Variations on a Theme: Riemannian Geometry in Infinite Dimensions
Algebraic Topology Special Session, BMC 2007

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Prelude

No wise fish would go anywhere without a porpoise.

Lewis Carroll

Finite Dimensional Riemannian Geometry

Infinite Dimensional Riemannian Geometry

Grand Scheme

Successful Theory

Potentially Useful Theory
(M, g)
\[ g_x : T_xM \times T_xM \rightarrow \mathbb{R} \]

All manifolds admit a Riemannian structure.

\[ (X, g) \]
\[ g_x : T_xX \times T_xX \rightarrow \mathbb{R} \]

\( \prod_{\text{box}} S^n \) does not admit a Riemannian structure.
<table>
<thead>
<tr>
<th>Result</th>
<th>Fact</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TM \cong T^\star M$</td>
<td>$TCP^\infty \not\cong T^\star CP^\infty$</td>
<td>$(TX, g)$ is (Strong if $TX \cong T^\star X$, Weak if $TX \not\cong T^\star X$).</td>
</tr>
</tbody>
</table>
Riemannian geometry works very well for strong Riemannian structures.

Only Hilbert manifolds can have strong Riemannian structures.

Not all infinite dimensional manifolds are Hilbert manifolds.

Not all infinite dimensional manifolds can be approximated by Hilbert manifolds.

Example: $\text{Diff}(M)$
### Co-Riemannian Structures

**Why is** $TM \to T^*M$ **an isomorphism?**

**Usual Proof**
It is injective.

**Unusual Proof**
It has an inverse $g^\star : T^*M \to TM$.

$(u, v) \to u(g^\star_x(v))$ is an inner product on $T^*_xM$.

<table>
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<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>A co-Riemannian structure is $(X, g^\star)$, $g^\star_x : T^<em>_xX \times T^</em>_xX \to \mathbb{R}$</td>
</tr>
</tbody>
</table>

<table>
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<th>Result</th>
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<tr>
<td>$X$ modelled on reflexive spaces, get $T^*X \to TX$</td>
</tr>
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</table>
Some constructions that only use $T^*X \to TX$:

1. Gradient vector field of a function,

2. Dirac operator of a spin manifold.

Existence results

- $LM$ admits a weak co-Riemannian structure,

- $\mathbb{C}P^\infty$ does not.
<table>
<thead>
<tr>
<th>Weak co-Riemannian</th>
<th>Weak Riemannian</th>
<th>Strong Riemannian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^\star X \rightarrow TX$</td>
<td>$TX \rightarrow T^\star X$</td>
<td>$TX \cong T^\star X$</td>
</tr>
</tbody>
</table>
Result

\((TM, g)\)
locally \textit{isometrically} trivial

What should this mean?

\textbf{Version 1:}

\[(TX, g)\]
locally is
\[(V, g_V)\]

\textbf{Remarks:}

- Distinguish between \((\overline{V}, g)\) and \(\overline{V}\).
- May need to \textit{enlarge} the structure group.
Second Variation

\[ X_2: \]
\[ \rightarrow (TX, g) \]
\[ \rightarrow (\overline{V}, g_V) \]

Remarks:
▶ Distinguish between \((V, g_V)\) and \(V\).
▶ May need to enlarge the structure group.
Second Variation

<table>
<thead>
<tr>
<th>X X 3:</th>
<th>compatibly with structure groups</th>
</tr>
</thead>
</table>

Remarks:

▶ Distinguish between \((V, g_V)\) and \(V\).

▶ May need to enlarge the structure groups.
### Cadenza: the Free Loop Space

<table>
<thead>
<tr>
<th>Manifold</th>
<th>$M$</th>
<th>$LM$</th>
<th>$LM$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bundle</strong></td>
<td>$TM$</td>
<td>$TLM$</td>
<td>$T^\ast LM$</td>
</tr>
<tr>
<td><strong>Fibre</strong></td>
<td>$\mathbb{R}^n$</td>
<td>$L\mathbb{R}^n$</td>
<td>$\mathcal{L}(L\mathbb{R}^n, \mathbb{R})$</td>
</tr>
<tr>
<td><strong>Group</strong></td>
<td>$\text{Gl}(\mathbb{R}^n)$</td>
<td>$L\text{Gl}(\mathbb{R}^n)$</td>
<td>$L\text{Gl}(\mathbb{R}^n)$</td>
</tr>
<tr>
<td><strong>Completion Group</strong></td>
<td>$\text{Gl}(\mathbb{R}^n)$</td>
<td>$L\text{Gl}(\mathbb{R}^n)$</td>
<td>$L_{\text{pol}}\text{Gl}(\mathbb{R}^n)$</td>
</tr>
<tr>
<td><strong>Orthogonal Group</strong></td>
<td>$O_n$</td>
<td>$LO_n$</td>
<td>$L_{\text{pol}}O_n$ (modified)</td>
</tr>
</tbody>
</table>
1. Riemannian and co-Riemannian structure depend only on Riemannian structure of $M$.

2. Co-Riemannian structure on $LM$ is sufficient to define a Dirac operator.