to V$ defined by $\phi := \alpha\beta$ depends only on the element of $P_{\text{per}}V$ and not on the choice of representative. This satisfies:

$$(\tau\phi)(t) = \phi(t+1) = \alpha(t+1)\beta(t+1) = g\alpha(t)\beta(t) = g\phi(t).$$

Hence $\tau\phi = g\phi$.

Conversely, suppose that $\tau\phi = g\phi$. Choose some $\alpha \in P_{\text{per}}G$ above g and define $\beta := \alpha^{-1}\phi$. Then $\beta(t+1) = \alpha^{-1}(t)g^{-1}g\phi(t) = \beta(t)$ so $\beta \in LV$. Changing α to $\alpha\gamma$ changes β to $\gamma^{-1}\beta$. Hence the element in $P_{\text{per}}V$ represented by (α, β) depends only on ϕ.

Now we consider the polynomial path space. We need to show that the fibre of $P_{\text{pol}}V$ above g is the union of the finite dimensional subspaces of the fibre of $P_{\text{per}}V$ that are D-invariant.

Let $\xi \in \mathfrak{g}$ be such that $\exp(\xi) = g$. We consider two actions of ξ on PV. The first is the isomorphism $\alpha \to \eta_{-\xi}\alpha$ which maps $P_{\text{per},g}V$ onto LV. The second is $\alpha \to \xi\alpha$, extending the action of $\mathfrak{g}$ on V to PV. As ξ is a finite dimensional operator, it has a minimum polynomial. This is true also of its action on PV. Therefore any finite dimensional subspace of PV is contained in a finite dimensional ξ-invariant subspace. Moreover, the action of ξ on PV commutes with that of D so any finite dimensional D-invariant subspace of PV is contained in a finite dimensional subspace that is both D-invariant and ξ-invariant.

Hence as ξ preserves both $P_{\mathrm{per},g}V$ and LV, when considering the union of finite dimensional D-invariant subspaces in either, it is sufficient to consider those that are in addition ξ-invariant.

We shall now show that $W \subseteq LV$ is ξ and D-invariant if and only if $\eta_\xi W$ is ξ and D-invariant. This will establish the result.

The ξ-invariance is straightforward since ξ commutes with η_ξ. Hence $W \subseteq LV$ is ξ-invariant if and only if $\eta_\xi W \subseteq P_{\mathrm{per},g}V$ is ξ-invariant.

If W is ξ and D-invariant, then consider $\alpha \in \eta_{\pm\xi}W$ (the $\pm$ allows us to consider both directions at once). This is of the form $\eta_{\pm\xi}\beta$ for some $\beta \in W$. Then:

$$D\alpha = (D\eta_{\pm\xi})\beta + \eta_{\pm\xi}(D\beta) = \eta_{\pm\xi}(\pm\xi\beta + D\beta) \in \eta_{\pm\xi}W.$$

Hence $\eta_{\pm\xi}W$ is D-invariant. □

An immediate corollary of this is that the fibres of $P_{\mathrm{per}}V$ and of $P_{\mathrm{pol}}V$ are D-invariant. For $P_{\mathrm{per}}V$ this follows from the fact that $\exp(D) = \tau$ so D and τ commute. If we wish to emphasise the fibre, we shall refer to D as D_g.

In the complex case, as D_g is skew-adjoint, any element of the fibre of $P_{\mathrm{pol}}V$ above g is thus the sum of eigenvectors of D_g.

When viewing a fibre of $P_{\mathrm{per}}V$ or $P_{\mathrm{pol}}V$ as a subspace of PV, the corresponding element $g \in G$ is not uniquely determined by any one path (contrast with the case of $P_{\mathrm{per}}G$ or $P_{\mathrm{pol}}G$). Thus to keep track of the fibre, we shall often use the notation (g, ϕ).

There is an action of G on $P_{\mathrm{per}}V$ and on $P_{\mathrm{pol}}V$ given by the following equivalent definitions:

$$\begin{aligned} g \cdot [\alpha, \beta] &= [g\alpha, \beta], \\ g \cdot [\alpha, \beta] &= [g\alpha g^{-1}, g\beta], \\ g \cdot (h, \phi) &= (ghg^{-1}, g\phi). \end{aligned}$$

We put in both of the top two descriptions to show that the two actions of G on $P_{\mathrm{per}}G$ (and thus on $P_{\mathrm{pol}}G$) define the same action on $P_{\mathrm{per}}V$ (and $P_{\mathrm{pol}}V$). This action preserves the sub-bundle $P_{\mathrm{pol}}V$ and sends the operator D_h to $D_{ghg^{-1}}$.

4 Dual Loop Bundles

The goal of this section is to construct the inner product and the Hilbert completion of the dual of the vector bundle $LE \to LM$, where $E \to M$ is a real or complex vector bundle. The first part of this construction involves defining the polynomial loop bundle, $L_{\mathrm{pol}}E \to LM$ and proving that it is a locally trivial vector bundle modelled on $L_{\mathrm{pol}}\mathbb{F}^n$, for $\mathbb{F}$ one of $\mathbb{R}$ or $\mathbb{C}$. Once this has been defined, we thicken it to a Hilbert bundle which is a sub-bundle of LE. This dualises to the required completion of L^*E. We show how to construct an inner product on this bundle by finding an isomorphism of the completion of LE with the completion of L^*E.

In section 4.3 we discuss the basic properties of the polynomial loop bundle, and thus of the Hilbert completion of L^*E. In particular we consider the action of the group of diffeomorphisms of the circle. The natural action on LE does not preserve the polynomial sub-bundle but it can be modified to an action which does.

The construction of the polynomial loop bundle relies on the holonomy operator coming from a connection. The holonomy map can be viewed as a variant of a classifying map for the original loop bundle. In section 4.4 we examine this idea.

4.1 Polynomial Loop Bundles

Let M be a smooth finite dimensional manifold without boundary. Let G be one of U_n, SU_n, or SO_n. Let $\mathbb{F}$ be the corresponding field. Let $Q \to M$ be a principal G-bundle. Let $E = Q \times_G \mathbb{F}^n$ be the corresponding vector bundle. As G preserves the inner product on $\mathbb{F}^n$, E carries a fibrewise inner product. Let ∇ be a covariant differential operator on E coming from a connection on Q.

We think of a point in a fibre Q_p as being an isometry $\mathbb{F}^n \to E_p$. We shall also use the *adjoint bundle* associated to Q, $Q^{\mathrm{ad}} := Q \times_{\mathrm{conj}} G$ where G acts on itself by conjugation. This is a bundle of groups. A point in a fibre Q^{ad}_p is an isometry of E_p to itself.

It is a standard result that the loop and path spaces of E form vector bundles over, respectively, the loop and path spaces of M with frame bundles the loop and path spaces of Q and adjoint bundles the loop and path spaces of Q^{ad}.

Recall from section 2.2 that for X each of E, Q, and Q^{ad}, the fibre of the bundle $P^M X$ above $p \in M$ is $P(X_p)$. Thus:

$$\begin{aligned} P^M E &= E \otimes C^\infty(\mathbb{R}, \mathbb{F}), & L^M E &= E \otimes L\mathbb{F}, \\ &= Q \times_G C^\infty(\mathbb{R}, \mathbb{F}^n), & &= Q \times_G L\mathbb{F}^n, \\ P^M Q &= Q \times_G PG, & L^M Q &= Q \times_G LG, \\ P^M Q^{\mathrm{ad}} &= Q \times_{\mathrm{conj}} PG, & L^M Q^{\mathrm{ad}} &= Q \times_{\mathrm{conj}} LG. \end{aligned}$$

As with M inside LM and PM, G sits inside LG and PG as the constant loops. In the middle line, the action is as a subgroup, in the third line the action is via conjugation.

Since G acts on $P_{\mathrm{per}}G$, $P_{\mathrm{pol}}G$, $P_{\mathrm{per}}\mathbb{F}^n$, and $P_{\mathrm{pol}}\mathbb{F}^n$, we can define corresponding bundles over M. For *abc* each of *per* or *pol*, let:

$$P^M_{\mathrm{abc}} E := Q \times_G P_{\mathrm{abc}}\mathbb{F}^n,$$

$$P^M_{\mathrm{abc}}Q := Q \times_G P_{\mathrm{abc}}G,$$
$$P^M_{\mathrm{abc}}Q^{\mathrm{ad}} := Q \times_{\mathrm{conj}} P_{\mathrm{abc}}G.$$

In the middle line, we use the left action of G on $P_{\mathrm{abc}}G$. In the last line, we use the conjugation action. As each of the model spaces for these bundles is itself a bundle over G and the actions on the total spaces induce the conjugation action on the base, for X each of E, Q, and Q^{ad}, $P^M_{\mathrm{abc}}X$ is a fibre bundle over Q^{ad} with fibre $L_{\mathrm{abc}}Y$, where Y is either $\mathbb{F}^n$ or G as appropriate.

The covariant differential operator defines a parallel transport operator. This defines three compatible families of bundle maps $\psi^t_X : X^t \to PX$, for X each of E, Q, and Q^{ad}. The properties of these maps are:

$$\psi^t_E(pqw) = \psi^t_{Q^{\mathrm{ad}}}(p)\psi^t_Q(q)w, \qquad p \in Q^{\mathrm{ad},t}_\gamma,\ q \in Q^t_\gamma,\ w \in \mathbb{F}^n \subseteq P\mathbb{F}^n. \tag{4.1}$$

$$\psi^t_X e_t \psi^s_X = \psi^s_X, \tag{4.2}$$

$$e_{s+1}\psi^{t+1}_X = e_s\psi^t_X, \qquad \text{over } LM. \tag{4.3}$$

For the second, note that $e_t\psi^s_X$ is a map from X^s to X^t. This compatibility relation is the statement that if one parallel transports from time s to time t and then on from time t to some-when else, it is the same as transporting straight from s to ones final time. For the last, over LM then $X^{t+1} = X^t$ so the domains and codomains of these maps are the same. This property is then an application of the fact that the parallel transport operator is intrinsic to M, therefore the parallel transport from X^t to X^s is the same as that from X^{t+1} to X^{s+1}.

These operators extend to bundle equivalences:

$$\Psi^t_X : P^{M,t}X \to PX, \tag{4.4}$$

with the property that $e_s(\Psi^t_X\alpha) = (e_s\psi^t_X)(\alpha(s))$. Note that these equivalences have been chosen such that $(\Psi^t_X\alpha)(s)$ always lies in X^s no matter which t was the starting point.

Recall that, for X each of E, Q, or Q^{ad}, LX sits inside P^LX. It is straightforward to recognise this submanifold: LX consists of those paths $\beta \in P^LX$ which are themselves periodic. Note that for any path β in P^LX then $\beta(t+1)$ and $\beta(t)$ both lie in the same fibre of $X \to M$.

Thus in the right-hand side of (4.4) (restricted to LM), it is straightforward to recognise the sub-bundles consisting of the loops. We wish to transfer this recognition principle to the left-hand side of (4.4). We do this using the *holonomy operator*.

Definition 4.1 *On LM, define the fibrewise operators $h_X : X^0 \to X^0$ by $h_X = e_1\psi^0_X$.*

Over PM, $e_1\psi^0_X$ is a map $X^0 \to X^1$. Over LM then $X^0 = X^1$ so h_X is as defined. The fibres of Q^{ad} act on each of E, Q, and Q^{ad}: on E the action is by definition, on Q and on Q^{ad} by composition.

Lemma 4.2 *The operator h_E is a section of $Q^{ad,0}$. The operators h_E, h_Q, and $h_{Q^{ad}}$ satisfy: $h_{Q^{ad}}(p)h_E = h_Ep$, and $h_Q(q) = h_Eq$. Thus h_E determines both h_Q and $h_{Q^{ad}}$.*

Proof. Since $e_1\psi^0_E$ is a fibrewise isometry $E^0 \to E^0$, it is a section of $Q^{\mathrm{ad},0}$. Then from (4.1), for $p \in Q^{\mathrm{ad},0}$, $q \in Q^0$, $v \in E^0$, and $w \in \mathbb{F}^n \subseteq P\mathbb{F}^n$:

$$(h_Ep)v = (e_1\psi^0_Ep)v$$

$$
\begin{aligned}
&= e_1(\psi^0_E(pv)) \\
&= e_1(\psi^0_{Q^{\mathrm{ad}}}(p)\psi^0_E(v)) && \text{by (4.1)} \\
&= (e_1\psi^0_{Q^{\mathrm{ad}}})(p)(e_1\psi^0_E)(v) \\
&= h_{Q^{\mathrm{ad}}}(p)h_E(v). \\
(h_E q)w &= (e_1\psi^0_E q)w \\
&= e_1(\psi^0_E(qw)) \\
&= e_1(\psi^0_Q(q)w) && \text{by (4.1)} \\
&= (e_1\psi^0_Q)(q)w \\
&= h_Q(q)w. && \square
\end{aligned}
$$

Lemma 4.3 $e_{t+1}\psi^0_X = e_t\psi^0_X h_X$.

Proof.

$$
\begin{aligned}
e_{t+1}\psi^0_X &= e_{t+1}\psi^1_X e_1\psi^0_X && \text{by (4.2)} \\
&= e_t\psi^0_X e_1\psi^0_X && \text{by (4.3)} \\
&= e_t\psi^0_X h_X. && \square
\end{aligned}
$$

Corollary 4.4 *Under the bundle isomorphism of* (4.4)*, the sub-bundle LX of $P^L X$ corresponds to:*

$$\{\alpha(t) \in P^{M,0}X : h_X\alpha(t+1) = \alpha(t)\}.$$

Proof. An element $\alpha \in P^{M,0}X$ is mapped to a loop in $P^L X$ if and only if $(\Psi^0_X\alpha)(t+1) = (\Psi^0_X\alpha)(t)$ for all $t \in \mathbb{R}$. The left-hand side of this simplifies to:

$$e_{t+1}(\Psi^0_X\alpha) = (e_{t+1}\psi^0_X)(\alpha(t+1)) = (e_t\psi^0_X)(h_X\alpha(t+1))$$

whilst the right-hand side simplifies to:

$$e_t(\Psi^0_X\alpha) = (e_t\psi^0_X)(\alpha(t)).$$

Since $e_t\psi^0_X : X^0 \to X^t$ is an isomorphism, this implies that $\Psi^0_X\alpha$ is a loop if and only if $h_X\alpha(t+1) = \alpha(t)$ for all $t \in \mathbb{R}$. $\square$

A section $\chi : LM \to Q^{\mathrm{ad},0}$ is the same thing as a map $\chi : LM \to Q^{\mathrm{ad}}$ which covers the map $e_0 : LM \to M$. For such a section, X each of Q, Q^{ad}, or E, and *abc* each of *pol* or *per*, let $L^{M,0,\chi}_{\mathrm{abc}}X \to LM$ be the pull-back of $P^M_{\mathrm{abc}}X \to Q^{\mathrm{ad}}$ via the map $\chi : LM \to Q^{\mathrm{ad}}$.

Corollary 4.5 *For X each of Q, Q^{ad}, and E, Ψ^0_X restricts to a bundle isomorphism $L^{M,0,h_E^{-1}}_{per}X \to LX$.*

Definition 4.6 *The* polynomial loop bundles, *$L_{pol}X$, for X each of Q, Q^{ad}, and E are defined to be the images in LX of $L^{M,0,h_E^{-1}}_{pol}X$ under the map Ψ^0_X.*

The following is immediate:

Proposition 4.7 *The polynomial loop bundles are locally trivial with $L_{pol}Q$ a $L_{pol}G$-principal bundle, $L_{pol}Q^{ad}$ a bundle of groups modelled on $L_{pol}G$, and $L_{pol}E$ a vector bundle modelled on $L_{pol}\mathbb{F}^n$. Moreover:*

$$L_{pol}Q^{ad} = L_{pol}Q \times_{conj} L_{pol}G,$$

$$
\begin{aligned}
L_{pol}E &= L_{pol}Q \times_{L_{pol}G} L_{pol}\mathbb{F}^n, \\
LQ &= L_{pol}Q \times_{L_{pol}G} LG, \\
LQ^{ad} &= L_{pol}Q \times_{conj} LG, \\
LE &= L_{pol}Q \times_{L_{pol}G} L\mathbb{F}^n.
\end{aligned}
$$

The bundle $L_{\text{pol}}E$ has a more concrete description in terms of the connection on E. For any path $\gamma : \mathbb{R} \to M$, the connection on E defines a covariant differential operator $D_\gamma : \Gamma_{\mathbb{R}}(\gamma^* E) \to \Gamma_{\mathbb{R}}(\gamma^* E)$; that is, $D_\gamma : P_\gamma E \to P_\gamma E$. As the map Ψ^0_E was constructed using parallel transport, it (rather, its inverse) takes D_γ to the operator $\frac{d}{dt}$ acting on $P^{M,0}E$. If γ happens to be a loop, D_γ restricts to an operator on $L_\gamma E$. As Ψ^0_E identifies $L_\gamma E$ with the fibre of $P_{\text{per}}E \to Q^{\text{ad},0}$ above $h_E^{-1}(\gamma)$, it takes D_γ to the operator $D_{h_E^{-1}(\gamma)}$.

Hence $L_{\text{pol},\gamma}E$ can be constructed from the action of D_γ on $L_\gamma E$ in the same fashion as $P_{\text{pol},g}\mathbb{F}^n$ from $P_{\text{per},g}\mathbb{F}^n$, namely as the union of the finite dimensional D_γ-invariant subspaces of $L_\gamma E$. In the complex case, $L_{\text{pol},\gamma}E$ is the span of the eigenvalues in $L_\gamma E$ of D_γ.

4.2 The Completion of the Cotangent Bundle

The Hilbert completion of the cotangent bundle is now straightforward. We merely need to select a Hilbert space that lies between $L_{\text{pol}}\mathbb{R}^n$ and $L\mathbb{R}^n$. The group $L_{\text{pol}}SO_n$ will act on this Hilbert space and thus we can construct a locally trivial bundle over the loop space LM with fibre a Hilbert space which sits naturally between $L_{\text{pol}}TM$ and LTM. Dualising this will yield a Hilbert space sitting naturally between L^*TM and $L^*_{\text{pol}}TM$. On fibres, this will be a Hilbert completion of $T^*LM = L^*TM$.

There are many Hilbert spaces between $L_{\text{pol}}\mathbb{R}^n$ and $L\mathbb{R}^n$. We choose $L^2_e\mathbb{R}^n$. The choice of e is dictated by the desire not to have unnecessary constants at a later stage. It is not overly significant. There is an obvious inner product on $L^2_e\mathbb{R}^n$ but as it is not preserved under the action of $L_{\text{pol}}SO_n$, we shall not spend any time discussing it.

Lemma 4.8 *Let G be one of SO_n, SU_n, or U_n. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ as appropriate. Then $L_{pol}G$ acts continuously on $L^2_e\mathbb{F}^n$.*

Proof. It is sufficient to show that $L_{\text{pol}}U_n$ acts on $L^2_e\mathbb{C}^n$ since $L_{\text{pol}}SU_n$ and $L_{\text{pol}}SO_n$ are subgroups of $L_{\text{pol}}U_n$ and the action of $L_{\text{pol}}SO_n$ on $L^2_e\mathbb{R}^n$ is well-defined if and only if its action on $L^2_e\mathbb{C}^n$ is well-defined.

To show that $L_{\text{pol}}U_n$ acts on $L^2_e\mathbb{C}^n$, it is sufficient to show that $L_{\text{pol}}M_n(\mathbb{C})$ acts. This is straightforward since it is generated as an algebra by $M_n(\mathbb{C})$ and z which both act on $L^2_e\mathbb{C}^n$. $\square$

Thus given a vector bundle $E \to M$ with structure bundle Q we obtain a Hilbert bundle $L^2_eE \to LM$ as $L_{\text{pol}}Q \times_{L_{\text{pol}}G} L^2_e\mathbb{F}^n$. There are maps $L_{\text{pol}}E \to L^2_eE \to LE$ which locally look like $L_{\text{pol}}\mathbb{F}^n \to L^2_e\mathbb{F}^n \to L\mathbb{F}^n$. Dualising this bundle yields the required completion of L^*E.

As remarked above, the natural inner product on $L^2_e\mathbb{F}^n$ is not preserved by the action of $L_{\text{pol}}G$. It is a simple matter to show that when the circle action is taken into account, the only Hilbert completion of $L_{\text{pol}}\mathbb{F}^n$ on which $L_{\text{pol}}G$ can act unitarily is $L^2\mathbb{F}^n$. Therefore, we shall have to find another route to an inner product on $L^2_e E$ (and thus its dual). The route we choose is to construct an isomorphism of Hilbert bundles $L^2_e E \to L^2 E$. This will allow us to pull-back the inner product on $L^2 E$ to $L^2_e E$.

Proposition 4.9 *There is a well-defined bundle map $L_{pol}E \to L_{pol}E$ given by:*

$$\alpha \to (\cos D_\gamma)\alpha$$

where:

$$\cos D_\gamma = \sum_{j=0}^{\infty} \frac{1}{(2j)!} D_\gamma^{2j}.$$

This extends to an isomorphism of Hilbert bundles $L^2_e E \to L^2 E$.

Proof. We start with the fibrewise situation. The fibre of $L_{\mathrm{pol}}E$ above a loop γ is the union of D_γ-invariant finite dimensional subspaces of $L_\gamma E$. On any finite dimensional space, the power series denoted by $\cos A$ converges for any operator A. Therefore, $\cos D_\gamma$ is well-defined on each finite dimensional D_γ-invariant subspace of $L_\gamma E$ and hence on $L_{\mathrm{pol}}E$.

When considering the Hilbert completions of $L_{\mathrm{pol},\gamma}E$, it is sufficient to assume that E is complex. In this case, there is a basis for $L_{\mathrm{pol},\gamma}E$ of eigenvectors of D_γ. We can choose this basis to have the following properties:

1. There are n eigenvectors $v_1, \dots, v_n \in L_\gamma E$ such that the corresponding eigenvalues are of the form $is_1, \dots, is_n$ with each $s_j \in [0, 2\pi i)$.

2. The other eigenvectors are of the form $z^k v_j$ for some $k \in \mathbb{Z}$.

The eigenvalue of $z^k v_j$ is $is_j + 2\pi i k$. Therefore, $(\cos D_\gamma) z^k v_j = \cosh(s_j + 2\pi k) z^k v_j$.

We wish to describe the sequences $(a^j_k)_{k \in \mathbb{Z}, j=1,\dots,n}$ such that $(a^j_k \cosh(s_j + 2\pi k))$ is square-summable. It is sufficient to consider each j in turn so we consider $(a_k)_{k\in\mathbb{Z}}$ such that $(a_k \cosh(s + 2\pi k))$ is square-summable for some $s \in [0, 2\pi)$.

Now elementary analysis shows that for all $x \in \mathbb{R}$:

$$\cosh(s) \geq \frac{\cosh(x+s)}{e^{|x|}} \geq \frac{1}{2}\min\{e^s, e^{-s}\}.$$

Therefore, $(a_k \cosh(s + 2\pi k))$ is square-summable if and only if $(a_k e^{2\pi|k|})$ is square-summable. This is precisely the condition that the loop corresponding to (a_k) extend analytically over an annulus of radii e and e^{-1} and be square-summable on the boundaries. □

Therefore, although we cannot transfer the standard inner product on $L^2_e \mathbb{F}^n$ to the fibres of $L^2_e E$, we can use the operator D_γ to define an inner product on the fibres of $L^2_e E$ which is equivalent to the standard inner product on $L^2_e \mathbb{F}^n$. This inner product is defined by:

$$(\alpha, \beta) = \langle (\cos D_\gamma)\alpha, (\cos D_\gamma)\beta \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2_\gamma E$. Thus $\alpha \to (\cos D_\gamma)\alpha$ is an isometry from $L^2_{e,\gamma} E$ to $L^2_\gamma E$.

Since an inner product on a Hilbert space defines a (conjugate-linear) isomorphism with its dual, we can extend the inclusion $L^*E \to L_e^{2,*}E$ to a triple $L^*E \to L_e^{2,*}E \cong L_e^2 E \to LE$. This mirrors the standard triple $LE \to L^2E \cong L^{2,*}E \to L^*E$ coming from the standard inner product on LE. Putting these together (and suppressing the isomorphisms) yields a non-commuting square:

$$\begin{array}{ccc} LE & \longrightarrow & L^2E \\ \uparrow & & \downarrow \\ L_e^2 E & \longleftarrow & L^*E. \end{array}$$

As mentioned above, the group $L_{\mathrm{pol}}G$ does not act on $L_e^2\mathbb{F}^n$ by isometries but does so act on $L^2\mathbb{F}^n$. Therefore the isomorphism $L_e^2 E \to L^2 E$ can be viewed as a careful alteration of the structure group of $L_e^2 E$ to one which acts by isometries. We can do the same with L^*E.

We have constructed a chain of bundle maps which on fibres looks like:

$$L_e^2\mathbb{F}^n \to L\mathbb{F}^n \to L^2\mathbb{F}^n \cong L^{2,*}\mathbb{F}^n \to L^*\mathbb{F}^n \to L_e^{2,*}\mathbb{F}^n.$$

The isometry $L_e^2 E \cong L^2 E$ takes this chain to one which ends in $L^{2,*}E$. On fibres this chain is obtained from the one above by adding a subscript e to each space. Using the identities $(L_e^2)_e\mathbb{F}^n = L_{e^2}^2\mathbb{F}^n$ and $(L_e^{2,*})_e\mathbb{F}^n = L^{2,*}\mathbb{F}^n$, the fibres of the vector bundles in this new chain are:

$$L_{e^2}^2\mathbb{F}^n \to L_e\mathbb{F}^n \to L_e^2\mathbb{F}^n \cong L_e^{(2,*)}\mathbb{F}^n \to L_e^*\mathbb{F}^n \to L^{2,*}\mathbb{F}^n.$$

In this picture, we have identified L^*E with the sub-bundle L_e^*E of LE and then taken the inner product and completion of L^*E to be $L^{2,*}E$. Since the action of $L_{\mathrm{pol}}G$ on L_e^*E is by isometries, this identification of L^*E with L_e^*E can be viewed as a careful alteration of the structure group of L^*E to one that acts by isometries.

In the appendix we shall consider this alteration of the structure group in more detail.

4.3 Properties of the Polynomial Bundle

The construction of the polynomial loop bundle started from a connection on the original bundle over M. However, it only actually used the map $\psi_X^0 : X^0 \to PX$ defined by the parallel transport operator. Thus as far as the polynomial loop bundle is concerned, having a connection is overkill. The connection is useful, though, as it implies that the polynomial loop bundle came from structure on the original manifold M and thus one can hope for more structure on the polynomial loop bundle than has yet been described. In this section, we shall investigate this. In the next, we shall give an interpretation of the maps ψ_X^0 in terms of classifying maps and twisted K-theory.

Before examining the interesting properties of the polynomial loop bundle, we list some basic ones that are fairly obvious:

Proposition 4.10 *Let M be a finite dimensional smooth manifold, $E_1, E_2 \to M$ finite dimensional vector bundles over the same field with inner products and connections compatible with the inner products.*

1. *Let $E = E_1 \oplus E_2$ orthogonally and equip E with the direct sum connection. Then $L_{pol}E = L_{pol}E_1 \oplus L_{pol}E_2$.*

2. *Suppose that E_1 is real, then $L_{pol}(E_1 \otimes \mathbb{C}) = (L_{pol}E_1) \otimes \mathbb{C}$.*

3. *Suppose that E_1 is complex, then $L_{pol}(E_{1\mathbb{R}}) = (L_{pol}E_1)_{\mathbb{R}}$.*

4. *Let $\psi : E_1 \to E_2$ be a bundle isomorphism which preserves the inner products and connections. Then ψ defines an isomorphism $L_{pol}\psi : L_{pol}E_1 \to L_{pol}E_2$.*

5. *Suppose that E_1 with its inner product is a sub-bundle of E_2 and that the covariant differential operator on E_1 is of the form $p\nabla$ where $p : E_2 \to E_1$ is the orthogonal projection and ∇ is the covariant differential operator on E_2. Then it is not necessarily the case that $L_{pol}E_1 = L_{pol}E_2 \cap LE_1$.*

Proof. Only the last of these is not immediate from the construction. Let E_2 be the bundle $S^1 \times \mathbb{C}^2$ and E_1 the bundle $S^1 \times \mathbb{C}^1$. Include E_1 in E_2 via the map $(t, 1) \to (t, \frac{1}{\sqrt{2}}(1, e^{2\pi i t}))$.

The loop space of E_1 is $LS^1 \times L\mathbb{C}$ and of E_2 is $LS^1 \times L\mathbb{C}^2$. The polynomial loop space of E_2 is $LS^1 \times L_{\mathrm{pol}}\mathbb{C}^2$. The inclusion $LE_1 \to LE_2$ is given by:

$$(\gamma, \beta) \to (\gamma, \frac{1}{\sqrt{2}}(\beta, e^{2\pi i \gamma(t)}\beta)).$$

Therefore $LE_1 \cap L_{\mathrm{pol}}E_2$ consists of those loops β such that both β and $e^{2\pi i\gamma}\beta$ are polynomials. We can choose γ such that whenever β is polynomial then $e^{2\pi i\gamma}\beta$ is not. Hence there is some γ such that above γ the fibres of LE_1 and $L_{\mathrm{pol}}E_2$ intersect trivially. □

The advantage of having the polynomial structure defined using a connection on the original bundle is the relationship with the diffeomorphism group of the circle. For $\sigma : S^1 \to S^1$ smooth (not necessarily a diffeomorphism), $\gamma : S^1 \to M$, and $\alpha \in L_\gamma E$, the following is a simple application of the chain rule:

$$D_{\gamma\circ\sigma}(\alpha \circ \sigma) = ((D_\gamma \alpha) \circ \sigma)\, \sigma', \tag{4.5}$$

where $\sigma' : S^1 \to \mathbb{R}$ is such that $d\sigma(\frac{d}{dt}) = \sigma' \frac{d}{dt}$.

From this formula, two results can be derived:

Proposition 4.11 *1. The action of* $\mathrm{Diff}(S^1)$ *on LE does not preserve the sub-bundle $L_{pol}E$. The subgroup of* $\mathrm{Diff}(S^1)$ *which does preserve the sub-bundle $L_{pol}E$ is $S^1 \rtimes \mathbb{Z}/2$ where the non-trivial element in the $\mathbb{Z}/2$-factor is the diffeomorphism $t \to -t$.*

2. Let ∇^a and ∇^b be two different connections on E. The two polynomial bundles so defined are different.

Proof. We shall consider the complex case so that we may talk about eigenvectors and eigenvalues of D_γ. The real case may be deduced from this.

1. For this, consider the situation over a constant loop. There, LE, resp. $L_{\mathrm{pol}}E$, is $E \otimes L\mathbb{C}$, resp. $E \otimes L_{\mathrm{pol}}\mathbb{C}$. The action of $\mathrm{Diff}(S^1)$ on LE is given

by its action on $L\mathbb{C}$. Thus if $\sigma \in \operatorname{Diff}(S^1)$ preserves $L_{\mathrm{pol}}E$ then it must preserve $L_{\mathrm{pol}}\mathbb{C}$ within $L\mathbb{C}$.

The map $t \to e^{2\pi i t}$ lies in $L_{\mathrm{pol}}\mathbb{C}$. It is also the identification of S^1 with $\mathbb{T}$. Under σ this transforms to $t \to e^{2\pi i \sigma(t)}$. As σ is a diffeomorphism of S^1, this map must still be an identification of S^1 with $\mathbb{T}$. The only polynomials which do this are those of the form $t \to \nu e^{\pm 2\pi i t}$ for $\nu \in \mathbb{T}$. Hence if $\sigma \in \operatorname{Diff}(S^1)$ preserves $L_{\mathrm{pol}}E$ within LE then $\sigma \in S^1 \rtimes \mathbb{Z}/2$.

The converse is direct from the equation 4.5 since if $\sigma \in S^1 \rtimes \mathbb{Z}/2$ then $\sigma' = \pm 1$ so:

$$D_{\gamma\circ\sigma}(\alpha \circ \sigma) = \pm(D_\gamma \alpha) \circ \sigma.$$

Hence σ maps eigenvectors of D_γ to eigenvectors of $D_{\gamma\circ\sigma}$ and thus preserves $L_{\mathrm{pol}}E$.

2. As ∇^a and ∇^b are different, there will be some loop γ such that D^a_γ and D^b_γ differ. The difference will be a section Φ of the bundle $\mathfrak{u}(\gamma^* E) \to S^1$, in other words an element of $L_\gamma \mathfrak{u}(E)$.

   If $L^a_{\mathrm{pol},\gamma}E = L^b_{\mathrm{pol},\gamma}E$ then both are preserved under D^a_γ and D^b_γ, hence under their difference. Thus Φ must be an element of $L_{\mathrm{pol}}\mathfrak{u}(E)$.

   By examining equation 4.5, we see that under the action of a smooth self-map σ of the circle, Φ transforms to $(\Phi \circ \sigma)\sigma'$. It is then a simple matter to find σ such that this is no longer a polynomial. Hence even if we were unlucky enough initially to choose a loop γ with $L^a_{\mathrm{pol},\gamma}E = L^b_{\mathrm{pol},\gamma}E$ then we can find some other loop $\gamma \circ \sigma$ over which the fibres of the polynomial bundles differ. □

It is straightforward to show that the result about the action of $\operatorname{Diff}(S^1)$ on $L_{\mathrm{pol}}E$ generalises to the statement that the subgroup of $\operatorname{Diff}(S^1)$ which preserves $L^?E$ is $\operatorname{Diff}(S^1) \cap L^?\mathbb{C}$ where the "?" represents some class of regularity of loop.

In the light of this result, it is perhaps surprising that there is an action of $\operatorname{Diff}(S^1)$ on $L_{\mathrm{pol}}E$ which covers the standard action of $\operatorname{Diff}(S^1)$ on LM. This comes about because the $\operatorname{Diff}(S^1)$-action preserves the parallel transport operator. Since all else was derived from that, we can make $\operatorname{Diff}(S^1)$ act on $L_{\mathrm{pol}}E$.

We start with the group $\operatorname{Diff}^+_0(S^1)$ of orientation and basepoint preserving diffeomorphisms. Since the whole diffeomorphism group is the semi-direct product of this with $S^1 \rtimes \mathbb{Z}/2$, an action of this group together with the above action of $S^1 \rtimes \mathbb{Z}/2$ will give an action of the whole diffeomorphism group.

An element of $\operatorname{Diff}^+_0(S^1)$ lifts canonically to an element of $\operatorname{Diff}^+_0(\mathbb{R})$. The image consists of those diffeomorphisms of $\mathbb{R}$ which satisfy $\sigma(t+1) = \sigma(t) + 1$. This allows $\operatorname{Diff}^+_0(S^1)$ to act on paths as well as loops.

Let $\sigma \in \operatorname{Diff}^+_0(S^1)$. Recall that the bundle $P^{M,0}E \to LM$ has fibre $P^{M,0}_\gamma E = P(E_{\gamma(0)})$. Thus as $\gamma \circ \sigma(0) = \gamma(0)$, the bundles $P^{M,0}E$ and $\sigma^*(P^{M,0}E)$ are genuinely the same bundle. The bundle $P^L E$, meanwhile, has fibre $P^L_\gamma E = \Gamma(\gamma^* E)$. Thus there is a natural isomorphism $P^L E \to \sigma^*(P^L E)$ given by $\alpha \to \alpha \circ \sigma$.

With these two isomorphisms, the square:

$$\begin{array}{ccc} P^{M,0}E & \xrightarrow{\Psi_E} & P^L E \\ \Big\| & & \sigma\Big\downarrow \\ P^{M,0}E & \xrightarrow{\Psi_E} & P^L E \end{array}$$

does not commute. To make it commute, we need to transfer one action of σ from one side to the other. Clearly, the action of σ on $P^L E$ restricts to the standard action on LE which we know does not preserve $L_{\text{pol}}E$.

It is also true that the action of σ on $P^{M,0}E$ preserves $L^{M,0,h_E^{-1}}E$ and $L^{M,0,h_E^{-1}}$. Thus is because the holonomy operator h_E is equivariant under the action of $\mathrm{Diff}_0^+(S^1)$. Therefore, the action of σ on $P^{M,0}E$ when transferred to $P^L E$ also restricts to an action on LE and on $L_{\text{pol}}E$.

In formulæ, the two actions of $\mathrm{Diff}_0^+(S^1)$ are as follows: any element of $P_\gamma E$ can be written as $\sum_j f^j \psi_E^0 v_j$ where $\{v_1, \dots, v_n\}$ is a basis for $E_{\gamma(0)}$. The usual action is:

$$\sigma\left(\sum_j f^j \psi_E^0 v_j\right) = \sum_j f^j \circ \sigma \psi_E^0 v_j$$

and the new action is:

$$\sigma\left(\sum_j f^j \psi_E^0 v_j\right) = \sum_j f^j \psi_E^0 v_j.$$

One way to make the distinction between the two actions is to have two views of the bundle $LE \to LM$. In one, a fibre $L_\gamma E$ is inextricably linked to the points of $\gamma(S^1)$. In the other, the fibre $L_\gamma E$ is linked only to the map γ. In the former, reparametrising the loop γ does not change $\gamma(S^1)$ and so the fibres $L_\gamma E$ and $L_{\gamma\circ\sigma}E$ are closely related. Any reasonable – in this view – group action must preserve this relationship. In the latter view, reparametrising the loop γ changes it and so there is no intrinsic relationship between the fibres $L_\gamma E$ and $L_{\gamma\circ\sigma}E$. Therefore there is no special relationship for a reasonable group action to preserve.

4.4 Loop Bundles and Twisted K-Theory

As mentioned in the previous section, the construction of the polynomial loop bundle started from a connection on the original bundle over M but a connection provides rather more structure than is needed. The vital piece was the section of the bundle h_E of $Q^{\text{ad},0} \to LM$ and the isomorphism of LE with $L_{\text{per}}^{M,0,h_E^{-1}}E$. Thus any pair (χ, Ψ) where χ is a section of $Q^{\text{ad},0} \to LM$ and Ψ is a bundle isomorphism of $L_{\text{per}}^{M,0,\chi}E$ with LE will do. However, if one wants χ and Ψ to come from structure on M, a connection is the simplest starting point. This ensures that the fibres of the polynomial loop bundle are related to the points on M over which they lie.

The section h_E (rather, h_E^{-1}) can be thought of as a type of classifying map of the bundle LE. It is not, strictly speaking, a classifying map as it does not land in BLG. Rather it classifies LE "up to constant loops". The basic idea of this viewpoint is that when considering infinite dimensional geometry, anything finite dimensional is relatively uninteresting or already well-understood. Therefore, saying that a bundle is trivial "up to constant loops" is saying that it is really a finite dimensional object that has been enhanced in some trivial way to make it appear infinite dimensional and therefore is of little interest. For example, with polynomial loops, the "polynomial" part is really defined for based loops. To get the free polynomial loops, one simply includes the constant loops in an appropriate way.

To make this slightly more mathematical, recall that the free loop group, LG, is the semi-direct product of the based loop group and the constant loops. That is, there is a split short exact sequence:

$$\Omega G \to LG \leftrightarrow G.$$

Thus within the class of LG-objects are those which come from G-objects via the inclusion $G \to LG$. For example, within the class of vector bundles over a space Y with fibre $L\mathbb{C}^n$ lie the vector bundles of the form $E \otimes L\mathbb{C}$ for some n-dimensional vector bundle $E \to Y$.

There is a similar sequence of classifying spaces. A particular choice of classifying spaces, used for example in [CS04], is:

$$G \to EG \times_{\text{conj}} G \to BG.$$

Given a classifying map $Y \to BG$ we can thus pull-back the G-bundle (which is not a principal bundle but rather a bundle of groups) over Y. We interpret this as a bundle over Y with fibre $B\Omega G$. Thus a section of this bundle defines a *twisted* principal ΩG-bundle over Y. A section of the $B\Omega G$-bundle is also a map from Y to $EG \times_{\text{conj}} G = BLG$ and thus classifies a principal LG-bundle.

Conversely, given a classifying map $Y \to BLG = EG \times_{\text{conj}} G$, we can project down onto BG and pull-back the G-bundle as above. The original classifying map then defines a section of this G-bundle and so a twisted principal ΩG-bundle over Y.

Hence a principal LG-bundle can be interpreted as a principal ΩG-bundle twisted by a principal G-bundle. The G-bundle that defines the twisting is the pull-back of the *principal* G-bundle from BG. The $B\Omega G$-bundle used above is the adjoint bundle of this principal G-bundle.

This is actually an unstable phenomenon, at least in the case of U_n. There is a group homomorphism $LU_n \to U_n \times \mathrm{Gr}_{\text{res}}(H)$ which is a homotopy equivalence in the stable range. Thus $EU_\infty \times_{\text{conj}} U_\infty \simeq BU_\infty \times U_\infty$. In fact, by choosing appropriate models for U_∞ and BU_∞ we can make this a homeomorphism. Let H be a complex, separable infinite dimensional Hilbert space. Let $U(H)$ be the unitary operators on H. Let $U_{\mathcal{K}}$ be the unitary operators on H of the form $1 + T$ for some compact operator T. Then by [Kui65] and [Pal65], $U(H)$ is contractible and $U_{\mathcal{K}} \simeq U_\infty$. The maps $U(H) \times_{\text{conj}} U_{\mathcal{K}} \leftrightarrow BU_{\mathcal{K}} \times U_{\mathcal{K}}$ are:

$$[q,p] \to ([q], qpq^{-1}), \quad ([q],p) \to [q, q^{-1}pq],$$

where we use the fact that $U_{\mathcal{K}}$ is normal in $U(H)$. Thus the twisting of an ΩU_∞-bundle by a U_∞-bundle is trivial.

We end by noting in passing that the splitting of BLU_∞ is related to the fact that LU_∞ is its own (based) loop space and hence its own classifying space. Thus, for example, it defines a ring spectrum and hence a generalised cohomology theory of period 1. It is not a very interesting theory as it is just $K^0 + K^{-1}$.

5 The Dirac Operator on the Loop Space

In this section we construct the Dirac operator on the loop space of an appropriate manifold. We start with a review of the theory of spin in infinite dimensions and its links to loop groups. We then turn to the question of what structure on the original manifold gives rise to a spin structure on the loop space. Finally, we construct the Dirac operator.

5.1 Spin Structures and Polarisations

In this section we shall review the essential details of the construction of the spin representation in infinite dimensions, also referred to as the Fock representation. This is gleaned mostly from [PR94] with the application to loop spaces coming from [PS86].

Let V be an infinite dimensional real vector space with a continuous inner product, $(\cdot,\cdot)$. Let J be a choice of unitary structure on V; that is, J is an orthogonal transformation on V such that $J^2 = -1$. Let V_J denote V with this complex structure and let $\langle\cdot,\cdot\rangle$ be the hermitian inner product on V_J defined by $\langle u, v\rangle = (u, v) + i\,(u, Jv)$.

Let $\mathbb{H}_J$ be the Hilbert space completion of $\Lambda^\bullet V_J$, the exterior power of V_J, with respect to the inner product:

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_l\rangle = \begin{cases} 0 & l \neq k \\ \det(\langle u_i, v_j\rangle) & l = k. \end{cases}$$

Recall that $\mathcal{L}(\mathbb{H}_J)$ is the Banach space of (complex) continuous linear maps from $\mathbb{H}_J$ to itself. Define operators $c : V \to \mathcal{L}(\mathbb{H}_J)$ and $a : V \to \mathcal{L}(\mathbb{H}_J)$ by:

$$\begin{aligned} c(v)u_1 \wedge \cdots \wedge u_k &= v \wedge u_1 \wedge \cdots \wedge u_k \\ a(v)u_1 \wedge \cdots \wedge u_k &= \sum_{j=1}^{k} (-1)^{j-1}\langle u_i, v\rangle u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_k. \end{aligned}$$

Let $\pi : V \to \mathcal{L}(\mathbb{H}_J)$ be the operator $c + a$.

Proposition 5.1 *The operator c is complex linear and a is conjugate linear, regarding V as V_J, and they satisfy the* canonical anti-commutation relations*:*

$$\begin{aligned} \{c(u), a(v)\} &= \langle u, v\rangle \\ \{c(u), c(v)\} = \{a(u), a(v)\} &= 0 \end{aligned}$$

where for operators X, Y, $\{X, Y\} = XY + YX$.

Hence π is real linear and satisfies $\pi(v)^2 = (v, v)\,I$.

The map π is called *Clifford multiplication*. The space $\mathbb{H}_J$ decomposes as $\mathbb{H}_J^+ \oplus \mathbb{H}_J^-$ with $\mathbb{H}_J^+$ the completion of $\Lambda^{\mathrm{ev}} V_J$ and $\mathbb{H}_J^-$ of $\Lambda^{\mathrm{odd}} V_J$. With respect to this grading, Clifford multiplication is of odd degree. That is, it interchanges the factors.

It is fairly obvious, and is described in [PR94, theorem 1.2.7], that the construction of the Fock representation factors through the Hilbert completion of V defined by the inner product. Thus if $\widetilde{V}$ is a subspace of V, possibly with a finer topology, that is dense in V with the inner product topology and such that

J restricts to a unitary structure on $\widetilde{V}$ then the Fock representations of (V, J) and $(\widetilde{V}, J)$ are the same.

The *implementation question* is the following: let $O(V)$ be the orthogonal group of V. For which $g \in O(V)$ is there some $U_g \in U(\mathbb{H}_J)$ such that $\pi(gv) = U_g \pi(v) U_g^{-1}$? It is answered by:

Theorem 5.2 ([PR94, ch 3]) *For $g \in O(V)$ there is some $U_g \in U(\mathbb{H}_J)$ such that $\pi(gv) = U_g \pi(v) U_g^{-1}$ if and only if $[g, J]$ is a Hilbert-Schmidt operator. Moreover, if U_g and U_g' both implement g then $U_g = \lambda U_g'$ for some $\lambda \in S^1$.*

An operator $T : H_1 \to H_2$ between Hilbert spaces is said to be *Hilbert-Schmidt* if for some, and hence every, orthogonal basis $\{e_i\}$ of H_1 then $(\|Te_i\|)$ is square summable. The subgroup of $O(V)$ consisting of g such that $[g, J]$ is Hilbert-Schmidt is written $O_J(V)$ in [PR94].

An intimately related problem is that of *equivalence*: given unitary structures J and K on V, what condition is equivalent to there being a unitary transformation $T : \mathbb{H}_J \to \mathbb{H}_K$ such that $\pi_J(v) = T\pi_K(v)T^{-1}$? The answer is given by:

Theorem 5.3 ([PR94, ch 3]) *Let J and K be unitary structures on V. The Fock representations $\mathbb{H}_J$ and $\mathbb{H}_K$ are unitarily equivalent if and only if $J - K$ is Hilbert-Schmidt.*

Thus one could define a *Fock structure* on V to be an equivalence class of unitary structures. The Fock representation would then only depend on this class, rather than the explicit choice of unitary structure. This idea provides a neat sequé from the theory of Fock representations to that of polarisations.

There are various equivalent definitions of a polarisation, we choose the one that is closest to the theory of unitary structures. The theory of polarisations and the relationship with loop groups is the subject of [PS86]. The following definitions are equivalent to those from [PS86, ch 6] although we have used notation similar to that of [PR94] for better comparison with the theory of unitary structures.

Definition 5.4 *Let H be a complex Hilbert space. A* polarising operator *on H is an operator $J \in \mathcal{L}(H)$ such that $J^2 + I$ is trace class and $J \pm iI$ are not finite rank.*

A polarisation *on H is an equivalence class of polarising operators under the relation $J_1 \sim J_2$ if and only if $J_1 - J_2$ is Hilbert-Schmidt.*

Let $\mathcal{J}$ be a polarisation on H. The restricted general linear group *of H with respect to $\mathcal{J}$, $\mathrm{Gl}_{\mathcal{J}}(H)$, is defined as the subgroup of $\mathrm{Gl}(H)$ consisting of those A for which $[A, J]$ is Hilbert-Schmidt for one, and hence all, $J \in \mathcal{J}$.*

In [PS86], the notation used is $\mathrm{Gl}_{\mathrm{res}}(H)$. The notation $\mathrm{Gl}_{\mathcal{J}}(H)$ emphasises the dependence on the polarisation $\mathcal{J}$. The operator used in the above definition is slightly different from the operator J used in [PS86, ch 6]. To get from the one to the other, multiply by $-i$.

Clearly a polarising operator J defines a polarisation by taking the equivalence class of J. Thus a unitary structure J on a real Hilbert space H gives rise to a polarisation on the complexification $H_{\mathbb{C}}$ by taking the equivalence class of J, extended to the complexification by linearity. With respect to this polarisation, it is evident that $O_J(H) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap O(H)$.

There are three equivalent definitions of a unitary structure given in [PR94, ch 2.1]. Using these correspondences, a careful examination of [PS86, ch 12] reveals that the standard unitary structure on $L^2(S^1, \mathbb{R}^{2n})$ is defined in the following way: Let $\{e_k\}$ be the standard basis for $\mathbb{R}^{2n}$. Let $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the complex structure $J_0 e_{2k} = e_{2k-1}$, $J_0 e_{2k-1} = -e_{2k}$. The unitary structure on $L^2(S^1, \mathbb{R}^{2n})$ is defined by the operator J which satisfies:

$$\begin{aligned} J(v \cos k\theta) &= v \sin k\theta \\ J(v \sin k\theta) &= -v \cos k\theta \\ J(v) &= J_0(v). \end{aligned}$$

Here we identify $\mathbb{R}^{2n}$ with the subspace of constant loops in $L^2(S^1, \mathbb{R}^{2n})$.

The standard polarisation operator J on $L^2(S^1, \mathbb{C}^m)$ satisfies the identity:

$$J(vz^k) = -(-1)^{\mathrm{sign}(k)} ivz^k.$$

Proposition 5.5 *The standard polarisation on $L^2(S^1, \mathbb{C}^{2n})$ is that defined by the standard unitary structure on $L^2(S^1, \mathbb{R}^{2n})$. If m is odd, the standard polarisation on $L^2(S^1, \mathbb{C}^m)$ does not contain a unitary structure for $L^2(S^1, \mathbb{R}^m)$.*

Proof. To distinguish the operators, let $J_{\mathbb{R}}$ denote the unitary structure on $L^2(S^1, \mathbb{R}^{2n})$ and also its extension to $L^2(S^1, \mathbb{C}^{2n})$. Let $J_{\mathbb{C}}$ be the polarising operator on $L^2(S^1, \mathbb{C}^m)$. The first part of the proposition follows from the observation that $J_{\mathbb{R}}$ and $J_{\mathbb{C}}$ agree on the subspace of $L^2(S^1, \mathbb{C}^{2n})$ consisting of loops orthogonal to the constant loops. This has finite codimension and so $J_{\mathbb{C}} - J_{\mathbb{R}}$ is finite rank. Thus $J_{\mathbb{R}}$ and $J_{\mathbb{C}}$ define the same polarisation on $L^2(S^1, \mathbb{C}^{2n})$.

Let m be odd. Let $\{e_k\}$ be the standard basis for $\mathbb{R}^m$. Let $J_0 : \mathbb{R}^m \to \mathbb{R}^m$ be the map $J_0 e_{2k} = e_{2k-1}$, $J_0 e_{2k-1} = -e_{2k}$, $J_0 e_m = 0$. Let $J_{\mathbb{R}}$ be the map on $L^2(S^1, \mathbb{R}^m)$ defined using J_0 as for the even dimensional case. This restricts to a unitary structure on the subspace $\langle e_m \rangle^{\perp}$. As before, $J_{\mathbb{R}}$ and $J_{\mathbb{C}}$ agree on the subspace of loops orthogonal to the constant loops and thus define the same polarisation on $L^2(S^1, \mathbb{C}^m)$.

Let K be a unitary structure on $L^2(S^1, \mathbb{R}^m)$. The space $L^2(S^1, \mathbb{C}^m)$ decomposes orthogonally according to the eigenspaces of $J_{\mathbb{R}}$ and of K. Corresponding to $J_{\mathbb{R}}$ we have $L^2(S^1, \mathbb{C}^m) = V_+ \oplus V_- \oplus \mathbb{C}$ as $\pm i$-eigenspaces and the 0-eigenspace. Corresponding to K we have $L^2(S^1, \mathbb{C}^m) = W_+ \oplus W_-$. Let Σ denote the operation of complex conjugation on $L^2(S^1, \mathbb{C}^m)$. Then $\Sigma W_{\pm} = W_{\mp}$, $\Sigma V_{\pm} = V_{\mp}$, and $\Sigma \mathbb{C} = \mathbb{C}$.

The identity map decomposes as the matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} : V_+ \oplus V_- \oplus \mathbb{C} \to W_+ \oplus W_-.$$

Here $a : V_+ \to W_+$ is the inclusion of V_+ followed by the projection onto W_+, and similarly for the other entries. Since the identity map commutes with complex conjugation, $d = \Sigma b \Sigma$, $e = \Sigma a \Sigma$, and $f = \Sigma c \Sigma$.

Now assume that $J_{\mathbb{R}} - K$ is Hilbert-Schmidt.

The operator $b : V_- \to W_+$ can be written as $\frac{1}{4}(I - iK)(I + iJ_{\mathbb{R}})P$ where $P : V_+ \oplus V_- \oplus \mathbb{C} \to V_+ \oplus V_-$ is the orthogonal projection. This expands to $\frac{1}{4}(I + KJ_{\mathbb{R}} + i(J_{\mathbb{R}} - K))P$. As $K^2 = -I$, $I + KJ_{\mathbb{R}} = K(J_{\mathbb{R}} - K)$ and therefore b is Hilbert-Schmidt. Similarly, d is Hilbert-Schmidt. Since c and f have domain

$\mathbb{C}$, they are finite rank. Thus the operator $a + e$ differs from the identity by a compact operator so is Fredholm of index zero.

Since a and e start from orthogonal subspaces and end in orthogonal subspaces, the fact that $a + e$ is Fredholm implies that both a and e are also Fredholm. The identity $e = \Sigma a \Sigma$ then implies that $\operatorname{Index} a = \operatorname{Index} e$. The matrix form of $a + e$ is:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \end{bmatrix}$$

from which it is evident that the index of $a + e$ is $\operatorname{Index} a + \operatorname{Index} e + 1$. This is incompatible with $\operatorname{Index} a = \operatorname{Index} e$ and so we deduce that $J_{\mathbb{R}} - K$ cannot be Hilbert-Schmidt. Hence there is no unitary structure for $L^2(S^1, \mathbb{R}^m)$ in the standard polarisation of $L^2(S^1, \mathbb{C}^m)$. □

For the record, we note the following properties of the groups associated to the standard polarisation on $L^2(S^1, \mathbb{C}^{2n})$ and the standard unitary structure on $L^2(S^1, \mathbb{R}^{2n})$.

Lemma 5.6 *Let $H = L^2(S^1, \mathbb{R}^{2n})$ and let J be the standard unitary structure on H. Let $H_{\mathbb{C}} = L^2(S^1, \mathbb{C}^{2n})$ be the complexification and $\mathcal{J}$ the standard polarisation on $H_{\mathbb{C}}$.*

1. *$O_J(H) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap O(H)$;*
2. *let $U_{\mathcal{J}}(H_{\mathbb{C}}) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap U(H_{\mathbb{C}})$, then $U_{\mathcal{J}}(H_{\mathbb{C}}) \to \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}})$ is a deformation retract;*
3. *let $\mathrm{Gl}_J(H) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap \mathrm{Gl}(H)$, then $O_J(H) \to \mathrm{Gl}_J(H)$ is a deformation retract; and*
4. *$U_{\mathcal{J}}(H_{\mathbb{C}}) \simeq \Omega U$, $O_J(H) \simeq \Omega O$.*

In [PS86, ch 6], it is shown that the natural action of LU_{2n} on $H_{\mathbb{C}}$ defines an inclusion $LU_{2n} \to U_{\mathcal{J}}(H_{\mathbb{C}})$. Since $LO_{2n} = LU_{2n} \cap O(H)$, it follows that the natural action of LO_{2n} on H defines an inclusion $LO_{2n} \to O_J(H)$.

The action of $O_J(H)$ on $\mathbb{H}_J$ is projective. That is, there is a central S^1-extension of $O_J(H)$, usually written $\mathrm{Pin}_J(H)$ (the identity component being $\mathrm{Spin}_J(H)$), which acts unitarily on $\mathbb{H}_J$. This central extension is classified by a generator of $H^2(O_J(H), \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$.

Examining LO_{2n}, we see that it has four components. The identity component is the semi-direct product $SO_{2n} \times \Omega \,\mathrm{Spin}_{2n}$ which has double cover $L\,\mathrm{Spin}_{2n}$. The central extension of $O_J(H)$ pulls back to a central S^1-extension of $L\,\mathrm{Spin}_{2n}$ written $\widetilde{L}\,\mathrm{Spin}_{2n}$. This is classified by a generator of $H^2(L\,\mathrm{Spin}_{2n}, \mathbb{Z})$, which is also isomorphic to $\mathbb{Z}$. Note also that the transgression map $\tau : H^\bullet(\mathrm{Spin}_{2n}, \mathbb{Z}) \to H^{\bullet-1}(L\,\mathrm{Spin}_{2n}, \mathbb{Z})$ is an isomorphism from degree 3 to degree 2.

The two components of Pin_J can be easily distinguished. Recall that $\mathbb{H}_J$ decomposes as $\mathbb{H}_J^+ \oplus \mathbb{H}_J^-$. The identity component of $\mathrm{Pin}_J(H)$, whence also $\widetilde{L}\,\mathrm{Spin}_{2n}$, preserves this decomposition. The other component swaps the factors.

Finally, the circle action on $L^2(S^1, \mathbb{R}^{2n})$ lies in $O_J(H)$ and has a canonical lift to $\mathrm{Pin}_J(H)$. This defines a circle action on $\mathbb{H}_J$. The circle action on $L\,\mathrm{Spin}_{2n}$ therefore lifts to $\widetilde{L}\,\mathrm{Spin}_{2n}$ and the action of $\widetilde{L}\,\mathrm{Spin}_{2n}$ on $\mathbb{H}_J$ is circle equivariant.

5.2 String Manifolds and Spin Connections

In this section we explain how a string structure on a manifold defines a connection on the spin bundle of the loop space. Let M be an oriented, Riemannian manifold of even dimension d. Let $R \to M$ be the principal SO_d-bundle determined by the metric and the orientation. Let $\omega : TR \to \mathfrak{so}_d$ be the Levi-Civita connection on M.

The group Spin_d is the connected double cover of SO_d, universal if $d > 2$. A *spin structure* on M is a principal Spin_d-bundle $Q \to M$ such that Q is a double covering of R and the following diagram commutes:

$$\begin{tikzcd} \operatorname{Spin}_d \times Q \arrow[r] \arrow[d] & Q \arrow[d] \\ SO_d \times R \arrow[r] & R. \end{tikzcd}$$

The manifold M admits a spin structure if and only if $w_2(M) = 0$; the set of isomorphism classes of spin structures is in bijective correspondence with $H^1(M; \mathbb{Z}_2)$.

In order that the loop space, LM, admit a spin structure the structure group of LM must lift from $L\operatorname{Spin}_d$ to $\widetilde{L}\operatorname{Spin}_d$. We would also like this to be S^1-equivariant. The $L\operatorname{Spin}_d$-principal bundle on LM is LQ. Thus we are asking for an S^1-bundle, equivalently a line bundle, over LQ with certain properties. The primary property is that on fibres it must pull-back to the fibration $S^1 \to \widetilde{L}\operatorname{Spin}_d \to L\operatorname{Spin}_d$.

As explained in [Bry93, ch VI], line bundles on loop spaces are closely related to gerbes on the original manifold. In particular, the central extension $\widetilde{L}\operatorname{Spin}_d$ of $L\operatorname{Spin}_d$ corresponds to the gerbe of Spin_d classified by the generator of $H^3(\operatorname{Spin}_d; \mathbb{Z})$ (recall that as a simply connected, simple Lie group, there is a canonical isomorphism of $H^3(\operatorname{Spin}_d; \mathbb{Z})$ with $\mathbb{Z}$ and hence a canonical generator). Rather than asking for a line bundle over LQ we therefore ask for a gerbe over Q. This has the considerable advantage that the line bundle defined by the gerbe will be $\operatorname{Diff}^+(S^1)$-equivariant.

We have to answer the following question: what is the obstruction to constructing a gerbe on Q which on fibres pulls-back to the fundamental gerbe on Spin_d? We can rephrase this question in cohomological terms where it becomes: when can we find an element $a \in H^3(Q; \mathbb{Z})$ such that if $i : \operatorname{Spin}_d \to Q$ is the inclusion of a fibre then $i^* a$ is the generator of $H^3(\operatorname{Spin}_d; \mathbb{Z})$?

To answer this we examine the Serre spectral sequence of the fibration $\operatorname{Spin}_d \to Q \to M$. The first part of the E_2-term is:

3	$H^3(\operatorname{Spin}_d; \mathbb{Z})$				
2	0	0	0		
1	0	0	0	0	
0	$H^0(M; \mathbb{Z})$	$H^1(M; \mathbb{Z})$	$H^2(M; \mathbb{Z})$	$H^3(M; \mathbb{Z})$	$H^4(M; \mathbb{Z})$
	0	1	2	3	4

This contains all the possible contributions to $H^3(Q; \mathbb{Z})$. The only part that might not persist to the E_∞-term is $H^3(\operatorname{Spin}_d; \mathbb{Z})$ in the $(0, 3)$ position. This persists until the E_4-term where the differential is $d_4 : H^3(\operatorname{Spin}_d; \mathbb{Z}) \to$

$H^4(M;\mathbb{Z})$. Let $\lambda \in H^4(M;\mathbb{Z})$ denote the image of the canonical generator of $H^3(\mathrm{Spin}_d;\mathbb{Z})$ under d_4. If $\lambda = 0$ then $H^3(Q;\mathbb{Z}) \cong H^3(M;\mathbb{Z}) \oplus H^3(\mathrm{Spin}_d;\mathbb{Z})$ and the inclusion of a fibre induces the projection $H^3(M;\mathbb{Z}) \oplus H^3(\mathrm{Spin}_d;\mathbb{Z}) \to H^3(\mathrm{Spin}_d;\mathbb{Z})$. If $\lambda \neq 0$ then $H^3(Q;\mathbb{Z}) = H^3(M;\mathbb{Z})$ and the inclusion of a fibre is the zero map on H^3. The class λ is known to satisfy $2\lambda = p_1(M)$ which has led to it being written as $p_1(M)/2$. This notation is somewhat misleading as λ depends on the choice of spin structure on M.

Definition 5.7 *A manifold M is a* string *manifold if it is an oriented, Riemannian, spin manifold such that $\lambda = 0$ together with a choice of* string structure. *That is, a choice of gerbe, $\mathcal{G}$, over the spin structure $Q \to M$ which on fibres is the fundamental gerbe on* Spin_d*.*

Once we have a string structure, there is a natural notion of a string connection.

Definition 5.8 *A* string connection *on a string manifold with string manifold with string structure $\mathcal{G}$ consists of the Levi-Civita connection on Q and a* Spin_d*-equivariant connective structure on the gerbe $\mathcal{G}$.*

Theorem 5.9 *A string connection on M defines a* $\mathrm{Diff}^+(S^1)$*-equivariant spin connection on LM.*

Compare this result with that of [Man02].

Proof. The Levi-Civita connection on M is a map $\omega : TR \to \mathfrak{so}_d$. As $\mathrm{Spin}_d \to SO_d$ is a covering map, it is a local diffeomorphism and so $\mathfrak{spin}_d = \mathfrak{so}_d$. Thus the Levi-Civita connection lifts to a connection on Q via $\omega' : TQ \to TR \xrightarrow{\omega} \mathfrak{so}_d = \mathfrak{spin}_d$. The loop of this is a $\mathrm{Diff}^+(S^1)$-equivariant map $L\omega' : TLQ \to L\mathfrak{spin}_d$. This is also a connection.

The gerbe with its connective structure defines a $\mathrm{Diff}^+(S^1)$-equivariant S^1-bundle $\widetilde{L}Q \to LQ$ with a connection $\alpha : T\widetilde{L}Q \to \mathbb{R}$. As the gerbe on M pulls back to the fundamental gerbe on fibres, so also $\widetilde{L}Q \to LQ$ pulls back to $\widetilde{L}\,\mathrm{Spin}_d \to L\,\mathrm{Spin}_d$ on fibres. Also, the connection is $L\,\mathrm{Spin}_d$-equivariant. Hence $L\omega' \oplus \alpha : T\widetilde{L}Q \to L\mathfrak{spin}_d \oplus \mathbb{R}$ is a connection on $\widetilde{L}Q$. □

5.3 The Dirac Operator

We can now construct the Dirac operator. Let M be a finite dimensional, simply connected, string manifold. The loop space LM thus has a spin structure with spin connection. The Levi-Civita connection on the tangent bundle of M defines the polynomial loop bundle, $L_{\mathrm{pol}}TM$, and thus the Hilbert completion of $T^*LM = L^*TM$.

The spinor bundles, S^+, S^-, of LM are constructed from L^2TM. As L^2TM is a real Hilbert bundle, it is canonically isomorphic to its dual, $L^{2,*}TM$. Thus we can view the spinor bundle as being constructed from either L^2TM or $L^{2,*}TM$ as seems appropriate. The point of having the two approaches is to distinguish between L^2TM as the completion of LTM and of L^*TM and thus determine which of the finite dimensional constructions we are generalising.

The spin connection on LM defines a covariant differential operator:

$$\nabla : \Gamma(S^+) \to \Gamma(\mathcal{L}(TLM, S^+)).$$

By taking the spinor bundles from $L^{2,*}TM$, we are considering this to be the completion of $L^*TM = T^*LM$. Thus we consider Clifford multiplication to be a fibrewise map $L^*TM \to \mathcal{L}(S)$. The following proposition is essentially the *remarkable isomorphism* and will enable us to compose this with the covariant differential operator to define the Dirac operator.

Proposition 5.10 *Let V be a complete nuclear reflexive space with a continuous inner product. Let J be a unitary structure on V. The map $\pi : V \to \mathcal{L}(\mathbb{H}_J)$ defines a continuous linear map $\pi : \mathcal{L}(V^*, \mathbb{H}_J) \to \mathbb{H}_J$.*

Before proving this, we show how this leads to the definition of the Dirac operator. We are considering S^+ and S^- to be constructed from the cotangent bundle, T^*LM. Therefore, we take V in the statement of the proposition to be $L^*\mathbb{R}^n$ which is a complete nuclear reflexive space. Since its dual is $L\mathbb{R}^n$, Clifford multiplication defines a fibrewise linear map:

$$\pi : \mathcal{L}(LTM, S^+) \to S^-.$$

Definition 5.11 *The Dirac operator, $\partial\!\!\!/ : \Gamma(S^+) \to \Gamma(S^-)$, on LM is the composition:*

$$\partial\!\!\!/ : \Gamma(S^+) \xrightarrow{\nabla} \Gamma(\mathcal{L}(TLM, S^+)) \xrightarrow{\pi} \Gamma(S^-).$$

Since every piece of structure that went into its construction is equivariant under rotations of the circle, the Dirac operator is similarly equivariant.

We conclude with the proof of proposition 5.10:

Proof of proposition 5.10. Let H denote the Hilbert space completion of V with respect to the inner product topology defined by the inner product on V. From [PR94, ch 2.4], we know that $\pi : V \to \mathcal{L}(\mathbb{H}_J)$ extends to an isometric inclusion $\pi : H \to \mathcal{L}(\mathbb{H}_J)$. The map $H \times \mathbb{H}_J \to \mathbb{H}_J$, $(x, \xi) \to \pi(x)\xi$, is therefore continuous. From [Sch71, ch III, §6], it extends to a continuous linear map with domain the projective tensor product $H \widetilde{\otimes} \mathbb{H}_J$.

The inclusion $V \to H$ induces a continuous linear map $V \widetilde{\otimes} \mathbb{H}_J \to H \widetilde{\otimes} \mathbb{H}_J$. From [Sch71, ch IV, §9.4], as V is a complete nuclear space then the space $V \widetilde{\otimes} \mathbb{H}_J$ is isomorphic to $\mathcal{L}_e(V^*_\tau, \mathbb{H}_J)$; where this denotes the space of linear maps from V^* to $\mathbb{H}_J$. The topology on V^* is the Mackay topology and the topology on the space of maps is that of uniform convergence on equicontinuous sets.

From [Sch71, ch IV, §5] we deduce that as V is reflexive, the Mackay topology on the dual agrees with the strong topology. Also as V is reflexive, it is barrelled and so equicontinuous sets in V^* are the same as bounded sets. Hence $\mathcal{L}_e(V^*_\tau, \mathbb{H}_J) = \mathcal{L}(V^*, \mathbb{H}_J)$. □

Appendix: Inner Products on the Space of Distributions

In this appendix we examine inner products on $L^*\mathbb{R}^n$. The goal is to classify the inner products on $L^*\mathbb{R}^n$ which have the following properties: the inner product is invariant under the circle action, the involution of reversing loops is orthogonal, and the operations of multiplication by $\cos\theta$ and $\sin\theta$ are continuous.

We shall actually work with $\mathcal{S}^*$, the dual of the space of rapidly decreasing, complex-valued, $\mathbb{Z}$-indexed sequences. As a sequence space, this is particularly simple to describe and therefore to work with. Taking Fourier coefficients defines an isomorphism $L\mathbb{C} \to \mathcal{S}$ which allows us to transfer information from $\mathcal{S}^*$ to $L^*\mathbb{C}$. Using the description of $L\mathbb{C}^n$ as $L\mathbb{C} \otimes \mathbb{C}^n$, we can extend the description to the dual of $L\mathbb{C}^n$, and thence to the dual of the underlying real space $L\mathbb{R}^n$.

As a preliminary, we shall prove a negative result. We shall show that there is no "natural" inner product on $L^*\mathbb{C}$. That is, if $L\mathbb{C}^\times = L(\mathbb{C}^\times)$ denotes the space of never-zero smooth loops in $\mathbb{C}$ then there is no inner product on $L^*\mathbb{C}$ such that the group $L\mathbb{C}^\times$ acts continuously with respect to the inner product topology. This is in stark contrast to the situation for $L\mathbb{C}$ where $L\mathbb{C}^\times$ does act continuously with respect to the standard inner product.

Theorem A.12 *Let $L_c\mathbb{C}$ be a class of loops in $\mathbb{C}$ with the following properties:*

1. *there are continuous inclusions $\mathbb{C} \to L_c\mathbb{C} \to L^{1,\infty}\mathbb{C}$, where $\mathbb{C}$ corresponds to the constant loops and $L^{1,\infty}\mathbb{C}$ is the space of continuously differentiable loops;*
2. *the class of loops is preserved under products; thus $L_c\mathbb{C}^\times$ acts on $L_c\mathbb{C}$ and hence, via the adjoint map, on $L_c^*\mathbb{C}$;*
3. *$L_c\mathbb{C}$ is reflexive;*
4. *$L_c\mathbb{C}$ cannot be given the structure of a Hilbert space;*

then for any inner product on $L_c^*\mathbb{C}$ there is some $\alpha \in L_c\mathbb{C}^\times$ which acts unboundedly on $L_c^*\mathbb{C}$ with respect to the inner product topology.

Proof. Let $\langle\cdot,\cdot\rangle$ be a continuous inner product on $L_c^*\mathbb{C}$. Let H denote the Hilbert space completion of $L_c^*\mathbb{C}$ with respect to $\langle\cdot,\cdot\rangle$. The dual of the inclusion of $L_c^*\mathbb{C}$ in H is a map $H^* \to L_c^{**}\mathbb{C} = L_c\mathbb{C}$.

Suppose that $L_c\mathbb{C}^\times$ acts continuously on $L_c^*\mathbb{C}$ with respect to the inner product topology. This implies that H^* is preserved in $L_c\mathbb{C}$ by $L_c\mathbb{C}^\times$. Suppose that $H^* \cap L_c\mathbb{C}^\times \neq \emptyset$. Because $L_c\mathbb{C}^\times$ is a group, this implies that $L_c\mathbb{C}^\times \subseteq H^*$. The linear span of $L_c\mathbb{C}^\times$ is $L_c\mathbb{C}$ so $H^* = L_c\mathbb{C}$. However, this implies that $H^* \to L_c\mathbb{C}$ is a continuous, linear bijection from a Hilbert space onto $L_c\mathbb{C}$ which contradicts the fourth assumption.

Thus we need to show that the other assumptions imply that $H^* \cap L_c\mathbb{C}^\times \neq \emptyset$. In other words, we need to show that there is an element in H^* which is never zero. To do this, we shall use the Banach-Steinhaus theorem as stated in [Sch71, III, §4.6]. As $L_c\mathbb{C}$ is reflexive, it is the dual of $L_c^*\mathbb{C}$. We shall write the evaluation of $\alpha \in L_c\mathbb{C}$ on $a \in L_c^*\mathbb{C}$ as $a(\alpha)$ rather than $\alpha(a)$ to avoid confusion with the notation $\alpha(\lambda)$ for the evaluation of α on $\lambda \in S^1$.

From the corollary to [Sch71, IV, §2.3], as the inclusion $L_c^*\mathbb{C} \to H$ is injective with weakly dense image, the map $H^* \to L_c^*\mathbb{C}$ is also injective with weakly dense image. Thus there is a sequence (α_n) in H^* which converges weakly to 1. That is, for all $a \in L_c^*\mathbb{C}$, $(a(\alpha_n))$ converges in $\mathbb{C}$ to $a(1)$. The space $L_c^*\mathbb{C}$ is reflexive, hence barrelled, and so the Banach-Steinhaus theorem applies. This states that (α_n) converges to 1 uniformly on each compact subset of $L_c^*\mathbb{C}$. We shall find a particularly convenient compact subset of $L_c^*\mathbb{C}$.

The norm on $L^{1,\infty}\mathbb{C}$ is $\|\gamma\|_{1,\infty} = \sup\{|\gamma(\lambda)|, |\gamma'(\lambda)|\}$. For $\lambda \in S^1$, there is an element e_λ of $L^{1,\infty}\mathbb{C}$ which evaluates a loop at time λ. If $\gamma \in L^{1,\infty}\mathbb{C}$ with $\|\gamma\|_{1,\infty} \leq 1$ then γ is Lipschitz with constant $K \leq 1$. Therefore $|e_\lambda(\gamma) - e_{\lambda'}(\gamma)|$ is less than or equal to the smaller angle between λ and λ'. Hence $\lambda \to e_\lambda$ is a continuous map from S^1 to $L^{1,\infty *}\mathbb{C}$. Composing this with the dual of the map $L_c\mathbb{C} \to L^{1,\infty}\mathbb{C}$ defines a continuous map $S^1 \to L_c^*\mathbb{C}$. Its image is thus compact and therefore $(\alpha_n) \to 1$ uniformly on $\{e_\lambda : \lambda \in S^1\}$.

Hence there is some N such that for $n \geq N$, $|e_\lambda(\alpha_n) - e_\lambda(1)| < 1$ for all $\lambda \in S^1$. Thus $|\alpha_N(\lambda) - 1| < 1$ so $\alpha_N(\lambda) \neq 0$ for all $\lambda \in S^1$. Hence H^* contains an element which is never zero. □

A.1 Inner Products on Distribution Space

In this section we investigate those inner products on $\mathcal{S}^*$ which, under the isomorphism $\mathcal{S}^* \cong L^*\mathbb{C}$, are invariant under the circle action and the involution of reversing loops, and such that multiplication by z is continuous in the inner product topology. In this investigation, we use $\mathcal{S}^*$ because it is a sequence space and so we have a good presentation of elements of $\mathcal{S}^*$ and of operators acting on it. We start by transferring the aforementioned operators from $L^*\mathbb{C}$ to $\mathcal{S}^*$.

Definition A.13 *Define the operators R_λ for $\lambda \in S^1$, ι, and z on $\mathcal{S}$ to be the operators corresponding under the Fourier isomorphism $\mathcal{S} \cong L\mathbb{C}$ to rotation by λ, reversal of the circle, and multiplication by z, respectively. We shall use the same notation for their adjoints which act on $\mathcal{S}^*$.*

The maps $\lambda \to R_\lambda \in \mathcal{L}(\mathcal{S})$ and $\lambda \to R_\lambda \in \mathcal{L}(\mathcal{S}^*)$ define an action of the circle on $\mathcal{S}$ and $\mathcal{S}^*$ respectively. We shall refer to ι as the *natural involution* on $\mathcal{S}$ and $\mathcal{S}^*$.

For $p \in \mathbb{Z}$, let $e^p \in \mathcal{S}$ and $e_p \in \mathcal{S}^*$ both denote the sequence with a 1 in the pth place and zero elsewhere. The sets $\{e^p\}$ and $\{e_p\}$ are topologically free bases for $\mathcal{S}$ and $\mathcal{S}^*$ respectively.

Lemma A.14 *In terms of the bases $\{e^p\}$ and $\{e_p\}$, the operators R_λ, ι, and z are given by the formulæ:*

$$
\begin{aligned}
R_\lambda e^p &= \lambda^p e^p & \iota e^p &= e^{-p} & z e^p &= e^{p+1} \\
R_\lambda e_p &= \lambda^{-p} e_p & \iota e_p &= e_{-p} & z e_p &= e_{p-1}
\end{aligned}
$$

Definition A.15 *Let $\mathcal{C}$ denote the cone of positive semi-definite, sesquilinear forms on $\mathcal{S}^*$ which are invariant under the action of the circle and under the action of the natural involution. Let $\mathcal{C}^+ \subseteq \mathcal{C}$ denote the sub-cone consisting of positive definite forms.*

Let $\mathcal{T}$ denote the cone of positive, rapidly decreasing sequences (a_p) such that $a_p = a_{-p}$ for all $p \in \mathbb{Z}$. Let $\mathcal{T}^+$ denote the sub-cone of strictly positive sequences.

Theorem A.16 *The map $(\cdot, \cdot) \to ((e_p, e_p))$ defines a bijection of cones from $\mathcal{C}$ to $\mathcal{T}$ such that $\mathcal{C}^+$ is carried onto $\mathcal{T}^+$.*

Proof. As the set $\{e_p : p \in \mathbb{Z}\}$ is a basis for $\mathcal{S}^*$, any sesquilinear form, $(\cdot,\cdot)$, on $\mathcal{S}^*$ is completely determined by the $\mathbb{Z}\times\mathbb{Z}$-indexed set of numbers $\{(e_p, e_q)\}$. We shall refer to this as the double sequence associated to $(\cdot,\cdot)$.

Suppose that $(\cdot,\cdot)$ is a sesquilinear form on $\mathcal{S}^*$ invariant under the action of R_λ for some $\lambda \in S^1$ not of finite order. Then for all $p, q \in \mathbb{Z}$, $(R_\lambda e_p, R_\lambda e_q) = (e_p, e_q)$. Using the formula from lemma A.14, the left-hand side of this equation is $\lambda^{q-p}(e_p, e_q)$. As λ is not of finite order, this implies that $(e_p, e_q) = 0$ for $p \neq q$. Thus the double sequence associated to $(\cdot,\cdot)$ is zero off the main diagonal.

Conversely, suppose that $(\cdot,\cdot)$ is a sesquilinear form on $\mathcal{S}^*$ such that the associated double sequence is zero off the main diagonal. For $a = (a^p) \in \mathcal{S}^*$, the number (a, a) is given by the formula $\sum |a^p|(e_p, e_p)$. Thus as $R_\lambda a = (\lambda^{-p} a^p)$, $(R_\lambda a, R_\lambda a) = (a, a)$ for any $\lambda \in S^1$. Hence $(\cdot,\cdot)$ is invariant under the circle action.

If, in addition, the natural involution acts unitarily – that is, the sesquilinear form is invariant under the action of the natural involution – then lemma A.14 shows that $(e_p, e_p) = (e_{-p}, e_{-p})$. The converse is immediate.

Let $(\cdot,\cdot)$ be a sesquilinear form which is invariant under the circle action and under the natural involution. Let $a_p = (e_p, e_p)$ for $p \in \mathbb{Z}$. The form $(\cdot,\cdot)$ is continuous and therefore defines a conjugate linear map $\mathcal{S}^* \to \mathcal{S}^{**} = \mathcal{S}$. Under this map, an element $b \in \mathcal{S}^*$ is taken to the sequence $((e_p, b))$. Let $\mathbb{1} \in \mathcal{S}^*$ denote the sequence consisting completely of 1s. Under the map $\mathcal{S}^* \to \mathcal{S}$ defined by the form, this element is taken to $((e_p, \mathbb{1})) = (a_p)$. Hence the sequence (a_p) is rapidly decreasing and thus the map in the statement of the theorem is well-defined.

Thus a sesquilinear form which is invariant under the circle action and under the natural involution is completely determined by the sequence $((e_p, e_p))$. It is simple to see that the sequence is positive if and only if the sesquilinear form is positive semi-definite, and that the sequence is strictly positive if and only if the sesquilinear form is positive definite. Thus the sequence is an element of $\mathcal{T}$ and is in $\mathcal{T}^+$ if and only if the original sesquilinear form were positive definite. Whence the map $\mathcal{C} \to \mathcal{T}$ is well-defined and injective. A simple check shows that this is a map of cones.

To show that the map is surjective, and hence a bijection, let $(a_p) \in \mathcal{T}$. Let $b = (b^p)$ and $c = (c^p)$ be elements of $\mathcal{S}^*$. There exist integers $m, n > 0$ such that $(p^{-m} b^p)$ and $(p^{-n} c^p)$ are bounded. As (a_p) is rapidly decreasing, the sequence $(p^{n+m+2} a_p)$ is bounded and hence $(p^{n+m} a_p)$ is summable. Hence $(b^p \overline{c^p} a_p)$ is a summable sequence and thus the formula:

$$(b, c) \to \sum_{p\in\mathbb{Z}} b^p \overline{c^p} a_p$$

is well-defined as a sesquilinear map $\mathcal{S}^* \times \mathcal{S}^* \to \mathbb{C}$. It is evidently positive semi-definite. To show continuity, it is sufficient to show that it is continuous when restricted to each space $\{(x^p) : (p^{-n} x^p) \text{ is bounded}\}$. Continuity of this restriction follows from the estimate:

$$\left| \sum_{p\in\mathbb{Z}} b^p \overline{c^p} a_p \right| \leq \sup\{|p^{-n} b^p|\} \sup\{|p^{-n} c^p|\} \sum_{p\in\mathbb{Z}} |p^{2n} a_p| .$$

Thus the sequence (a_p) defines a sesquilinear form on $\mathcal{S}^*$. It is clear that the associated double sequence for this form is zero off the main diagonal and on the main diagonal is (a_p). Thus it is invariant under the circle action and the natural involution and so is an element of $\mathcal{C}$. It is the preimage of (a_p) under the map $\mathcal{C} \to \mathcal{T}$ showing that the map is a bijection. $\square$

Any continuous inner product on $\mathcal{S}^*$ defines a Hilbert space completion, but the map from inner products to Hilbert space completions is not injective. Two inner products define the same Hilbert space completion if and only if the identity map on $\mathcal{S}^*$ extends to an isomorphism between the completions. This condition can be stated elegantly in terms of the sequences in $\mathcal{T}^+$ associated to the given inner products:

Lemma A.17 *Let $(a_p), (b_p) \in \mathcal{T}^+$. The Hilbert space completions defined by the inner products associated to (a_p) and (b_p) are equivalent if and only if the sequences (a_p/b_p) and (b_p/a_p) are bounded.*

We now turn to the operator z and determine the answer to the following question: for which inner products on $\mathcal{S}^*$ is the operator z continuous with respect to the inner product topology?

Proposition A.18 *Let $(a_p) \in \mathcal{T}^+$. Let $(\cdot, \cdot)$ be the associated inner product on $\mathcal{S}^*$. The operator z is continuous with respect to the inner product topology if and only if the sequence of ratios (a_p/a_{p+1}) is bounded.*

In this case, $\|z\|^2 = \sup\{a_p/a_{p+1}\}$.

Notice that as $a_p = a_{-p}$, the sequence (a_p/a_{p-1}) is just (a_p/a_{p+1}) in reverse order. Moreover, we cannot have z acting unitarily as this would imply that (a_p) is constant, contradicting the fact that it is rapidly decreasing.

Proof. Let $\|\cdot\|$ be the norm defined by the inner product. Suppose that z is continuous with respect to the inner product topology on $\mathcal{S}^*$. In particular, $\|ze_{p+1}\| \leq \|z\| \|e_{p+1}\|$ for all p. From lemma A.14, $ze_{p+1} = e_p$. Thus for $p \in \mathbb{Z}$, $\sqrt{a_p} \leq \|z\| \sqrt{a_{p+1}}$. Hence the sequence (a_p/a_{p+1}) is bounded above by $\|z\|^2$.

Conversely, suppose that (a_p/a_{p+1}) is bounded above by, say, M. Let $b = (b^p) \in \mathcal{S}^*$, then:

$$\|zb\|^2 = \sum \left|b^{p+1}\right| a_p \leq \sum \left|b^{p+1}\right| M a_{p+1} = M \|b\|^2 .$$

Thus z is continuous with respect to $\|\cdot\|$ and so extends to a continuous linear operator on H. Moreover, $\|z\|^2 \leq M$.

Combining the two relationships for $\|z\|$ shows that $\|z\|^2 = \sup\{a_p/a_{p+1} : p \in \mathbb{Z}\}$ when either side exists. $\square$

Corollary A.19 *Let $(a_p) \in \mathcal{T}$ be such that (a_p/a_{p+1}) is bounded. For each $q \in \mathbb{Z}$, the operator z^q is continuous with respect to the inner product topology and $\|z^q\|^2 = \sup\{a_p/a_{p+q}\}$.*

Corollary A.20 *Let $\mathcal{C}_z$ be the subset of $\mathcal{C}$ consisting of those inner products for which the operation of multiplication by z is continuous, $\mathcal{T}_z$ the corresponding sub-cone of $\mathcal{T}$. Then $\mathcal{C}_z$ is a non-empty sub-cone of $\mathcal{C}^+$.*

Proof. The set $\mathcal{T}_z$ consists of those sequences $(a_p) \in \mathcal{T}^+$ for which (a_p/a_{p+1}) is bounded. This is non-empty as the sequence $(2^{-|p|})$ lies in $\mathcal{T}_z$.

Clearly, if $(a_p) \in \mathcal{T}_z$ then for any $t > 0$, $(ta_p) \in \mathcal{T}_z$. If $(a_p), (b_p) \in \mathcal{T}_z$ then there exist $M, N > 0$ such that $a_p/a_{p+1} \leq M$ and $b_p/b_{p+1} \leq N$ for all

p. Equivalently, $a_p \leq Ma_{p+1}$ and $b_p \leq Nb_{p+1}$. Let $R = \max\{M, N\}$, then $a_p + b_p \leq R(a_{p+1} + b_{p+1})$ so $((a_p + b_p)/(a_{p+1} + b_{p+1}))$ is bounded, hence lies in $\mathcal{T}_z$.

Therefore $\mathcal{T}_z$ is a sub-cone of $\mathcal{T}^+$ and so $\mathcal{C}_z$ is a sub-cone of $\mathcal{C}^+$. □

These inner products transfer to $L^*\mathbb{C}$ via the isomorphism $L^*\mathbb{C} \to \mathcal{S}^*$. We can find a formula which is more natural on $L\mathbb{C}$.

Proposition A.21 *Let $(\cdot,\cdot) \in \mathcal{C}$. Let $(a_p) \in \mathcal{T}$ be the associated sequence. Thinking of $\mathcal{T}$ as a subset of $\mathcal{S}$, let $\gamma_a \in L\mathbb{C}$ be the image of (a_p) under the isomorphism $\mathcal{S} \cong L\mathbb{C}$.*

Under the isomorphism $\mathcal{S}^* \cong L^*\mathbb{C}$, the form $(\cdot,\cdot)$ is given by the formula $(b, c) \to b(\overline{c} \diamond \gamma_a)$ where $c \diamond \gamma_a \in L\mathbb{C}$ is the map $\lambda \to c(R_{\lambda^{-1}}\gamma_a)$.

Proof. The map $S^1 \times S^1 \to \mathbb{C}$ defined by $(\lambda, \mu) \to \gamma_a(\lambda^{-1}\mu)$ is the composition of smooth maps hence is smooth. Therefore by the exponential law for smooth maps, [KM97, I.3], its adjoint, $\lambda \to R_{\lambda^{-1}}\gamma_a$, is a smooth map $S^1 \to L\mathbb{C}$. The element $c \in L^*\mathbb{C}$ is a continuous linear map $L\mathbb{C} \to \mathbb{C}$, hence is smooth, so the map $\lambda \to c(R_{\lambda^{-1}}\gamma_a)$ is a smooth map $S^1 \to \mathbb{C}$. Thus the formula $(b, c) \to b(\overline{c} \diamond \gamma_a)$ makes sense. It is also evident that the map $c \to c \diamond \gamma_a$ is continuous and so $(b, c) \to b(\overline{c} \diamond \gamma_a)$ is at least separately continuous and thus completely determined by its effect on a basis.

Using the isomorphisms $L\mathbb{C} \cong \mathcal{S}$ and $L^*\mathbb{C} \cong \mathcal{S}^*$, we transfer these operators to $\mathcal{S}$ and $\mathcal{S}^*$. Under these isomorphisms, $R_{\lambda^{-1}}\gamma_a$ becomes the sequence $(\lambda^{-p}a_p)$ and so $e_q(R_{\lambda^{-1}}\gamma_a) = \lambda^{-q}a_q$. Thus $e_q \diamond \gamma_a$ is the sequence corresponding to the function $\lambda \to \lambda^{-q}a_q$ which is $a_q e^{-q}$. Therefore, $\Sigma e_p(e_q \diamond \gamma_a) = e_{-p}(a_q e^{-q}) = a_q \delta_p^q$.

Hence the sesquilinear form on $L^*\mathbb{C}$, $(b, c) \to b(\overline{c} \diamond \gamma_a)$, corresponds to the original sesquilinear form on $\mathcal{S}^*$, $((b^p), (c^p)) \to \sum b^p \overline{c^p} a_p$. □

The inner products we consider on $\mathcal{S}^*$ and $L^*\mathbb{C}$ arise as inner products on the underlying real spaces and therefore give a classification of inner products on $L^*\mathbb{R}$ which are invariant under the circle action and the natural involution, and also of those for which the operations of multiplication by $\cos\theta$ and by $\sin\theta$ are continuous.

Proposition A.22 *The sesquilinear forms on $\mathcal{S}^*$ and $L^*\mathbb{C}$ considered above are the complexifications of sesquilinear forms on the underlying real spaces of both $\mathcal{S}^*$ and $L^*\mathbb{C}$.*

Proof. For $\mathcal{S}^*$, this is evident from the formula. For $L^*\mathbb{C}$, it follows from the invariance under ι together with the fact that ι intertwines the complex conjugation operators arising from $\mathcal{S}^*$ and $L^*\mathbb{C}$ (note that the isomorphism $L^*\mathbb{C} \to \mathcal{S}^*$ does not induce an isomorphism of real structures and thus the complex conjugation operators differ). □

A.2 Polarisations

In this section we examine how the theory of polarisations, and thus of unitary structures, interacts with these inner products on the space of distributions. We examine an inner product on $L^*\mathbb{C}^n$ determined by a sequence in $\mathcal{T}_z$. To pass from an inner product on $L^*\mathbb{C}$ to one on $L^*\mathbb{C}^n$, we use the isomorphism $L^*\mathbb{C}^n \cong L^*\mathbb{C} \otimes \mathbb{C}^n$ together with the the standard inner product on $\mathbb{C}^n$.

Lemma A.23 *Let J be the operator on $\mathcal{S}^*$ defined by $Je_p = -(-1)^{sign(p)} ie_p$. Let $(\cdot, \cdot) \in \mathcal{C}^+$ be an inner product on $\mathcal{S}^*$ and let H be the corresponding Hilbert space completion. Then the operator J defines a polarisation of H.*

This extends in a natural way to a polarisation of the Hilbert completion of $L^*\mathbb{C}^n$.

Proof. Let $(a_p) \in \mathcal{T}^+$ be the sequence corresponding to the inner product. For $b = (b^p) \in \mathcal{S}^*$, it follows straight from the formula for J that $(Jb, Jb) = (b, b)$ and therefore J extends to a unitary operator on H. It satisfies $J^2 = -1$ and $J \pm iI$ are not finite rank. Therefore, it defines a polarisation on H.

To extend this to the Hilbert completion of $L^*\mathbb{C}^n$, we observe that this completion is naturally isomorphic to $H \otimes \mathbb{C}^n$. The polarising operator J on H defines one on $H \otimes \mathbb{C}^n$ by taking $J \otimes I_n$. □

Definition A.24 *The polarisation $\mathcal{J}$ so defined on the completion of $L^*\mathbb{C}^n$ is called the* standard polarisation.

Proposition A.25 *Let $(a_p) \in \mathcal{T}_z$. Let H be the associated Hilbert space completion of $L^*\mathbb{C}^n$. The polynomial loop group $L_{pol}U_n$ acts continuously on H and preserves the polarisation.*

Proof. Let J be the polarising operator on H as defined in lemma A.23. Let $\mathcal{L}_{\mathcal{J}}(H)$ be the set of all bounded linear operators A on H such that $[A, J]$ is Hilbert-Schmidt. It is clear that $\mathrm{Gl}_{\mathcal{J}}(H) = \mathrm{Gl}(H) \cap \mathcal{L}_{\mathcal{J}}(H)$. The norm of an element $A \in \mathcal{L}_{\mathcal{J}}(H)$ is the sum of the operator norm of of A and the Hilbert-Schmidt norm of $[A, J]$.

There is an isometry $M_n(\mathbb{C}) \to \mathcal{L}(H)$ given by $A(a \otimes v) = a \otimes Av$, thinking of H as the completion of $L^*\mathbb{C} \otimes \mathbb{C}^n$. This maps continuously into $\mathcal{L}_{\mathcal{J}}(H)$ since $A \in M_n(\mathbb{C})$ commutes with J.

The operator z acts continuously on H and $[J, z]$ is finite rank. It therefore lies in $\mathcal{L}_{\mathcal{J}}(H)$. Thus the image of $L_{\mathrm{pol}}M_n(\mathbb{C})$ in $\mathcal{L}(H)$ lies in $\mathcal{L}_{\mathcal{J}}(H)$. Thus the image of $L_{\mathrm{pol}}U_n$ lies in $\mathrm{Gl}_{\mathcal{J}}(H)$. □

Proposition A.26 *The inclusion $L_{pol}U_n \to \mathrm{Gl}_{\mathcal{J}}(H)$ is homotopic to the standard inclusion which factors through LU_n.*

Proof. Let $T : H \to L^{2,*}\mathbb{C}^n$ be the isometry which takes e_p to $\sqrt{a_p}e_p$. This identifies $\mathrm{Gl}_{\mathcal{J}}(H)$ with $\mathrm{Gl}_{\mathcal{J}}(L^{2,*}\mathbb{C}^n)$ and so defines the map $L_{\mathrm{pol}}U_n \to \mathrm{Gl}_{\mathcal{J}}(L^{2,*}\mathbb{C}^n)$.

Let $\zeta_t : L^{2,*}\mathbb{C}^n \to L^{2,*}\mathbb{C}^n$ be the map defined by $\zeta_t(e_p) = (a_{p-1}/a_p)^{t/2} e_{p-1}$. As a_p is positive for all p, the formula makes sense. Since (a_p) lies in $\mathcal{T}_z$, (a_{p-1}/a_p) is bounded above and below so ζ_t is an isomorphism of Hilbert spaces.

The map ζ_0 is the (adjoint of the) map z. The map ζ_1 is the map TzT^{-1}. Therefore the two inclusions of $L_{\mathrm{pol}}U_n$ in $\mathrm{Gl}_{\mathcal{J}}(H)$ are $\sum z^q A_q \to \sum \zeta_0^q A_q$ and $\sum z^q A_q \to \sum \zeta_1^q A_q$. The required homotopy is $F(\sum z^q A_q, t) = \sum \zeta_t^q A_q$. □

This homotopy equivalence is closely related to the deformation retract $\mathrm{Gl}(H) \to U(H)$. This retract uses the polar decomposition: any invertible linear operator on H can be written in the form $A = Q\,|A|$ where $|A|$ is a self-adjoint operator with strictly positive eigenvalues and Q is a unitary operator. The retraction maps A to Q and the homotopy equivalence is given by contracting the eigenvalues of $|A|$ to 1.

This retract and homotopy equivalence restricts to give retracts and homotopy equivalences of the various standard subgroups of $\mathrm{Gl}(H)$ onto their unitary counterparts. Writing $H_{\mathbb{R}}$ for an underlying real Hilbert space of H, the map $\mathrm{Gl}(H) \to U(H)$ also maps:

$$\begin{aligned}
\mathrm{Gl}(H_{\mathbb{R}}) &\to O(H_{\mathbb{R}}), \\
\mathrm{Gl}_{\mathcal{J}}(H) &\to U_{\mathcal{J}}(H), \\
\mathrm{Gl}_{J}(H_{\mathbb{R}}) &\to O_{J}(H_{\mathbb{R}}), \\
\mathrm{Gl}_n(\mathbb{C}) &\to U_n, \\
\mathrm{Gl}_n(\mathbb{R}) &\to O_n.
\end{aligned}$$

The relation with the homotopy equivalence above comes about because in the polar decomposition of ζ_1, the unitary operator is z. The homotopy ζ_t is the homotopy which contracts the positive part to the identity.

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