

Variations on a Theme: Riemannian Geometry in Infinite Dimensions

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Abstract

Infinite dimensional Riemannian manifolds have traditionally been divided into two types: strong and weak. One can generalise many of the standard constructions of Riemannian geometry to strong Riemannian manifolds but not to weak ones. In this talk I shall examine some of the basic results of Riemannian geometry in order to refine this classification. The purpose is to show that some constructions can be generalised to certain weak Riemannian manifolds. I shall conclude by describing which level of structure is available for the free loop space of a finite dimensional manifold and by explaining how this can be used in applications.

1 Prelude

No wise fish would go anywhere without a porpoise.

Lewis Carroll

The purpose of this talk is to introduce several variations on the theme of Riemannian geometry in infinite dimensions. Given the length of this talk, I shall not spend any time explaining why one would wish to do Riemannian geometry in infinite dimensions. I am sure that many people here know better than I do why one should wish to do this, and those that don't are unlikely to be enlightened by what little I could say in a couple of sentences. Hopefully all are willing to accept, at least for the next half hour, that it is something one might wish to do.

Doing *geometry* in infinite dimensions is an example of a broader technique. We take some theory that works well in finite dimensions and attempt to apply it to infinite dimensional objects, in this case manifolds. Sometimes the adaptation to infinite dimensions is straightforward, such as in homotopy theory where we eat infinite dimensional objects for breakfast. As we add in more rigid structure then it becomes harder and harder to make the adaptation and we find that things that "just work" in finite dimensions no longer do so in the infinite case and considerable effort has to be expended to make them work, or to figure out conditions necessary to make them work. Geometry is

a particularly good area for this generalisation as the adaptation is generally possible but sufficiently difficult to be of intrinsic interest.

The process of taking a construction or theory from finite dimensional geometry and applying it in infinite dimensions is more of an art than a science; the intricacies of how to modify it to make it work are unique to the specific concept being generalised. However, there are certain basic results in geometry that turn up again and again, primarily because they are so basic, and these can be studied on their own without needing to know how they are going to be applied.

The plan is to take two of these fundamental results of Riemannian geometry and see how to make them work in infinite dimensions, or if they don't work then to see how to classify their failure. The two results are: the isomorphism between the tangent and cotangent bundles and the local triviality of the orthogonal structure.

This talk is based on ideas originating in [Sta].

2 Main Theme: Riemannian Structures

Before we start, let us observe that the basic object of Riemannian geometry easily generalises to infinite dimensions. As I am sure everyone knows, the fundamental object of Riemannian geometry is a Riemannian manifold and this consists of a manifold M together with a smooth choice g of inner products on the tangent spaces of M .¹

$$(M, g)$$

$$g_x: T_x M \times T_x M \rightarrow \mathbb{R}.$$

When we play this theme in infinite dimensions we find no dis-harmonies in the definition. Given a smooth manifold X of possibly infinite dimension we define a Riemannian structure on X to be a smooth choice g of inner products on the tangent spaces of X .

$$(X, g)$$

$$g_x: T_x X \times T_x X \rightarrow \mathbb{R}.$$

However, even here we start to see why things might be different in infinite dimensions.

All manifolds admit a Riemannian structure.

$\prod_{\text{box}} S^n$ does *not* admit a Riemannian structure.

This is the product of an infinite number of spheres with the *box* topology².

¹In this talk, M will always be a finite dimensional manifold.

²The Tychanoff topology on an infinite product is not usually a manifold topology.

3 First Variation: the Tangent Isomorphism

One of the most basic results of finite dimensional Riemannian geometry is the identification of the tangent and cotangent bundles. This isomorphism is used in many constructions of geometry, often through the associated identification of one-forms with vector fields.

3.1 Weak and Strong Riemannian Structures

The statement in finite dimensions is the following.

Result. $TM \cong T^*M$

Unfortunately in infinite dimensions then there are many Riemannian manifolds which do not have this property. A simple one is the full complex projective space.

Fact. $TC\mathbb{P}^\infty \not\cong T^*\mathbb{C}\mathbb{P}^\infty$ ($\mathbb{C}\mathbb{P}^\infty = \bigcup \mathbb{C}\mathbb{P}^n$)

As this basic result cannot be guaranteed to hold in infinite dimensions, we divide up Riemannian manifolds into two types: those where it does hold and those where it does not. The task then becomes to determine which Riemannian manifolds are of which type.

Definition. (TX, g) is

Strong if $TX \cong T^*X$,

Weak if $TX \not\cong T^*X$.

Where, of course, we mean that the map $TX \rightarrow T^*X$ is the map induced by the Riemannian structure.

The division of Riemannian structures into weak and strong is an old one and the following facts are well-known.

- Riemannian geometry works very well for *strong* Riemannian structures.
- Only *Hilbert* manifolds can have strong Riemannian structures. And, moreover, not all Riemannian structures on Hilbert manifolds are strong Riemannian structures.
- Not all infinite dimensional manifolds are Hilbert manifolds.
- Not all infinite dimensional manifolds can be approximated by Hilbert manifolds.

Example: $\text{Diff}(M)$

So our first variation is the observation that infinite dimensional Riemannian manifolds split into “strong” and “weak”. With a strong structure then we can easily mirror much of finite dimensional Riemannian geometry. With a weak structure then we have issues.

3.2 Co-Riemannian Structures

Requiring a strong Riemannian structure is incredibly restrictive. It rules out most naturally occurring infinite dimensional manifolds. It is true that sometimes when we have an infinite dimensional manifold then we can find a “nearby” one which is modelled on a Hilbert space and has a strong Riemannian structure. However, this is not always possible and even when it is it may not be desirable.

On the other hand, a weak Riemannian structure does not give us anything beyond the injective vector bundle morphism from the tangent to the cotangent bundle. This does still allow some constructions to proceed, but not many. Basically, any construction that whenever it uses the isomorphism $TM \cong T^\star M$ only actually uses the fact that there is an injective vector bundle morphism $TM \rightarrow T^\star M$ has as much likelihood of success with weak Riemannian structures as with strong ones (though there may be other obstructions to other parts of the generalisation to infinite dimensional manifolds).

By examining the proof of the fact that the map $TM \rightarrow T^\star M$ is an isomorphism we may be able to glean out intermediate types of Riemannian structure lying between weak and strong.

Why is $TM \rightarrow T^\star M$ an isomorphism?

Usual Proof. *It is injective.*

This proof does not generalise to infinite dimensions as it depends heavily on dimension. However, although this is the usual proof of this particular result, it is not the usual way that one proves something to be an isomorphism. A more usual way of proving that type of result is to exhibit an inverse.

Unusual Proof. *It has an inverse*

$$g^\star : T^\star M \rightarrow TM.$$

This inverse map has a very nice property.

$$(u, v) \rightarrow u(g_x^\star(v))$$

is an inner product on $T_x^\star M$.

That is to say, the map $g^\star : T^\star M \rightarrow TM$ is the map induced by a smooth choice of inner product on $T^\star M$. That this inner product is actually that induced by the inner product on TM via the isomorphism $T^\star M \cong TM$ is neither here nor there. As we can't generalise the isomorphism to infinite dimensions, we might at least be able to generalise one of its consequences.

Definition. *A co-Riemannian structure is (X, g^\star) , $g_x^\star : T_x^\star X \times T_x^\star X \rightarrow \mathbb{R}$.*

Result. *If X is modelled on reflexive spaces, we get an injective vector bundle map*

$$T^\star X \rightarrow TX.$$

To see that this is a useful definition, observe that there are constructions that, when purportedly using the isomorphism $TM \cong T^\star M$, actually only use the map $T^\star M \rightarrow TM$.

Some constructions that only use $T^\star X \rightarrow TX$:

1. Gradient vector field of a function,
2. Dirac operator of a spin manifold.

For a manifold to admit a weak co-Riemannian structure it is sufficient that the fibres admit an inner product and that the base admit partitions of unity. We can therefore see that

- LM admits a weak co-Riemannian structure,
- $\mathbb{C}P^\infty$ does not.

In summary, we have been considering the isomorphism between the tangent and cotangent bundles in finite dimensions.

$$TM \cong T^*M$$

From this we get two types of structure in infinite dimensions – Riemannian and co-Riemannian structures – and each comes in two flavours – strong and weak. Removing redundancies we get three objects.

Strong Riemannian

$$TX \cong T^*X$$

Weak Riemannian

$$TX \rightarrow T^*X$$

Weak co-Riemannian

$$T^*X \rightarrow TX$$

We should issue a warning: many manifolds admit both Riemannian and co-Riemannian structures but these will not, in general, be compatible in the same fashion as in finite dimensions. We must be aware of the potential for dis-harmony.

4 Second Variation: Local Triviality

We now turn to our other fundamental result of Riemannian geometry that we wish to investigate. Many constructions in finite dimensional geometry involve constructing some sort of bundle out of the initial data. To ensure that this is possible, we need to know that the Riemannian structure can be locally trivialised. This is always possible in finite dimensions.

Result. (TM, g) is locally isometrically trivial.

Before we can go anywhere with this idea in infinite dimensions we need to know what the corresponding statement would look like in infinite dimensions. What should this mean?

Version 1:

$$(TX, g) \text{ locally is } (V, g_V).$$

Here g_V is some fixed inner product on V , the model space of TX . Now unless we are on a Hilbert manifold, g_V will induce a topology on V which is weaker than its given one. It is highly likely that V will not be complete with respect to this topology and so there is a strong temptation to complete it to the nearest Hilbert space, say \bar{V} . This makes it highly unlikely that (TX, g) will be locally like (\bar{V}, g) but it may be that we can transfer this completion back to the manifold.

Version 2:

$$(TX, g) \rightarrow (\overline{TX}, g) \text{ locally is } (V, g_V) \rightarrow (\bar{V}, g_V).$$

With infinite dimensional vector bundles it is frequently the case that we do not want to use the full general linear group as our structure group – usually it either is not a Lie group or is contractible. We therefore want to involve two structure groups in the above: one for the tangent bundle and one for its completion.

Version 3:

$$(TX, g) \rightarrow (\overline{TX}, g) \text{ locally is } (V, g_V) \rightarrow (\bar{V}, g_V) \text{ compatibly with structure groups.}$$

There are a couple of remarks that are worth making about this but which in finite dimensions are not worth mentioning. The really difficult bit in the above is finding the bundle \overline{TX} . Once we have done that, a suitable inner product is much simpler to find. In finite dimensions, of course, this is trivial as we simply recover the original bundle. Also, the role of structure groups is much increased in infinite dimensions and so we might wish to allow some modifications in the structure group of the original bundle providing these don't change the resulting bundle of Hilbert spaces by too much. Since we are generally not using the whole general linear group of the typical fibre these modifications may *enlarge* the group as well as reduce it. Of course, the less extra structure that we have to add in then the stronger is our situation.

It is possible to construct manifolds which do not admit any Riemannian structure which can be locally trivialised. Here is one example. Any loop in $L\mathbb{R}$ decomposes uniquely as the sum of a constant loop and one which integrates to zero. Using this we construct a quotient of $L\mathbb{R}$ by the relation

$$(\lambda, \gamma) \sim \left(\lambda + 1, \frac{d\gamma}{dt} \right)$$

this is essentially an infinite dimensional cylinder such that going round the cylinder once replaces a loop by its derivative. This admits Riemannian structures, but none that can be isometrically trivialised because such a trivialisation would mean that taking the norm of the derivative would be equivalent (as a norm) to taking the norm of the original loop and this cannot be so.

It is possible to give a list of requirements for such a structure to exist, together with relaxations of these requirements depending on what modifications we allow, but it would probably be more illustrative to give an example.

5 Cadenza: the Free Loop Space

Let M be a finite dimensional Riemannian manifold, $LM := C^\infty(S^1, M)$ its free loop space. This is not modelled on Hilbert spaces but is modelled on Fréchet spaces which do admit inner products. Also, the dual of the model spaces admits inner products. Therefore we can easily see that LM admits weak Riemannian and co-Riemannian structures.

In table 1, we list the bundles together with various groups. The first is the “natural” structure group. The second is the group which acts on the Hilbert completion of the bundle. For the tangent spaces of M and LM this is the same as the natural structure group but for the cotangent spaces of LM then the loop group $L\text{Gl}(\mathbb{R}^n)$ does not act on any Hilbert completion of the model space so we have to reduce it to the (homotopy equivalent) polynomial loop group. The third group in the list is the group which acts by isometries. For M this is the usual orthogonal group, and for the tangent spaces of LM this is the expected loop group. For the cotangent spaces we need to take the polynomial loops in the orthogonal group but we need to modify the action slightly (by a homotopy). That is, we need to enlarge the structure group slightly to find a suitable subgroup that acts by isometries.

Manifold	M	LM	LM
Bundle	TM	TLM	$T^\star LM$
Fibre	\mathbb{R}^n	$L\mathbb{R}^n$	$\mathcal{L}(L\mathbb{R}^n, \mathbb{R})$
Group	$\text{Gl}(\mathbb{R}^n)$	$L\text{Gl}(\mathbb{R}^n)$	$L\text{Gl}(\mathbb{R}^n)$
Completion Group	$\text{Gl}(\mathbb{R}^n)$	$L\text{Gl}(\mathbb{R}^n)$	$L_{\text{pol}}\text{Gl}(\mathbb{R}^n)$
Orthogonal Group	O_n	LO_n	$L_{\text{pol}}O_n$ (modified)

Table 1: Riemannian Structure of the Loop Space

In conclusion, we make the following remarks about this structure.

1. The Riemannian and co-Riemannian structure – and all associated structure – of LM depend only on Riemannian structure of M .
2. The co-Riemannian structure on LM is sufficient to define a Dirac operator.
3. For more details, see [Sta]

References

- [Sta] Andrew Stacey. The Geometry of the Loop Space and a Construction of a Dirac Operator, arXiv:math.DG/0505077.