Elliptic Cohomology for Beginners: Introducing the $\widehat{A}$-genus.

Andrew Stacey

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1 Introduction

Before I begin with the actual seminar, I should like to say something about the level I will be talking at. The overall aim of this seminar is to help people to understand John Jones's series of seminars on Elliptic Cohomology. The aim is not to explain everything so well that he has nothing left to talk about, but to provide an introduction to possibly unfamiliar terms so that when John comes to explaining the theory, you have at least met some of the words before. Because of this, some people may find parts of this seminar unbelievably basic.

Most of my definitions will be "rough and ready", my aim is for them to be just sufficient to allow understanding without swamping you in details. In fact, I've gone out of my way to avoid any theory that I think will be covered in John's talks.

Right, the plan is to take you through an example of Elliptic Cohomology in action. The example chosen is called the $\widehat{A}$-genus. It is actually a degenerate case and so is not strictly an elliptic genus. However, it is still a good introductory example.

2 The Diagram

When I arrived at Warwick last summer, I met John before term started to see if there was anything I could be getting on with and he gave me a copy of his lecture notes on Elliptic Cohomology to go away and understand. Back then, I thought I knew what an ellipse was but hadn't even met cohomology. One of the things that made it slightly harder to understand was a lack of diagrams, specifically the diagram John drew up last week. In the end, I drew a diagram myself. Figure 1 is an amalgamation of his and my diagrams. As you can see, mine is a little more detailed. I've put in some of what's going on between the links.

What I plan to do is to follow the $\widehat{A}$-genus around this diagram and see what happens.

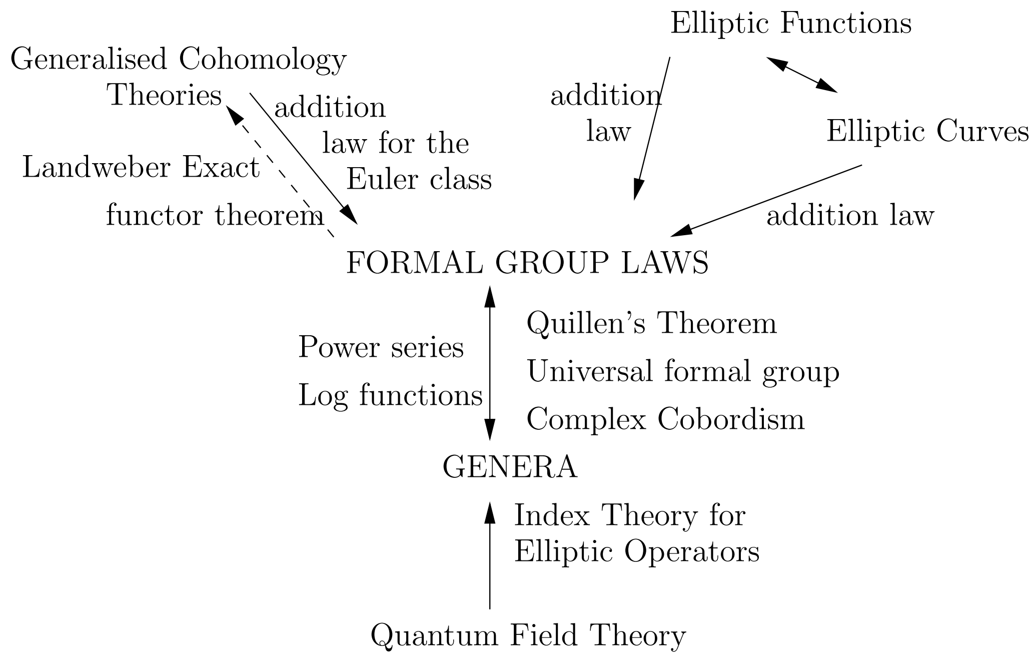

Figure 1: The links between the topics

3 The Formal Group Law of Sinh

I think I ought to start at the centre of my diagram and introduce Formal Group Laws. I should also say a word about formal power series. There is a slight abuse of notation here. Take the function $\frac{1}{1-x}$ of real or complex numbers. This has a power series expansion about 0 which is $1 + x + x^2 + x^3 + \ldots$. This is valid for $|x| < 1$. There is also the Formal Power Series $1 + x + x^2 + x^3 + \ldots$ which exists in the ring of formal power series over $\mathbb{Z}$ (well, any ring with a 1 actually). However, in order to make the notation easier to write and say, we talk about $\frac{1}{1-x}$ in both cases. Then a formal group law is a formal power series in two variables over some ring satisfying certain conditions and, in some cases, this can also be written in terms of elementary functions so we do so. All the operations we will be doing are the same whether we do them to the functions or to the power series so we usually don't bother noting the difference.

The reason why we use *formal* power series is because we don't worry about convergence at this stage. That is because whenever we do finally substitute something in place of the indeterminate, it will have finite order, that is there is some power x^n which is zero so all higher terms are zero. One of the places for further work is looking at what happens when we can no longer assume this fact.

We're going to see the following formula quite a lot, so I thought it best to go through the actual calculation at the beginning so that I can quote it later.

Consider the function $f(z) = 2\sinh(\frac{1}{2}z)$. This belongs to the class of meromorphic functions with a single zero at zero. Thus near zero it is invertible. Equivalently, its Taylor series at zero has leading term z and so the formal power series so defined has an inverse with respect to composition.

Now for indeterminates u, v (when we want to use complex variables then we assume u, v to be sufficiently close to zero so that f^{-1} is well-defined) then let x, y be such that $u = f(x)$ and $v = f(y)$. Let the function or power series F be defined by:

$$F(u, v) = f(x + y)$$

Then we have:

$$\begin{aligned} F(u,v) &= 2\sinh(\frac{1}{2}(x+y)) \\ &= 2\sinh(\frac{1}{2}x)\cosh(\frac{1}{2}y) + 2\sinh(\frac{1}{2}y)\cosh(\frac{1}{2}x) \\ &= 2\sinh(\frac{1}{2}x)\sqrt{1+\frac{1}{4}(2\sinh(\frac{1}{2}y))^2} \\ &\quad +2\sinh(\frac{1}{2}y)\sqrt{1+\frac{1}{4}(2\sinh(\frac{1}{2}x))^2} \\ &= f(x)\sqrt{1+\frac{1}{4}f(y)^2} + f(y)\sqrt{1+\frac{1}{4}f(x)^2} \\ &= u\sqrt{1+\frac{1}{4}v^2} + v\sqrt{1+\frac{1}{4}u^2} \end{aligned}$$

The Taylor expansions of these functions are:

$$\begin{aligned} f(x) &= x + \frac{1}{24}x^3 + \frac{1}{1920}x^5 + \frac{1}{322560}x^7 + \frac{1}{92897280}x^9 + \mathrm{O}(x^{10}) \\ f^{-1}(x) &= x - \frac{1}{24}x^3 + \frac{3}{640}x^5 - \frac{5}{7168}x^7 + \frac{35}{294912}x^9 + \mathrm{O}(x^{10}) \\ F(u,v) &= u + v + \frac{1}{8}u^2v + \frac{1}{8}v^2u - \frac{1}{128}u^4v - \frac{1}{128}v^4u + \frac{1}{1024}u^6v + \frac{1}{1024}v^6u \\ &\quad -\frac{5}{32768}v^8u - \frac{5}{32768}v^8u + \mathrm{O}(u^{10}, v^{10}) \end{aligned}$$

Notice that the denominators in the expansion of F are all powers of 2. So the formal power series F is defined over $\mathbb{Z}[\frac{1}{2}]$, or any ring containing an image of $\mathbb{Z}[\frac{1}{2}]$.

4 Spin Manifolds

As this is a geometry seminar, the best place to start is naturally with the Quantum Field Theory. Fortunately, all I know about that is that it is a fairly good attempt to link Quantum Theory with Relativity and that Physicists keep making predictions or finding interesting objects from it that then send the Mathematicians into a flurry of activity trying to base them in rigorous mathematics.

One such interesting object is the concept of a spin manifold. I believe that this arose out of Dirac trying to quantise the electron. The thing about an electron is that it is a fermion and so has spin one half. A rough and ready definition of that is that when you think you've rotated it through 360 degrees, you haven't gotten back where you started and you need to rotate a further 360 degrees to recover your starting point.

This lead to Dirac trying to square root the Laplace operator. The Dirac operator is such a square root. Problems in Quantum Field Theory often lead to problems in Geometry so it was then natural to ask if we could define this operator on a compact smooth manifold.

Now there are two views of a vector bundle useful to us here. One is that we have a vector space assigned to each point on the manifold in such a way that as we move smoothly over the manifold then the vector space twists smoothly. Another view is that locally a vector bundle is trivial so the important parts are the transition functions. These must lie in the automorphism group of the vector space, but we have special cases when they lie in particular subgroups.

Thus an orientable vector bundle is one for which we can choose a frame which varies smoothly over the manifold and is always a frame, or an orientable vector bundle is one whose transition functions lie in the special orthogonal group.

We look at the construction of a spin bundle on a manifold from both perspectives.

There is an algebra associated with $\mathbb{R}^n$ called the Clifford Algebra. This is constructed as follows:

Take a basis $\{e_i\}$ for $\mathbb{R}^n$. Let Cl_n be the algebra generated by the e_i subject to the relations that the square of any e_i is 1 and that the e_i anti-commute, so $e_i e_j = -e_j e_i$. There is an obvious basis consisting of the products of the e_i. There is also a clear injective linear map from $\mathbb{R}^n$ to Cl_n.

There are various useful bits of structure we can add to Cl_n.

First, we can introduce a $\mathbb{Z}_2$-grading by looking at even and odd products of the e_i. Call these Cl_n^0 and Cl_n^1.

$$\begin{aligned} Cl_n^0 &= \langle e_{i_1} \dots e_{i_{2r}} : i_1 < \dots < i_{2r},\, 0 \leq 2r \leq n \rangle \\ Cl_n^1 &= \big\langle e_{i_1} \dots e_{i_{2r+1}} : i_1 < \dots < i_{2r},\, 0 \leq 2r+1 \leq n \big\rangle \end{aligned}$$

Then we can define an involution on the basis by reversing the order of multiplication, to get the right result we also multiply by $(-1)^r$ where r is the number of e_i in the basis element.

$$\epsilon(e_{i_1} \dots e_{i_r}) = (-1)^r e_{i_r} \dots e_{i_1}$$

Using the injection from $\mathbb{R}^n$ into Cl_n we can look at those elements of the multiplicative group of units of Cl_n which fix $\mathbb{R}^n$, acting by conjugation.

$$x(v) = xvx^{-1}$$

Then $\mathrm{Spin}(n)$ is defined to be the subgroup of the group of units of Cl_n^0 which fix $\mathbb{R}^n$ as above and for each element, its involute is its inverse. So for $x \in \mathrm{Spin}(n), v \in \mathbb{R}^n$, then:

$$x \in Cl_n^0$$
$$x(v) \in \mathbb{R}^n$$
$$x\epsilon(x) = 1$$

As it acts on $\mathbb{R}^n$ then there is a homomorphism from $\mathrm{Spin}(n)$ into $Gl(n)$. It turns out that the image is $SO(n)$ (given that the original basis is orthogonal with respect to the metric) and the kernel is just $\{1, -1\}$. Thus $\mathrm{Spin}(n)$ is a double cover of $SO(n)$. That is, the following sequence is exact:

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 0$$

Let Cl_n^c denote the complexified Clifford algebra of $\mathbb{R}^n$. Then, after some basic algebra, we have the following algebra isomorphisms:

$$Cl_{2n}^c \cong M_{2^n}(\mathbb{C})$$

$$Cl_{2n+1}^c \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$$

Thus we have a representation of algebras. As $\mathrm{Spin}(n)$ is a subgroup of Cl_n^c then these representations give group representations for the spin groups. In the case $2n$ then we have one algebra-irreducible representation, which splits into two irreducible representations for the spin group. In the case $2n+1$ we have two algebra-irreducible representations but as representations of the spin group they are isomorphic and irreducible. We call the spin representation S and if it splits, it splits into S^+ and S^-. Note that S is a complex vector space of *complex* dimension 2^n in both cases Also note that S can be thought of as a subspace of Cl_n^c since $\mathbb{C}^{2^n}$ can be mapped into $M_{2^n}(\mathbb{C})$ as the first column, so we have a concrete construction of S here.

Now we turn to the first view of the spin bundle. What we do is apply the above construction fibre-wise to the cotangent bundle. There is a condition which needs to be satisfied to ensure that this is possible, but that's not important here.

We then end up with three bundles, the cotangent bundle (identified with the tangent bundle via a Riemannian metric), the Clifford bundle and the spin bundle. Sections of the spin bundle are called *spinors* and of the sub-bundles, positive and negative spinors.

The spin and cotangent bundles can be viewed as sub-bundles of the Clifford bundle so we can multiply elements of them together. The result stays in the spin bundle and so we can define Clifford multiplication:

$$T^*X \otimes S_X \to S_X$$

Since an element of T^*X lies in the 1-graded part of the Clifford bundle, in even dimensions this multiplication swaps the two spin sub-bundles:

$$T^*X \otimes S_X^+ \to S_X^-, \quad T^*X \otimes S_X^- \to S_X^+$$

This can of course be extended to sections of the relevant bundles.

Now we need our other view of vector bundles.

In the above, we had a homomorphism from $\mathrm{Spin}(n)$ to $SO(n)$ with kernel $\{+1, -1\}$. Our cotangent bundle has transition functions in $SO(n)$ (we are dealing with orientable bundles here) so we ask, when can we lift them to transition functions in $\mathrm{Spin}(n)$?

Recall that transition functions have to satisfy the cocycle condition:

$$g_{ij} g_{jk} g_{ki} = 1$$

So our lifts of the transition functions must also satisfy this condition. When this is possible, we can attach a spin bundle by specifying that its transition functions are the lifts of the transition functions of the cotangent bundle.

Thus we also get a spin bundle and, in even dimensions, two sub-bundles.

With this construction, it is easier to see that objects defined on the cotangent bundle can be lifted to the spin bundle. In particular, if we have a connection on the cotangent bundle, we can lift it to the spin bundle.

Of course, I'm ignoring a lot of work here as it is not immediately obvious that such lifts do work globally. However, in this case, they do.

So now we have a connection on spinors arising from the connection on the (co)tangent bundle:

$$\nabla : C^\infty(X; S_X) \to C^\infty(X; T^*X \otimes S_X)$$

and a multiplication:

$$C : C^\infty(X; T^*X \otimes S_X) \to C^\infty(X; S_X)$$

composing these we get a differential operator:

$$D : C^\infty(X; S_X) \to C^\infty(X; S_X)$$

In even dimensions we get two operators D^+, D^- which start in the relevant space of positive or negative spinors and end up in the other one.

Locally, we have the formula:

$$D(\sigma) = e_i \cdot \nabla_i \sigma$$

Where the product is Clifford multiplication. It so happens that the square of this operator is the Laplacian on spinors.

4.1 The Dirac Operator

This operator is called the Dirac operator, though the name is also used for its restrictions to positive or negative spinors.

It is an *Elliptic Operator*. This is occurrence one of the word "Elliptic". So far as I'm aware, it is not the one from which Elliptic Cohomology got its name.

In this case, what elliptic means is that the operator has finite dimensional kernel and cokernel (range over image). Thus there is a well-defined local constant, the index of D, being the difference of their dimensions.

In odd dimension, then the Dirac operator is self-adjoint so its index is zero. In even dimensions, we work with the operator D^+. Its adjoint is D^- so we have that the index of D^+ is the dimension of its kernel minus the dimension of the kernel of D^-.

4.2 The Atiyah-Singer Index Theorem

I now want to introduce you to Messers Atiyah and Singer. They started looking at elliptic operators and their indices. The formula they arrived at related the index of the operator to characteristic cohomology classes.

As characteristic cohomology classes are quite an important part of Elliptic Cohomology, I hope you won't mind me giving a brief beginners guide to them now.

Recall that for a manifold, there are many functors used to study it. Two important ones are the graded rings, $K(X)$ and $H^*(X)$, being the K-theory and the cohomology rings.

K-theory is defined for a particular field and can be thought of as follows. Over any manifold we have a well-defined notion of a vector bundle over the field. We can add and multiply any two by fibre-wise taking the direct sum or tensor product. To get from this to the K-theory, we introduce a notion of equivalence sufficient to allow us to formally define additive inverses. Then we add in the inverses.

A cohomology ring over a ring of coefficients is a graded ring satisfying the Eilenberg-Steenrod axioms of cohomology. These include such things as behaviour under maps of the base manifold, invariance under homotopy and the most important two (for our purposes) which are the Exact sequence for a relative pair and the dimension axiom, which says that the cohomology of a point is concentrated in the grade zero part.

Given these, it is possible to show that for a certain class of objects, of which compact manifolds are part, then the actual method of defining cohomology is irrelevant. So if De Rham is your hero, think of differential forms, otherwise singular or cellular is equally fine. In one sense, Elliptic Cohomology is what happens when we relax the dimension axiom.

Similarly, homology can be axiomatised (the difference being that the arrows go the other way) and there is a natural pairing between cohomology and homology theories which allows us to evaluate a cohomology element on a homology element. This is often written like this:

$$\langle \alpha, a \rangle \quad \text{for } \alpha \in H^*, a \in H_*$$

For orientable manifolds, the orientation gives rise to an element in homology, often written as $[X]$ for a manifold X.

A characteristic class is just a map from the K-theory for a given field to the cohomology ring for a given coefficient ring.

Important ones are the Chern classes for field $\mathbb{C}$ and ring $\mathbb{Z}$ and the Steifel-Whitney classes for field $\mathbb{R}$ and ring $\mathbb{Z}_2$.

These maps are not homomorphisms. It is in fact how they behave on sums and products which is so important in this theory. The most common are exponential classes which convert sums into products. The following may show why this is so important:

If M, N are two manifolds and $X = M \times N$ is their direct product, then $TX \cong TM \oplus TN$ so if we have an exponential class e and we define $e(Y) = e(TY)$ then $e(M \times N) = e(M)e(N)$.

There is one further piece of information that you need about such things. There is a principle called the splitting principle which says that even though a typical vector bundle is not a sum of line bundles, for the purposes of characteristic classes, we may behave as if it were. Thus if we know how a class behaves with respect to addition and multiplication of line bundles, we know how it behaves full stop.

What Atiyah and Singer and others did was to find a way of writing the index of an elliptic operator in terms of such classes evaluated on the fundamental class of some manifold.

For the Dirac operator, what we get is:

$$\mathrm{Index} D^+ = (-1)^k \left\langle \frac{\mathrm{ch}(S_X^+ - S_X^-)}{\mathrm{e}(TX)} \mathrm{Td}(TX \otimes \mathbb{C}), [X] \right\rangle$$

Where $\mathrm{ch}, \mathrm{Td}, \mathrm{e}$ are all particular characteristic classes. For those interested, they are the Chern character, the Todd class and the Euler class. k is half the dimension of X.

After much simplification, if we imagine that by the splitting principle then TX could be written as a sum of complex line bundles (remember, we're in even dimensions here), then this reduces to the formula:

$$\mathrm{Index} D^+ = (-1)^k \left\langle \prod_{i=1}^{k} \frac{x_i}{2\sinh(\frac{1}{2}x_i)}, [X] \right\rangle$$

Where x_i is the first chern class of the i^{th} line bundle in the split of TX.

Now we have something new. The characteristic class $\prod_{i=1}^{k} \frac{x_i}{2\sinh(\frac{1}{2}x_i)}$ is defined independently of whether X is a spin manifold or not. It is clear from the definition that it is an exponential characteristic class. We define

$$\widehat{A}(E) = \prod_{i=1}^{k} \frac{x_i}{2\sinh(\frac{1}{2}x_i)}(E)$$

Where E is any vector bundle over X which can be given a type of complex structure called a complex orientation. Exactly what this complex structure is will become clear, I hope, in John's seminars but it is definable even when the dimension of the bundle is odd.

Now recall that we say a manifold is orientable if its tangent bundle is orientable and in the same vein, we define the genus $\widehat{A}$ of an orientable manifold to be the pairing of $\widehat{A}$ of its tangent bundle with the fundamental orientation class. Thus we can define $\widehat{A}(X)$ for all manifolds whose tangent bundle can be given a complex orientation. Note that this definition does not depend on the manifold being a spin manifold, but when it is a spin manifold we have the result:

$$\widehat{A}(X) = \text{index } D_X^+$$

There may be a factor of $(-1)^k$ for some suitably defined k occuring in some of the above formula, but that isn't important at this stage.

Our earlier example should have convinced you that this respects products of manifolds, it won't surprise you, I'm sure, to learn that it also is additive on disjoint unions and zero on boundaries. Thus we have the $\widehat{A}$-genus.

5 Cohomology

Now we look at the cohomology theory given by this genus. This is a generalised cohomology theory so satisfies all the axioms with the possible exception of the dimension axiom. Although we often look for the smallest ring of coefficients (with a particular characteristic), it won't harm matters by taking $\mathbb{Q}$ here. We grade $\mathbb{Q}$ by giving everything grade 0, thus if $A^*(X)$ is our cohomology theory then $A^*(pt) = \mathbb{Q}$. So this does satisfy the dimension axiom and thus we have $A^*(X) = H^*(X, \mathbb{Q})$, standard cohomology with coefficients in $\mathbb{Q}$.

As there is an injection $\mathbb{Z} \to \mathbb{Q}$ then there is similarly a map from cohomology with coefficients in $\mathbb{Z}$ to cohomology with coefficients in $\mathbb{Q}$, thus the chern classes are well defined.

For a general genus which can be defined by the formula:

$$\Phi(X) = \left\langle \prod_{i=1}^{n} \frac{x_i}{f(x_i)}(TX), [X] \right\rangle$$

then the Euler class in the corresponding cohomology is given by $e(L) = f(c_1(L))$ on a line bundle L.

So our Euler class is:

$$e_{\widehat{A}}(L) = 2\sinh(\frac{1}{2}c_1(L))$$

Now for two line bundles L_1, L_2 the formula for $c_1(L_1 \otimes L_2)$ is simply the sum $c_1(L_1) + c_1(L_2)$. So the formula for $e_{\widehat{A}}(L_1 \otimes L_2)$ is given as follows:

$$\begin{aligned}
e_{\widehat{A}}(L_1 \otimes L_2) &= 2\sinh(\frac{1}{2}c_1(L_1 \otimes L_2)) \\
&= 2\sinh(\frac{1}{2}(c_1(L_1) + c_1(L_2))) \\
&= e_{\widehat{A}}(L_1)\sqrt{1 + \frac{1}{4}e_{\widehat{A}}(L_2)^2} + e_{\widehat{A}}(L_2)\sqrt{1 + \frac{1}{4}e_{\widehat{A}}(L_1)^2}
\end{aligned}$$

And this is our favourite formal group law.

6 Elliptic Functions

Now we turn to another point on our diagram and look at elliptic functions.

An elliptic function is simply a bi-periodic meromorphic function on the complex plane. Equivalently, it is a holomorphic map from the torus to the sphere. I'm sure you've all seen this representation of the torus as a quotient of the complex plane, it shouldn't be too hard to see the equivalence of the above definitions from figure 2.

The function I want to look at is $f(z) = 2\sinh(\frac{1}{2}z)$. This is singly periodic, but don't let that worry you. If, in the definition of the torus, we let one point go off to infinity then the limiting object is an infinite cylinder and singly periodic meromorphic functions can be thought of as holomorphic maps from this cylinder to the sphere.

This cylinder is a manifold and a quotient group of $\mathbb{C}$. Thus it has an identity, and around that identity we can find charts that map the identity to 0 and are local holomorphic bijections. This is a coordinate function.

Given any meromorphic periodic function with a simple zero at zero, we can make it into a coordinate function by composing it with the standard chart map given by the quotient map from $\mathbb{C}$.

This is where it matters what coordinate we choose.

This coordinate is probably not a local group homomorphism. However, we can view it as a kind of deformation of the group structure. In our example, I'm sure you can guess what this deformation is:

$$f(x+y) = f(x)\sqrt{1 + \frac{1}{4}f(y)^2} + f(y)\sqrt{1 + \frac{1}{4}f(x)^2}$$

Putting $u = f(x), v = f(y), F(u,v) = f(x+y)$ gives:

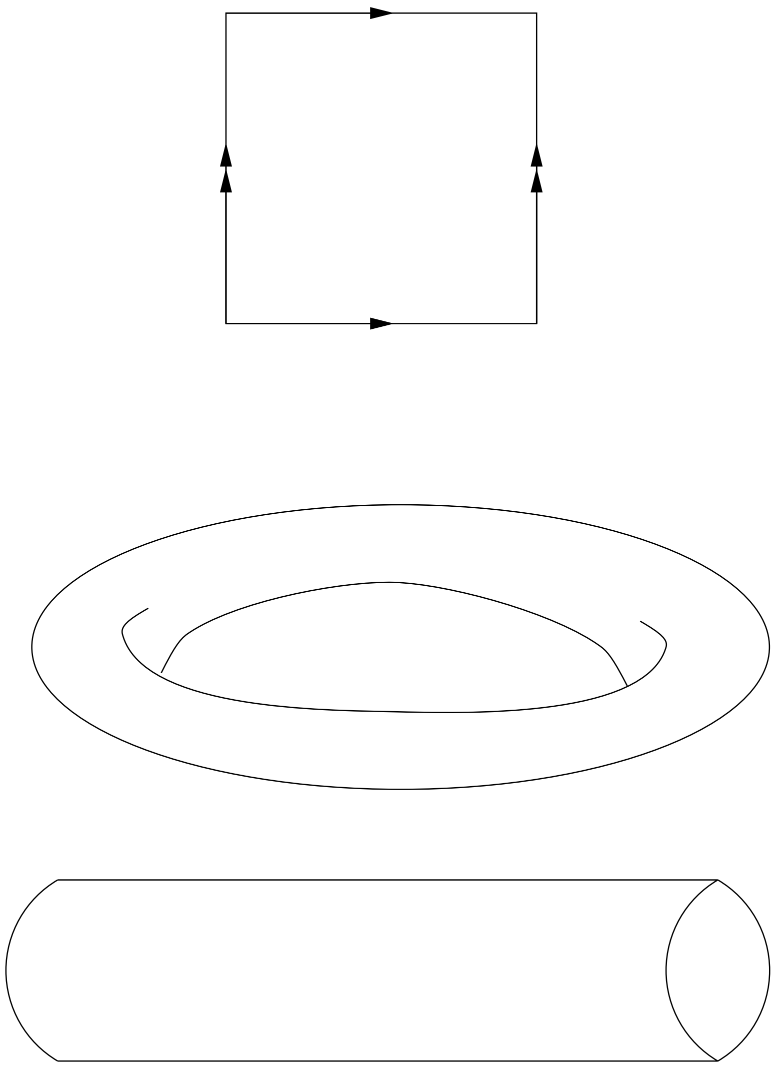

Figure 2: Two pictures of a torus and a cylinder

$$F(u,v) = u\sqrt{1+\frac{1}{4}v^2} + v\sqrt{1+\frac{1}{4}u^2}$$

I hope you're getting used to this formal group law by now.

7 Elliptic Curves

Finally we turn to elliptic curves. An elliptic curve is a locus of points in $\mathbb{R}^2$ or $\mathbb{C}^2$ specified by an equation of degree three. Sometimes we embed this curve in the relevant projective space by making it homogeneous in three variables. This is often useful as it adds a "point at infinity" to the curve.

Now several classes of curves can be turned into commutative groups. The actual formula often depends on a choice of origin.

Figure 3 illustrates the following addition laws.

A straight line has an obvious group structure. Take $y = mx + c$ with origin $(0, c)$, then for $(x_1, y_1), (x_2, y_2)$ lying on the line, their sum is the point $(x_1 + x_2, y_1 + y_2 - c)$.

The addition law for a conic is also easily explained. Take a conic section and a choice of origin, say O. Then for points A, B lying on the conic, take the straight line joining them (or tangent if the points are the same) and draw the parallel line through O. This intersects the conic at a third point, C, and define the sum $A + B$ to be C.

The addition law for an elliptic curve is also easy to understand. It is characterised by the statement that three points add up to zero if and only if they are collinear, with the proviso that if a line touches the curve tangentially then that point is counted twice. We then chose our origin at an inflection point to ensure that it behaves correctly. Then given two points, we take the line joining them and find its third point of intersection with the curve. Then this point is the inverse of the sum so we find the third point on the line through this point and the origin.

These addition laws give rise to a formula of the form:

$$(x_3, y_3) = (F_x(x_1, y_1, x_2, y_3), F_y(x_1, y_1, x_2, y_3))$$

But as (x_1, y_1) is a point on the curve, we can write y_1 in terms of x_1, hopefully, and similarly for y_2 so we get:

$$x_3 = F(x_1, x_2)$$

And similar for y_3 in terms of y_1 and y_2.

Looks familiar, doesn't it?

Consider the curve given by:

$$y^2 = 1 + \frac{1}{4}x^2$$

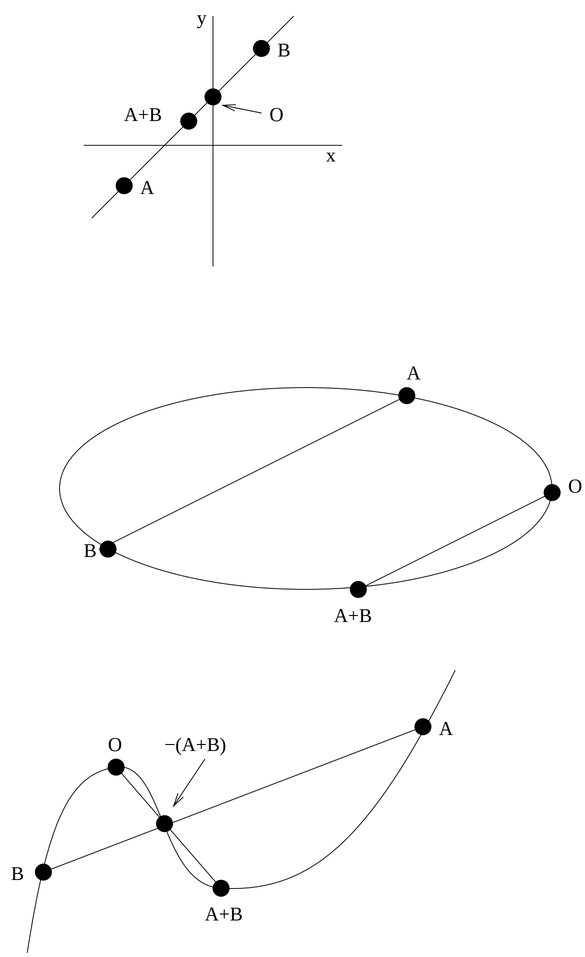

Figure 3: Addition laws on curves

All along we've been dealing with degenerate cases so it shouldn't surprise you to learn that instead of an elliptic curve we here get a conic section.

Suppose $(x_1, y_1), (x_2, y_2)$ are two points on the curve. Then the line joining them has tangent:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

If we take our origin to be $(0, 1)$ then this leads to the addition formula:

$$(x_3, y_3) = (x_1 y_2 + x_2 y_1, \frac{1}{4} x_1 x_2 + y_1 y_2)$$

Then substituting for y_1, y_2 we get that

$$x_3 = x_1 \sqrt{1 + \frac{1}{4} x_2^2} + x_2 \sqrt{1 + \frac{1}{4} x_1^2}$$

Which should come as no surprise to anyone.

Finally, just note that the map $z \to (f(z), f'(z))$ maps $\mathbb{C}$ onto this curve with the point 0 going to our chosen origin $(0, 1)$. What we've seen is that this is a group isomorphism.

8 Conclusion

We have seen several areas of mathematics where formal group laws come into play. In our example, everything was chosen so that the same formal group law came up over and over again. What Elliptic Cohomology says is that this is no coincidence.

However, there's still a lot to be done. There are still several links missing in the diagram, and some of the links that are there are unsatisfactory. As we will see later, the theorems that give the links in some cases seem to ignore the geometrical aspects of the theory altogether and just use algebraic computations.

I hope that this example has been illustrative of the theory. I realise that I've given little or no details, but that was deliberate. What algebra there was is fairly easy to do, especially if you know what the answer should be!