

2008-06-13

Variations on a Theme: Riemannian Geometry in Infinite

Dimensions

└ Prelude

└ Prelude

Prelude

No wise fish would go anywhere without a purpose.

Lewis Carroll

Finite Dimensional
Riemannian Geometry

Infinite Dimensional
Riemannian Geometry

Grand Scheme

Successful
Theory

→

Potentially
Useful
Theory

Apology for change of talk.

Purpose: introduce variations of Riemannian geometry in infinite dimensions.

Not going to explain why.

Example of general technique.

Geometry: simple enough to do, difficult enough to be intrinsically interesting.

Basic results recur: isomorphism and triviality

Variations on a Theme: Riemannian Geometry in Infinite Dimensions

└ Main Theme: Riemannian Structures

└ Main Theme

$$\begin{array}{c} (M, g) \\ g: T_x M \times T_x M \rightarrow \mathbb{R} \end{array}$$

All manifolds admit a Riemannian structure.

$$\begin{array}{c} (X, g) \\ g: T_x X \times T_x X \rightarrow \mathbb{R} \end{array}$$

$\prod_{\text{bas}} S^1$ does **not** admit a Riemannian structure.

Fundamental object easily generalises

Not all plain sailing, however

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Variations on a Theme: Riemannian Geometry in Infinite Dimensions

└ First Variation: the Tangent Isomorphism

└ Weak and Strong Riemannian Structures

└ First Variation

First Variation

Result

$$TM \cong T^*M$$

Fact

$$T\mathbb{C}P^n \neq T^*\mathbb{C}P^n \\ (\mathbb{C}P^n = \bigcup \mathbb{C}P^1)$$

Definition

(TX, g) is
Strong if $TX \cong T^*X$,
Weak if $TX \not\cong T^*X$.

Isomorphism result in finite dimensions

Does not necessarily hold in infinite dimensions.

Definition to distinguish cases.

Map is always obvious one.

Variations on a Theme: Riemannian Geometry in Infinite Dimensions

└ First Variation: the Tangent Isomorphism

└ Weak and Strong Riemannian Structures

└ Strong Riemannian Structures

- Riemannian geometry works very well for **strong** Riemannian structures.
 - Only **Hilbert** manifolds can have strong Riemannian structures.
 - Not all infinite dimensional manifolds are Hilbert manifolds.
 - Not all infinite dimensional manifolds can be approximated by Hilbert manifolds.
- Example: $\text{Diff}(M)$

Well-known facts.

Even then, not necessarily strong.

Hideki Omori, 1978

Can't get isomorphism, can we still do better than just "weak"?

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Variations on a Theme: Riemannian Geometry in Infinite

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└ First Variation: the Tangent Isomorphism

└└ Co-Riemannian Structures

└└└ Co-Riemannian Structures

Why is $TM \rightarrow T^*M$ an isomorphism?

Usual Proof

It is injective.

Unusual Proof

It has an inverse

$$g^*: T^*M \rightarrow TM.$$

$(u, v) \rightarrow u(g^*(v))$
is an inner product on T_x^*M .

Definition

A co-Riemannian structure is

$$(X, g^*),$$

$$g^*: T_x^*X \times T_x^*X \rightarrow \mathbb{R}$$

Result

X modelled on reflexive spaces, get

$$T^*X \rightarrow TX$$

Look at proof to get ideas.

In fact, equivalent to inner product.

Generalise the existence of this map.

Slight subtlety, only get map if model space is reflexive. Less restrictive than Hilbertian.

└ First Variation: the Tangent Isomorphism

└└ Co-Riemannian Structures

└└└ Remarks

Some constructions that only use $T^*X \rightarrow TX$:

1. Gradient vector field of a function,
2. Dirac operator of a spin manifold.

Existence results

- LM admits a weak co-Riemannian structure,
- CP^n does not.

Fortunately, co-Riemannian structures are almost as easy to find as Riemannian structures

Moreover, this isn't a spurious definition as it now allows us to generalise more concepts than before.

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└ First Variation: the Tangent Isomorphism

└└ Co-Riemannian Structures

└└└ Conclusion

Conclusion

Strong Riemannian

$$TX \cong T^*X$$

Weak Riemannian

$$TX \rightarrow T^*X$$

Weak co-Riemannian

$$T^*X \rightarrow TX$$

$$TM \cong T^*M$$

One finite dimensional concept leads to three infinite dimensional ones.

Strong co-Riemannian same as strong Riemannian

Variations on a Theme: Riemannian Geometry in Infinite Dimensions

└ Second Variation: Local Triviality

└ Second Variation

Result

(TM, g)
locally isometrically trivial

What should this mean?

Version 3:

$(TX, g) \rightarrow (TX, g)$
locally is
 $(V, g_V) \rightarrow (V, g_V)$
compatibly with structure
groups

Remarks:

- Distinguish between (\overline{V}, g) and \overline{V} .
- May need to **enlarge** the structure group.

Local triviality: needed for bundle-like constructions.

Not completely obvious what this should generalise to.

Remark on topology and completions.

Remark on structure groups.

Completion as half-way mark; structure groups may need to be enlarged as well as reduced.

There are manifolds on which it is not possible to locally trivialise any Riemannian structure.