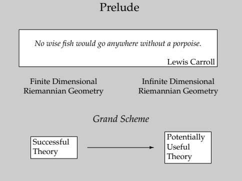


Apology for change of talk.

Purpose: introduce variations of Riemannian geometry in infinite dimensions.

Not going to explain why.

Example of general technique.

Geometry: simple enough to do, difficult enough to be intrinsically interesting.

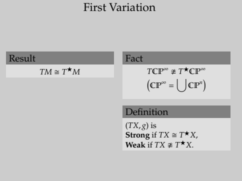
Basic results recur: isomorphism and triviality

Main Theme

(M, g)	(X, g)
$g_x \colon T_xM \times T_xM \to \mathbb{R}$	$g_x \colon T_xX \times T_xX \to \mathbb{R}$
All manifolds admit a Riemannian structure.	$\prod_{\text{box}} S^n$ does **not** admit a Riemannian structure.

Fundamental object easily generalises

Not all plain sailing, however



Isomorphism result in finite dimensions

Does not necessarily hold in infinite dimensions.

Definition to distinguish cases.

Map is always obvious one.

Strong Riemannian Structures

- Riemannian geometry works very well for **strong** Riemannian structures.
- Only **Hilbert** manifolds can have strong Riemannian structures.
- Not all infinite dimensional manifolds are Hilbert manifolds.
- Not all infinite dimensional manifolds can be approximated by Hilbert manifolds.

  Example: Diff(M)

Well-known facts.

Even then, not necessarily strong.

Hideki Omori, 1978

Can't get isomorphism, can we still do better than just "weak"?

Co-Riemannian Structures

Why is $TM \to T^\star M$ an isomorphism?

Usual Proof

It is injective.

Unusual Proof

It has an inverse

$$g^\star : T^\star M \to TM.$$

$$(u, v) \to u(g_x^\star(v))$$

is an inner product on $T_x^\star M$.

Definition

A **co-Riemannian structure** is

$$(X, g^\star),$$
$$g_x^\star : T_x^\star X \times T_x^\star X \to \mathbb{R}$$

Result

X modelled on **reflexive** spaces, get

$$T^\star X \to TX$$

Look at proof to get ideas.

In fact, equivalent to inner product.

Generalise the existence of this map.

Slight subtlety, only get map if model space is reflexive. Less restrictive than Hilbertian.

Remarks

Some constructions that only use $T^\star X \to TX$:

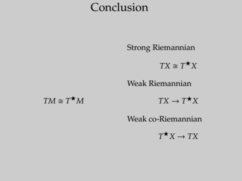
1. Gradient vector field of a function,
2. Dirac operator of a spin manifold.

Existence results

- LM admits a weak co-Riemannian structure,
- $\mathbb{CP}^\infty$ does not.

Fortunately, co-Riemannian structures are almost as easy to find as Riemannian structures

Moreover, this isn't a spurious definition as it now allows us to generalise more concepts than before.



One finite dimensional concept leads to three infinite dimensional ones.

Strong co-Riemannian same as strong Riemannian

Second Variation

Result

(TM, g)
locally **isometrically** trivial

What should this mean?
Version 3:

$(TX, g) \to (\overline{TX}, g)$
locally is
$(V, g_V) \to (\overline{V}, g_V)$
compatibly with structure groups

Remarks:

- Distinguish between $(\overline{V}, g)$ and $\overline{V}$.
- May need to **enlarge** the structure group.

Local triviality: needed for bundle-like constructions.

Not completely obvious what this should generalise to.

Remark on topology and completions.

Remark on structure groups.

Completion as half-way mark; structure groups may need to be enlarged as well as reduced.

There are manifolds on which it is not possible to locally trivialise any Riemannian structure.