Review Sheet for Math53 : Course Summary

October 18, 2006

1 Linear Algebra

Basic concepts: vector space, subspace, linear transformations, dimension.

Possible new topics: infinite dimensional vector spaces.

Fundamental idea: a vector space is a set of objects (called "vectors") which can be added together and can be scaled by either real or complex numbers. They are the abstraction of the additive structure of $\mathbb{R}^n$.

Examples:

1. $\mathbb{R}^n$ for any n, $\mathbb{C}^n$ for any n,
2. the space of $n \times m$ matrices with componentwise addition and scalar multiplication,
3. the space $L(E, F)$ of linear transformations from E to F,
4. $C(X, E)$ the space of (continuous) functions from a (metric) space X to a vector space E,
5. l^1, the space of absolutely convergent series.

Essentials for this course: the spaces $L(E, F)$ and $C(X, E)$.

2 Metric Spaces and Norms

Basic concepts: metric space, open set, open set definition of continuity, sequences, Cauchy sequences, convergent sequences, completeness, closed sets, compact sets, bounded sets, normed vector space, equivalence of norms in finite dimensions.

Possible new topics: all of them!

Fundamental idea: metric spaces are the "right" spaces in which to consider continuity and sequence convergence. They are the abstraction of the distance structure of $\mathbb{R}^n$. When the metric is a norm, the metric structure and vector space structure of a given space interact in a manner reminiscent of that of $\mathbb{R}^n$.

Examples:

1. $\mathbb{R}^n$ with norm $\|x\|_1 = \sum |x_i|$,
2. $\mathbb{R}^n$ with norm $\|x\|_2 = \left(\sum |x_i|^2\right)^{\frac{1}{2}}$,
3. $\mathbb{R}^n$ with norm $\|x\|_\infty = \max\{|x_i|\}$,
4. any subset $W \subseteq \mathbb{R}^n$ with the induced metric,
5. $C(K, E)$ with norm $\|f\|_\infty = \sup \|f(x)\|_E$ where K is a compact space and $(E, \|\cdot\|_E)$ is a normed vector space,

6. $L(E, F)$ with norm $\|T\| = \sup\{\|Tx\|_F : \|x\|_E = 1\}$ where $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are finite dimensional vector spaces.

Essentials for this course: $L(E, F)$ and $C(K, E)$ and properties thereof.

3 Calculus

Basic concepts: definition of the derivative, interpretation as local approximation and as direction finder, C^r-functions, curves, integral of a curve.

Possible new topics: C^r-functions, concept of tangent space.

Fundamental idea: differentiability quantifies continuity: continuity says that if x is near y then $f(x)$ will be near $f(y)$, differentiability says how near $f(x)$ will be to $f(y)$.

Essentials for this course: rth tangent vector of a curve, concept of derivative as pointwise "direction finder", integral of a curve.

Note: it is expected that you are familiar with the derivatives, integrals, and standard formulæ of polynomials, exponential functions, and trigonometric functions.

4 Ordinary Differential Equations

Basic concepts: definition of an ODE and of a solution of an ODE, types of ODE: rth order, autonomous, linear, homogeneous, proper form.

Possible new topics: the idea that an ODE is "merely" a function and that the subtlety is in the definition of a solution.

Fundamental idea: the idea that an ODE is "merely" a function and that the subtlety is in the definition of a solution.

Examples:

1. Classical mechanics:
   (a) inverse square force: $x_2 + x_0/\|x_0\|^3$,
   (b) simple harmonic motion: $x_2 + kx_0$,
   (c) air resistance: $x_2 + g + k\|x_1\| x_1$,
   (d) spin: $x_2 - kx_1 \times \omega$.
2. Circuit theory, e.g. $(x_1 - y_0, y_1 + x_0 + f(y_0))$.
3. Predator-prey: $(x_1 - Ax_0 + Bx_0y_0, y_1 - Cx_0y_0 + Dy_0)$.

Essentials for this course: definition of an ODE.

5 Picard's Theorem

Basic concepts: integral equations, initial data, Picard's theorem.

Possible new topics: integral equations, the fact that all ODEs are first order.

Fundamental idea: Given, fairly reasonable, conditions on the function defining the ODE, there exist solutions and they are unique.

Key steps in proof: set up a sequence of approximations to a solution using the integral formulation of the ODE, show that this is Cauchy and hence convergent, show that the limit is a solution of the ODE, show that any other solution must be arbitrarily close, hence equal.

6 Properties of the Solution

Basic concepts: extendibility of solutions, maximal solutions, their definition and uniqueness, definition of flow, differentiability of solutions and flow in terms of the function defining the original ODE.

Possible new topics: the concept that solutions with large domains can be obtained by "gluing together" solutions with small domains.

Fundamental idea: The result of applying Picard's theorem – that solutions to certain ODEs exist locally and are unique – can be extended to yield a global unique solution in a region where the defining function remains locally Lipschitz.

7 Predator-Prey Models

Basic concepts: development of the predator-prey ODE from simple assumptions, method of separation of variables for solving ODEs of the form $x_1 - G(x_0)H(t)$, equilibrium points, types of stability, phase diagrams for ODEs corresponding to two species.

Possible new topics: Proof of the technique of separation of variables, exact definitions of stability of equilibrium points.

Fundamental idea: Given a system at equilibrium, what happens if it is disturbed by a small amount?

8 Linear Theory

Basic concepts: linear theory as a tool to examine stability of equilibria of general ODEs, exponential of a matrix as solution of linear ODE, determination of behaviour in terms of eigenvalues.

Possible new topics: exponential of a matrix.

Fundamental idea: linear systems are "simple" and if there is a linear system "close" to a more general ODE certain properties of the linear system carry over to the general case, in particular we can determine the stability of *hyperbolic* equilibrium points.

9 Classical Mechanics

Basic concepts: integrals, Liapunov functions.

Possible new topics: Liapunov functions.

Fundamental idea: auxiliary functions can be used either to reduce the complexity of an ODE or as another tool to classify the equilibria.