

Variations on a Theme:
Riemannian Geometry in Infinite
Dimensions

Algebraic Topology Special Session, BMC 2007

Andrew Stacey

University of Sheffield

18th April 2007

Prelude

No wise fish would go anywhere without a porpoise.

Lewis Carroll

Finite Dimensional
Riemannian Geometry

Infinite Dimensional
Riemannian Geometry

Grand Scheme

Successful
Theory

Potentially
Useful
Theory

Main Theme

$$(M, g)$$

$$g_x: T_x M \times T_x M \rightarrow \mathbb{R}$$

All manifolds admit a Riemannian structure.

$$(X, g)$$

$$g_x: T_x X \times T_x X \rightarrow \mathbb{R}$$

$\prod_{\text{box}} S^n$ does **not** admit a Riemannian structure.

First Variation

Result

$$TM \cong T^{\star}M$$

Fact

$$T\mathbb{C}P^{\infty} \not\cong T^{\star}\mathbb{C}P^{\infty}$$
$$(\mathbb{C}P^{\infty} = \bigcup \mathbb{C}P^n)$$

Definition

(TX, g) is

Strong if $TX \cong T^{\star}X$,

Weak if $TX \not\cong T^{\star}X$.

Strong Riemannian Structures

- ▶ Riemannian geometry works very well for **strong** Riemannian structures.
- ▶ Only **Hilbert** manifolds can have strong Riemannian structures.
- ▶ Not all infinite dimensional manifolds are Hilbert manifolds.
- ▶ Not all infinite dimensional manifolds can be approximated by Hilbert manifolds.

Example: $\text{Diff}(M)$

Co-Riemannian Structures

Why is $TM \rightarrow T^{\star}M$ an isomorphism?

Usual Proof

It is injective.

Unusual Proof

It has an inverse

$$g^{\star} : T^{\star}M \rightarrow TM.$$

$(u, v) \rightarrow u(g_x^{\star}(v))$
is an inner product on $T_x^{\star}M$.

Definition

A **co-Riemannian structure** is

$$(X, g^{\star}),$$
$$g_x^{\star} : T_x^{\star}X \times T_x^{\star}X \rightarrow \mathbb{R}$$

Result

X modelled on **reflexive** spaces, get

$$T^{\star}X \rightarrow TX$$

Remarks

Some constructions that only use $T^*X \rightarrow TX$:

1. Gradient vector field of a function,
2. Dirac operator of a spin manifold.

Existence results

- ▶ LM admits a weak co-Riemannian structure,
- ▶ $\mathbb{C}P^\infty$ does not.

Conclusion

$$TM \cong T^\star M$$

Strong Riemannian

$$TX \cong T^\star X$$

Weak Riemannian

$$TX \rightarrow T^\star X$$

Weak co-Riemannian

$$T^\star X \rightarrow TX$$

Second Variation

Result

(TM, g)
locally **isometrically** trivial

What should this mean?

Version 1:

(TX, g)
locally is
 (V, g_V)

Remarks:

- ▶ Distinguish between (\bar{V}, g) and \bar{V} .
- ▶ May need to **enlarge** the structure group.

Second Variation

X 2:

$$\rightarrow (\overline{TX}, g)$$

$$\rightarrow (\overline{V}, g_V)$$

Second Variation


$X \times X^3$:

compatibly with structure
groups

Cadenza: the Free Loop Space

| Manifold | M | LM | LM |
|-------------------------|--------------------|----------------------|--|
| Bundle | TM | TLM | T^*LM |
| Fibre | \mathbb{R}^n | $L\mathbb{R}^n$ | $\mathcal{L}(L\mathbb{R}^n, \mathbb{R})$ |
| Group | $Gl(\mathbb{R}^n)$ | $L Gl(\mathbb{R}^n)$ | $L Gl(\mathbb{R}^n)$ |
| Completion Group | $Gl(\mathbb{R}^n)$ | $L Gl(\mathbb{R}^n)$ | $L_{\text{pol}} Gl(\mathbb{R}^n)$ |
| Orthogonal Group | O_n | LO_n | $L_{\text{pol}}O_n$ (modified) |

Conclusion

1. Riemannian and co-Riemannian structure depend only on Riemannian structure of M .
2. Co-Riemannian structure on LM is sufficient to define a Dirac operator.
3. For more details, see
 Andrew Stacey.
The Geometry of the Loop Space and a Construction of a Dirac Operator,
arXiv:math.DG/0505077.