

Variations on a Theme: Riemannian Geometry in Infinite Dimensions

Algebraic Topology Special Session, BMC 2007

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Prelude

No wise fish would go anywhere without a porpoise.

Lewis Carroll

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Finite Dimensional
Riemannian Geometry

Infinite Dimensional
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Grand Scheme

Successful
Theory



Potentially
Useful
Theory

Main Theme

(M, g)

$$g_x: T_x M \times T_x M \rightarrow \mathbb{R}$$

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Main Theme

$$(M, g)$$

$$g_x: T_x M \times T_x M \rightarrow \mathbb{R}$$

All manifolds admit a
Riemannian structure.

$$(X, g)$$

$$g_x: T_x X \times T_x X \rightarrow \mathbb{R}$$

Main Theme

$$(M, g)$$

$$g_x: T_x M \times T_x M \rightarrow \mathbb{R}$$

All manifolds admit a Riemannian structure.

$$(X, g)$$

$$g_x: T_x X \times T_x X \rightarrow \mathbb{R}$$

$\prod_{\text{box}} S^n$ does **not** admit a Riemannian structure.

First Variation

Result

$$TM \cong T^{\star}M$$

First Variation

Result

$$TM \cong T^*M$$

Fact

$$T\mathbb{C}P^\infty \not\cong T^*\mathbb{C}P^\infty$$
$$(\mathbb{C}P^\infty = \bigcup \mathbb{C}P^n)$$

First Variation

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Fact

$$T\mathbb{C}P^\infty \not\cong T^*\mathbb{C}P^\infty$$
$$(\mathbb{C}P^\infty = \bigcup \mathbb{C}P^n)$$

Definition

(TX, g) is

Strong if $TX \cong T^*X$,

Weak if $TX \not\cong T^*X$.

Strong Riemannian Structures

- ▶ Riemannian geometry works very well for **strong** Riemannian structures.

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Strong Riemannian Structures

- ▶ Riemannian geometry works very well for strong Riemannian structures.
- ▶ Only Hilbert manifolds can have strong Riemannian structures.
- ▶ Not all infinite dimensional manifolds are Hilbert manifolds.
- ▶ Not all infinite dimensional manifolds can be approximated by Hilbert manifolds.

Example: $\text{Diff}(M)$

Riemannian Structures

Why is $TM \rightarrow T^*M$ an isomorphism?

Usual Proof

It is injective.

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It has an inverse

$$g^* : T^*M \rightarrow TM.$$

Riemannian Structures

Why is $TM \rightarrow T^{\star}M$ an isomorphism?

Usual Proof

It is injective.

Unusual Proof

It has an inverse

$$g^{\star} : T^{\star}M \rightarrow TM.$$

$(u, v) \rightarrow u(g_x^{\star}(v))$
is an inner product on $T_x^{\star}M$.

Co-Riemannian Structures

Why is $TM \rightarrow T^{\star}M$ an isomorphism?

Usual Proof

It is injective.

Unusual Proof

It has an inverse

$$g^{\star} : T^{\star}M \rightarrow TM.$$

$(u, v) \rightarrow u(g_x^{\star}(v))$
is an inner product on $T_x^{\star}M$.

Definition

A **co-Riemannian structure** is

$$(X, g^{\star}),$$
$$g_x^{\star} : T_x^{\star}X \times T_x^{\star}X \rightarrow \mathbb{R}$$

Co-Riemannian Structures

Why is $TM \rightarrow T^{\star}M$ an isomorphism?

Usual Proof

It is injective.

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$$g^{\star} : T^{\star}M \rightarrow TM.$$

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Definition

A **co-Riemannian structure** is

$$(X, g^{\star}), \\ g_x^{\star} : T_x^{\star}X \times T_x^{\star}X \rightarrow \mathbb{R}$$

Result

X modelled on **reflexive** spaces, get

$$T^{\star}X \rightarrow TX$$

Remarks

Some constructions that only use $T^*X \rightarrow TX$:

1. Gradient vector field of a function,
2. Dirac operator of a spin manifold.

Existence results

- ▶ LM admits a weak co-Riemannian structure,
- ▶ $\mathbb{C}P^\infty$ does not.

Conclusion

$$TM \cong T^\star M$$

Strong Riemannian

$$TX \cong T^\star X$$

Weak Riemannian

$$TX \rightarrow T^\star X$$

Weak co-Riemannian

$$T^\star X \rightarrow TX$$

Second Variation

Result

(TM, g)
locally **isometrically** trivial

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What should this mean?

Version 1:

(TX, g)
locally is
 (V, g_V)

Second Variation

Result

$$(TM, g)$$

locally **isometrically** trivial

What should this mean?

Version 2:

$$(TX, g) \rightarrow (\overline{TX}, g)$$

locally is

$$(V, g_V) \rightarrow (\overline{V}, g_V)$$

Second Variation

Result

(TM, g)
locally **isometrically** trivial

What should this mean?

Version 3:

$$(TX, g) \rightarrow (\overline{TX}, g)$$

locally is

$$(V, g_V) \rightarrow (\overline{V}, g_V)$$

compatibly with structure
groups

Second Variation

Result

(TM, g)
locally **isometrically** trivial

What should this mean?

Version 3:

$$(TX, g) \rightarrow (\overline{TX}, g)$$

locally is

$$(V, g_V) \rightarrow (\overline{V}, g_V)$$

compatibly with structure
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Remarks:

- ▶ Distinguish between (\overline{V}, g) and \overline{V} .
- ▶ May need to **enlarge** the structure group.

Cadenza: the Free Loop Space

Manifold	<i>M</i>	<i>LM</i>	<i>LM</i>

Cadenza: the Free Loop Space

Manifold	M	LM	LM
Bundle	TM	TLM	T^*LM

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Group	$Gl(\mathbb{R}^n)$	$LGl(\mathbb{R}^n)$	$LGl(\mathbb{R}^n)$

Cadenza: the Free Loop Space

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Group	$Gl(\mathbb{R}^n)$	$L Gl(\mathbb{R}^n)$	$L Gl(\mathbb{R}^n)$
Completion Group	$Gl(\mathbb{R}^n)$	$L Gl(\mathbb{R}^n)$	$L_{\text{pol}} Gl(\mathbb{R}^n)$

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Orthogonal Group	O_n	LO_n	$L_{\text{pol}} O_n$

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Completion Group	$Gl(\mathbb{R}^n)$	$L Gl(\mathbb{R}^n)$	$L_{\text{pol}} Gl(\mathbb{R}^n)$
Orthogonal Group	O_n	LO_n	$L_{\text{pol}} O_n$ (modified)

Conclusion

1. Riemannian and co-Riemannian structure depend only on Riemannian structure of M .
2. Co-Riemannian structure on LM is sufficient to define a Dirac operator.
3. For more details, see



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The Geometry of the Loop Space and a Construction of a Dirac Operator,
arXiv:math.DG/0505077.