Operations in the First Morava K–Theory

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Abstract

We describe the families of degree zero operations in the first Morava K–theory, and consequently in mod p K–theory, in terms of Adams' operations.

Contents

1 Introduction

We study operations in the first Morava K–theory, primarily of degree zero, with a view to giving a complete comprehensive description of the various types of operation: stable, additive, and unstable. In the unstable and additive realms we mean by "degree zero" operations from degree zero to degree zero. In light of the close relation of the first Morava K–theory to ordinary K–theory with mod p coefficients, our results extend to that case. Here p is an odd prime which we fix now for the rest of the paper.

That relationship is one of the key tools in our analysis. Amongst other things it allows us to define "Adams' operations". As these are well studied we use these as our primary operations from which all else will be derived. Our first descriptions of the additive and stable operations express these as formal power series rings on a small number of Adams' operations:

Theorem (4.11, 4.13) *For $k \in \mathbb{Z}$ let Ψ^k denote the kth Adams' operation. Let $q \in \mathbb{Z}$ be primitive modulo p^2 and let $\tilde{q} = q^{p-1}$. The algebra of stable operations of degree zero in $K(1)^*(-)$ is:*

$$\mathbb{F}_p[\![\Psi^{\tilde{q}} - 1]\!].$$

The algebra of additive operations of degree zero is:

$$\mathbb{F}_p[\![\Psi^0, \Psi^p, \Psi^{\tilde{q}} - 1]\!]/(\Psi^0\Psi^k - \Psi^0).$$

This description uses as few Adams' operations as possible. At the other extreme we can use all the Adams' operations. We shall see that there is an additive Adams' operation for each p–adic integer.

Theorem (4.17) *The algebra of stable operations of degree zero in $K(1)^*(-)$ is isomorphic to:*

$$\mathbb{F}_p\langle\!\langle {\mathbb{Z}_p}^\times \rangle\!\rangle^{{\mathbb{F}_p}^\times} \cong \mathbb{F}_p\langle\!\langle 1 + p\mathbb{Z}_p \rangle\!\rangle.$$

The algebra of additive operations of degree zero is isomorphic to:

$$\mathbb{F}_p\langle\!\langle \mathbb{Z}_p \rangle\!\rangle^{{\mathbb{F}_p}^\times}.$$

Here the double brackets indicate a particular completion of the usual group ring of the p–adic units or integers. More details of this completion will be given in section 4.3. The ${\mathbb{F}_p}^\times$ action is derived from that of the $(p-1)$th roots of unity lying within ${\mathbb{Z}_p}^\times$. The multiplicative structure of the algebras comes from the multiplication in $\mathbb{Z}_p$; the addition is ignored altogether.

The unstable operations of degree zero have the following description:

Theorem (5.15) *The algebra of unstable operations $K(1)^*(\underline{K(1)}_0, o)$ is the free completed $K(1)^*$–algebra on the additive group of all stable operations.*

To complete our description we give an account of the odd degree stable operations in terms of transgressions of Adams' operations.

Our strategy in deriving these descriptions is to sandwich the algebras of operations between two well-known things: the algebras of operations of ordinary K–theory and the algebras of co-operations in Morava K–theory. The link between the first Morava K–theory and ordinary K–theory with mod p coefficients provides the butter on that slice of bread whilst the low-fat spread on the other slice comes from the fact that for the Morava K–theories operations are dual to co-operations. We find that our three types of operation are sequentially placed in this sandwich: our analysis for stable operations depends almost entirely on ordinary K–theory, that for additive operations relies on both parts, whilst for all unstable operations we practically forget ordinary K–theory altogether.

Outline: This paper proceeds as follows:

Conventions: We work in the homotopy category of spaces equivalent to a CW–complex and in the associated homotopy category of spectra. We are somewhat lax in our language and will refer to morphisms in our categories as just "maps". As cohomology does not really "see" the difference between a space and its suspension spectrum, neither will we and shall trust to context to distinguish if necessary. The only place where cohomology does sees the difference is in the question of absolute versus reduced cohomology; to ensure that there is no confusion we shall follow [Boa95] and redundantly write the cohomology and homology of spectra in the notation of the reduced theory.

Following [Boa95] and [BJW95] we grade homology *negatively* so that $E_k(X)$ is in degree $-k$. This is to allow us to grade cohomology positively.

We regard the equivalence between cohomology classes and maps into the representing space or spectrum as being so tight as to be, in light of other equivalences, essentially an equality; which indeed it is for spectra. However we shall employ the languages of both maps and classes as seems more natural to a specific occurrence.

For a multiplicative generalised cohomology theory $E^*(-)$ we shall write the coefficient ring as E^*, the representing spectrum as E, and the representing spaces as $\underline{E}_k$.

For an H–space X with H–map μ we define the *primitives*:

$$PE^*(X) := \{\alpha \in E^*(X) : \mu^*\alpha = {p_1}^*\alpha + {p_2}^*\alpha \text{ in } E^*(X \times X)\}$$

and the *indecomposables*:

$$QE_*(X) := \operatorname{coker}[\mu_* - {p_1}_* - {p_2}_* : E_*(X \times X) \to E_*(X)].$$

For a suitable multiplicative generalised cohomology theory $E^*(-)$ we identify operations, classes, and linear functionals. :

1. Cohomology operations:

$$r : E^*(-) \to E^{*+h}(-), r_k : E^k(-) \xrightarrow{+} E^{k+h}(-), r_k : E^k(-) \to E^{k+h}(-);$$

2. Cohomology classes:

$$\rho \in E^k(E, o), \rho_k \in PE^{k+h}(\underline{E}_k), \rho_k \in E^{k+h}(\underline{E}_k);$$

3. Linear functionals on co-operations:

$$\mathfrak{r} : E_*(E, o) \to E^*, \mathfrak{r}_k : QE_*(\underline{E}_k) \to E^*, \mathfrak{r}_k : E_*(\underline{E}_k) \to E^*.$$

"Suitable" cohomology theories include ordinary K–theory, K–theory with mod p coefficients, and the Morava K–theories.

2 From Atiyah to Morava

In this section we consider the route from ordinary K–theory to the first Morava K–theory.

2.1 Mod p K–Theory

As $\mathbb{F}_p$ is not flat as a $\mathbb{Z}$–module we cannot just tensor ordinary K–theory with $\mathbb{F}_p$ to produce K–theory with coefficients in $\mathbb{F}_p$. Therefore we have to resort to the more general method of introducing coefficients: smashing with a Moore spectrum. Fortunately the Moore spectrum for $\mathbb{F}_p$ is simple and amenable to analysis.

Definition 2.1 *Let K denote the spectrum for ordinary K–theory. Let $M\mathbb{F}_p$ be the Moore spectrum for the finite field $\mathbb{F}_p$. Define the spectrum $Kp := M\mathbb{F}_p \wedge K$ with associated cohomology and homology theories:*

$$Kp^*(X, o) := [X, Kp]^*,$$
$$Kp_*(X, o) := \pi_*^S(Kp \wedge X).$$

An important relationship between this and ordinary K–theory is summed up in the following result:

Proposition 2.2 *There are natural long exact sequences:*

$$\to K^i(X, o) \xrightarrow{p} K^i(X, o) \to Kp^i(X, o) \to K^{i+1}(X, o) \xrightarrow{p} K^{i+1}(X, o) \to,$$
$$\to K_i(X, o) \xrightarrow{p} K_i(X, o) \to Kp_i(X, o) \to K_{i-1}(X, o) \xrightarrow{p} K_{i-1}(X, o) \to .$$

Hence there are natural short exact sequences:

$$K^i(X, o)/p \to Kp^i(X, o) \to \ker\left(p : K^{i+1}(X, o) \to K^{i+1}(X, o)\right),$$
$$K_i(X, o)/p \to Kp_i(X, o) \to \ker\left(p : K_{i-1}(X, o) \to K_{i-1}(X, o)\right).$$

Proof. A free resolution of $\mathbb{F}_p$ is $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p$. Hence the Moore spectrum of $\mathbb{F}_p$ is defined as the cofibre of the map $S \xrightarrow{p} S$. Smashing the cofibre sequence $S \xrightarrow{p} S \to M\mathbb{F}_p$ with K results in the cofibre sequence:

$$K \xrightarrow{p} K \to Kp.$$

Since, for spectra, cofibration sequences are the same as fibration sequences up to sign, mapping into this sequence products a long exact sequence which, by definition, is the first long exact sequence. For homology we smash the cofibre sequence with E to get a cofibre sequence:

$$K \wedge E \xrightarrow{p} K \wedge E \to Kp \wedge E$$

which, upon taking stable homotopy, results in the homology long exact sequence. As both of these hold for spectra, they also hold for spaces. □

From these exact sequences we easily deduce the following:

Corollary 2.3 *1. If $K^*(X,o)$, resp. $K_*(X,o)$, is free then $Kp^*(X,o) \cong K^*(X,o)/p$, resp. $Kp_*(X,o) \cong K_*(X,o)/p$.*

2. If $K^*(X,o)$, resp. $K_*(X,o)$, is an $\mathbb{F}_p$–vector space then there is a short exact sequence:

$$K^i(X,o) \to Kp^i(X,o) \to K^{i+1}(X,o),$$
$$\text{resp. } K_i(X,o) \to Kp_i(X,o) \to K_{i-1}(X,o).$$

3. The coefficient ring for Kp^* is $\mathbb{F}_p[u,u^{-1}]$ with $|u| = -2$.

2.2 The First Morava K–Theory

The cohomology theory $Kp^*(-)$ splits via the Adams' idempotents into $(p-1)$–copies of a single theory called the *first Morava K–theory*, written $K(1)^*(-)$. This has coefficient ring:

$$K(1)^* = \mathbb{F}_p[v_1, {v_1}^{-1}]$$

with $|v_1| = -2(p-1)$. This is identified with a subring of Kp^* by setting $u^{p-1} = v_1$. The splittings of $Kp^*(X,o)$ and $Kp_*(X,o)$ can be succinctly expressed as isomorphisms of left Kp^*–modules:

$$Kp^*(X,o) \cong Kp^* \otimes_{kl^*} K(1)^*(X,o), \qquad (2.1)$$
$$Kp_*(X,o) \cong Kp^* \otimes_{kl^*} K(1)_*(X,o). \qquad (2.2)$$

For cohomology, this is an isomorphism of Kp^*–algebras.

At the spectrum level there is an equivalence:

$$Kp \simeq \bigvee_{j=0}^{p-2} \Sigma^{2j} K(1). \qquad (2.3)$$

The periodicity elements in Kp^* and $K(1)^*$ induce equivalences ξu and ξv_1 on the corresponding spectra. For clarity in the following we shall write these as degree zero maps $\xi u : \Sigma^2 Kp \to Kp$ and $\xi v_1 : \Sigma^{2(p-1)} K(1) \to K(1)$. These are related as follows: ξu is the composition:

$$\Sigma^2 K(1) \vee \cdots \vee \Sigma^{2(p-2)} K(1) \vee \Sigma^{2(p-1)} K(1) \xrightarrow{1 \vee \cdots \vee 1 \vee \xi v_1} \Sigma^2 K(1) \vee \cdots \vee \Sigma^{2(p-2)} K(1) \vee K(1)$$
$$\to K(1) \vee \cdots \vee \Sigma^{2(p-2)} K(1).$$

2.3 Stable Operations and Co-operations

The definition of the spectrum Kp provides an obvious way to make stable operations of ordinary K–theory into operations of Kp–theory:

$$\{K,K\}^j \to \{Kp,Kp\}^j, \quad \alpha \to 1_{M\mathbb{F}_p} \wedge \alpha.$$

It is obvious that this must factor through the mod p reduction of $\{K,K\}^j$ and that it is an algebra map with composition as multiplication.

Theorem 2.4 *The induced map $K^0(K,o)/p \to Kp^0(Kp,o)$ is an isomorphism of $\mathbb{F}_p$–algebras.*

Proof. We take the diagram $K \xrightarrow{\alpha} K$ and smash it with the diagram $S \to M\mathbb{F}_p$. This results in the commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & K \\ \downarrow & & \downarrow \\ Kp & \xrightarrow{1_{M\mathbb{F}_p} \wedge \alpha} & Kp \end{array}$$

This shows that the image of α under the natural map $K^*(K, o) \to Kp^*(K, o)$ is the same as the image of $1_{M\mathbb{F}_p} \wedge \alpha$ under the natural map $Kp^*(Kp, o) \to Kp^*(K, o)$.

In other words, the following diagram commutes:

$$\begin{array}{ccc} K^*(K, o) & \longrightarrow & Kp^*(K, o) \\ & \searrow & \downarrow \\ & & Kp^*(Kp, o) \end{array}$$

wherein the vertical and horizontal maps come from the natural map $K \to Kp$ and the diagonal map is $\alpha \to 1_{M\mathbb{F}_p} \wedge \alpha$.

The horizontal and vertical maps in this diagram fit into various exact sequences. As $K^*(K, o)$ is free the horizontal becomes an isomorphism after reduction mod p. The lower map is part of the long exact sequence:

$$\to Kp^{i-1}(K, o) \xrightarrow{p} Kp^{i-1}(K, o) \to Kp^i(Kp, o) \to Kp^i(K, o) \xrightarrow{p} Kp^i(K, o) \to$$

coming from the cofibre sequence $K \xrightarrow{p} K \to Kp$. As the p–map is zero on $Kp^*(-)$ this splits into short exact sequences:

$$Kp^{i-1}(K, o) \to Kp^i(Kp, o) \to Kp^i(K, o).$$

Finally, as $K^*(K, o)$, and thus also $Kp^*(K, o)$, is concentrated in even degree, the map $Kp^0(Kp, o) \to Kp^0(K, o)$ is an isomorphism. Hence the map $K^0(K, o)/p \to Kp^0(Kp, o)$ is an isomorphism. □

There is a similar result for co-operations. This time the map is more direct: the map $K \to Kp$ defines

$$K_*(K, o) = \pi_*^S(K \wedge K) \to \pi_*^S(Kp \wedge Kp) = Kp_*(Kp, o).$$

If we apply the two maps $K \to Kp$ sequentially rather than simultaneously we see that this map factors through $Kp_*(K, o)$ or, equivalently, $K_*(Kp, o)$. These maps preserve all the structure of stable co-operations.

Theorem 2.5 *The map $K_0(K, o)/p \to Kp_0(Kp, o)$ is an isomorphism.*

Proof. As for cohomology, $K_*(K, o)/p \cong Kp_*(K, o)$. The Whitehead trick allows us to identify this with $K_*(Kp, o)$ which we feed back into corollary 2.3 to see that there is a short exact sequence:

$$K_i(Kp, o) \to Kp_i(Kp, o) \to K_{i-1}(Kp, o).$$

As $K_*(K, o)$, and thus also $K_i(Kp, o)$, is concentrated in even degrees the map $K_0(Kp, o) \to Kp_0(Kp, o)$ is thus an isomorphism as required. □

2.4 Additive Operations and Co-operations

We would like to prove an analogous result for additive co-operations, namely that additive co-operations of degree zero in mod p K–theory come from ordinary K–theory. This is true, but the argument is slightly more complicated than for the stable situation.

Firstly we record the following straightforward result:

Lemma 2.6 *The map $K \to Kp$ induces an isomorphism $(QK_*(\underline{K}_k))/p \to QKp_*(\underline{K}_k)$ for all k.*

Proof. As $K_*(\underline{K}_k)$ is free, $Kp_*(\underline{K}_k) \cong K_*(\underline{K}_k)/p$. The result follows since quotienting to indecomposables commutes with quotienting by the p–ideal. □

The map $K \to Kp$ defines maps of representing spaces: $\underline{K}_k \to \underline{Kp}_k$. The maps are maps of H–spaces and so we have ring maps:

$$Kp_*(\underline{K}_k) \to Kp_*(\underline{Kp}_k)$$

and hence a map of indecomposables:

$$QKp_*(\underline{K}_*) \to QKp_*(\underline{Kp}_*).$$

Following [BJW95] we regrade the additive co-operations of a generalised cohomology theory $E^*(-)$ writing $Q(E)_i^k$ for $QE_i(\underline{E}_k)$ but with total degree $k - i$. This clearly makes no difference for $\underline{E}_0$. This notation together with lemma 2.6 allows us to rewrite the above map as a bialgebra map:

$$Q(K)_*^*/p \to Q(Kp)_*^*.$$

Theorem 2.7 *This map is an isomorphism in even degree.*

To prove this we will need to sandwich Kp–theory between K–theory and $K(1)$–theory. Essentially we shall look at the map $Q(K)_*^*/p \to Q(K(1))_*^*$ and use that to deduce the result we need. The first stage, therefore, is relating $Q(K(1))_*^*$ to $Q(Kp)_*^*$.

Proposition 2.8 *There is an isomorphism of bialgebras and Kp^*–bimodules:*

$$Q(Kp)_*^* \cong Kp^* \otimes_{K(1)^*} Q(K(1))_*^* \otimes_{K(1)^*} Kp^*.$$

Proof. Applying the zeroth space functor to the identity (2.3) converts the wedge to a director product yielding an equivalence of H–spaces:

$$\underline{Kp}_k \simeq \underline{K(1)}_k \times \underline{K(1)}_{k+2} \times \cdots \times \underline{K(1)}_{k+2(p-2)}.$$

The periodicity map has the same form as before.

For an H–space X, the indecomposables of $Kp_*(X)$ and of $K(1)_*(X)$ are related via:

$$QKp_*(X) \cong Kp^* \otimes_{K(1)^*} QK(1)_*(X).$$

Also we know from [BJW95, Lemma 4.5] that the functor Q preserves finite products. These show that:

$$\begin{aligned} QKp_*(\underline{Kp}_k) &\cong Kp^* \otimes_{K(1)^*} QK(1)_*(\underline{Kp}_k) \\ &\cong Kp^* \otimes_{K(1)^*} \Big(\bigoplus_{j=0}^{p-2} QK(1)_*(\underline{K(1)}_{k+2j}) \Big). \end{aligned}$$

Hence:

$$Q(Kp)^k_* \cong Kp^* \otimes_{K(1)^*} \Big(\bigoplus_{j=0}^{p-2} Q(K(1))^{k+2j}_* \Big).$$

Now the right action of $u \in Kp^*$ acts by the right action of v_1 on the last factor in the direct sum and then cyclically permuting all the factors. Taking into account the various degrees involved yields:

$$Q(Kp)^*_* \cong Kp^* \otimes_{K(1)^*} \bigoplus_{j=0}^{p-2} Q(K(1))^*_* u^j$$

Hence as Kp^*–bimodules:

$$Q(Kp)^*_* \cong Kp^* \otimes_{K(1)^*} Q(K(1))^*_* \otimes_{K(1)^*} Kp^*.$$

This is also an isomorphism of bialgebras as all the structure on Kp derives from that on $K(1)$. □

The other piece of the jigsaw is the notion of a universal Chern class in a multiplicative generalised cohomology theory, $E^*(-)$. A universal Chern class is an element $x \in E^2(\mathbb{CP}^\infty)$ which restricts to the canonical generator of $E^*(S^2)$. If such exists lots of pleasant consequences follow.

The first pleasant consequence is an isomorphism $E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!]$. The second pleasant consequence is the description of $E_*(\mathbb{CP}^\infty)$ as the free E^*–module on generators $\beta_i \in E_{2i}(\mathbb{CP}^\infty)$, dual to x^i. The element $x \in E^2(\mathbb{CP}^\infty)$, thought of as a map $\mathbb{CP}^\infty \to \underline{E}_2$, pushes these forward to define classes $b^E_i \in E_{2i}(\underline{E}_2)$ after which they quotient to $Q(E)^2_{2i}$.

All of this structure is natural in the theory $E^*(-)$. Moreover there is a certain amount of independence of choice: the Chern class $x \in E^2(\mathbb{CP}^\infty)$ is not unique but any other will have the form:

$$\tilde{x} = x + c_1 x^2 + c_2 x^3 + \cdots$$

for some $c_j \in E^{-2j}$. The conditions on a Chern class force the coefficients of x^0 and x to be as written. Dualising this shows that $\tilde{\beta}_i = \beta_i$ modulo lower terms. From this we deduce:

Lemma 2.9 *For a given n, the sub-bimodule of $Q(E)^*_*$ generated by products of the b^E_i for $i \leq n$ is independent of the choice of Chern class.*

We consider two Chern classes for K–theory with mod p coefficients. The first comes from ordinary K–theory and so the map $Kp_*(\mathbb{CP}^\infty) \to Q(Kp)^*_*$ makes the following diagram commute:

$$\begin{array}{ccc} K_*(\mathbb{CP}^\infty) & \longrightarrow & Q(K)^2_* \\ \downarrow & & \downarrow \\ Kp_*(\mathbb{CP}^\infty) & \longrightarrow & Q(Kp)^2_*. \end{array}$$

It is a well-known result that the upper horizontal line is an isomorphism. The first vertical map becomes an isomorphism after quotienting by the p–ideal.

The second Chern class for K–theory with mod p coefficients comes from the first Morava K–theory. This therefore makes the following diagram commute:

$$\begin{array}{ccc} K(1)_*(\mathbb{CP}^\infty) & \longrightarrow & Q(K(1))^2_* \\ \downarrow & & \downarrow \\ Kp_*(\mathbb{CP}^\infty) & \longrightarrow & Q(Kp)^2_*. \end{array}$$

We can beef-up the top line to make the coefficient ring Kp^* rather than $K(1)^*$:

$$\begin{array}{ccc} Kp^* \otimes_{K(1)^*} K(1)_*(\mathbb{CP}^\infty) & \longrightarrow & \left(Kp^* \otimes_{K(1)^*} Q(K(1))^*_* \otimes_{K(1)^*} Kp^*\right)^2_* \\ \downarrow & & \\ Kp_*(\mathbb{CP}^\infty) & \longrightarrow & Q(Kp)^2_*. \end{array}$$

For this we know that both vertical maps are isomorphisms.

We shall use lemma 2.9 to compare these two squares. In order to use it we need to determine the sub-bimodules spanned by products of the b_i in each case. For this we use the following pieces of information from [BJW95, Section 16]:

1. In ordinary K–theory, the product of two of the $b_i b_j$ is given by:

$$b_i b_j = \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i} \binom{i}{k} u^k b_{i+j-k} v^{-1}.$$

2. In $K(1)$–theory, the b_i for i not a prime power can be written in terms of b_j of lower order; whilst the $b_{(i)} := b_{p^i}$ satisfy:

$$b_{(i)}{}^{p-1} = v_1{}^{p^i} b_{(i)} w_1{}^{-1}.$$

   These are the only relations amongst the $b_{(i)}$ themselves. There are other relations but these involve other elements and so do not affect the sub-bimodule generated by the b_i.

3. In $K(1)$–theory, the even part of $Q(K(1))^*_*$ is spanned by products of the b_i.

Lemma 2.10 *Let $n \in \mathbb{N}$. The sub-bimodules of $Q(K)^*_*/p$ and of $Q(K(1))^*_*$ generated by the family $\{b_1, \dots, b_{p^n-1}\}$ have the same rank.*

Proof. We start with ordinary K–theory. We shall show that the product of two terms in the set $\{b_1, \dots, b_{p^n-1}\}$ is contained within the linear span of that set. As this set is linearly independent – since it is in $K_*(\mathbb{CP}^\infty)/p$ – this will prove that the sub-bimodule spanned by products of this family has rank p^n (we include the empty product to get the unit).

The coefficient in the expression for $b_i b_j$ is:

$$\frac{(i+j-k)!}{(i-k)!(j-k)!k!}.$$

For $i, j < p^n$ as $k \leq \min(i, j)$ we have $(i-k), (j-k), k < p^n$. Therefore, by corollary A.4, if $i + j - k \geq p^n$ the coefficient is divisible by p and so this term does not contribute.

In $K(1)$–theory, the sub-bimodule of $Q(K(1))^*_*$ spanned by powers of $\{b_1, \ldots, b_{p^n-1}\}$ is the same as that spanned by powers of $\{b_{(0)}, \ldots, b_{(n-1)}\}$ since each b_i for i not a power of p can be expressed in terms of lower order b_j and thus of lower order b_j with j a power of p. The relations show that a basis for this sub-bimodule of $Q(K(1))^*_*$ is:

$$\prod_{j=0}^{n-1} {b_{(j)}}^{m_j}$$

with $0 \leq m_j \leq p-1$. There are p^n elements in this basis and hence the sub-bimodule has rank p^n. □

From this we can prove theorem 2.7:

Proof of theorem 2.7. From the description of $Q(Kp)^*_*$ in terms of $Q(K(1))^*_*$ we see that the even part of $Q(K(1))^*_*$ determines that of $Q(Kp)^*_*$. Thus the even part of $Q(Kp)^*_*$ is spanned by powers of the b_i. Combining lemma 2.9 with lemma 2.10 we see that the map $Q(K)^*_* \to Q(Kp)^*_*$ induces an isomorphism on the sub-bimodules generated by products of $\{b_1, \ldots, b_{p^n-1}\}$. As the union of these is the whole thing on each side in even degree the map is an isomorphism. □

3 Numerical Polynomials

Now that we know how additive and stable co-operations in K–theory with mod p coefficients relate to those in ordinary K–theory we can begin our analysis of these objects.

As is well-known, stable and additive co-operations of degree zero in ordinary K–theory are closely linked to *numerical polynomials*:

Definition 3.1 *The bialgebras of* numerical polynomials, *A, and* stably numerical Laurent polynomials, *B, are defined as:*

$$A := \operatorname{Int}(\mathbb{Z}) = \{f(w) \in \mathbb{Q}[w] : f(\mathbb{Z}) \subseteq \mathbb{Z}\},$$
$$B := A[w^{-1}] = \{f(w) \in \mathbb{Q}[w, w^{-1}] : f(k) \in \mathbb{Z}[\frac{1}{k}], k \in \mathbb{Z} \setminus \{0\}\}.$$

The bialgebra structure comes from $\mathbb{Q}[w, w^{-1}]$ with the standard multiplication and the co-multiplication with w group-like; i.e. $\Delta w = w \otimes w$.

The relationship between these rings and K–theory is:

Proposition 3.2 *There are isomorphisms of bialgebras:*

$$A \cong Q(K)^0_0,$$
$$B \cong K_0(K, o).$$

From which we deduce:

Corollary 3.3 *There are isomorphisms of bialgebras:*

$$A/p \cong Q(Kp)^0_0,$$
$$B/p \cong Kp_0(Kp, o).$$

The description of $Q(K)^0_0$ as A provides a convenient language for describing how operations act. An additive operation $\phi : \underline{K}_0 \to \underline{K}_0$ defines a degree zero additive map of homotopy groups: $\phi_* : \pi_*(\underline{K}_0) \to \pi_*(\underline{K}_0)$. Since $\pi_k(\underline{K}_0)$ is $\mathbb{Z}$ in even degrees and zero in odd the map ϕ_* is completely determined by the integer sequence $(\lambda_k(\phi))_{k\in\mathbb{Z}}$ where ϕ_* acts on $\pi_{2k}(\underline{K}_0)$ by multiplication by $\lambda_k(\phi)$. For K–theory, additive operations are dual to additive co-operations so ϕ also defines a $\mathbb{Z}$–linear map $A \cong Q(K)^0_0 \to \mathbb{Z}$. This map satisfies $\phi(w^k) = \lambda_k(\phi)$. That this completely determines the linear map comes from the fact that A is free and its rationalisation has basis $\{w^k\}$.

Once we know that A/p is the additive co-operations of degree zero in K–theory with mod p coefficients we shall want to play the same game. To do so we must replace every occurrence of $\mathbb{Z}$ by $\mathbb{F}_p$ in the above. This proves problematic at the last stage because the monomials do not form a basis for A/p over $\mathbb{F}_p$, and we cannot play a similar trick to the integral case as we are already over a field. To get around this problem we factor through the p–localisation.

As $\mathbb{Z}_{(p)}$ is flat over $\mathbb{Z}$ taking ordinary K–theory and localising the groups with respect to p does produce a generalised cohomology theory. The additive co-operations of degree zero in this theory are thus the p–localisation of the additive co-operations of ordinary K–theory: $A \otimes \mathbb{Z}_{(p)}$. As $\mathbb{Z}_{(p)}$ is a subring of $\mathbb{Q}$ we can play the same game with additive operations in K–theory with p–local coefficients as with ordinary K–theory. Thus the λ–sequence of an additive operation determines the $\mathbb{Z}_{(p)}$–linear map $A_{(p)} \to \mathbb{Z}_{(p)}$.

The operation that we are interested in is the Adams' idempotent which defines the Adams' splitting of K–theory with p–local coefficients. The effect of this operation on the homotopy groups is simple: it leaves alone those groups corresponding to degree divisible by $2(p-1)$ and kills all others.

Now the dual of a coalgebra acts on the coalgebra via:

$$X' \otimes X \xrightarrow{1\otimes\Delta} X' \otimes X \otimes X \xrightarrow{\langle\cdot,\cdot\rangle\otimes 1} R \otimes X \to X.$$

Applying this to the Adams' idempotent defines the following map $A_{(p)} \to A_{(p)}$:

$$w^k \to \begin{cases} w^k & \text{if } p-1 \text{ divides } k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the Adams' summand of $A_{(p)}$ is $A_{(p)} \cap \mathbb{Q}[w^{p-1}]$. Since quotienting by the p–ideal factors through localising with respect to it we get a similar splitting of A/p. We do not yet know that this is the additive co-operations in K–theory with mod p coefficients but once we do we will be interested in the sub-bialgebra obtained by quotienting $A \cap \mathbb{Q}[w^{p-1}]$. Therefore we define:

Definition 3.4 *Define the bialgebras $\mathcal{A}$ and $\mathcal{B}$ by:*

$$\begin{aligned} \mathcal{A} &:= A \cap \mathbb{Q}[w^{p-1}], \\ \mathcal{B} &:= B \cap \mathbb{Q}[w^{p-1}, w^{1-p}] = A[w^{1-p}]. \end{aligned}$$

The preceding discussion shows that:

Proposition 3.5 *There are isomorphisms of bialgebras:*

$$\begin{aligned} \mathcal{A}/p &\cong Q(K(1))_0^0, \\ \mathcal{B}/p &\cong K(1)_0(K(1), o). \end{aligned}$$

As we examine A/p and B/p we shall keep an eye on the sub-bialgebras $\mathcal{A}/p$ and $\mathcal{B}/p$.

We shall write the equivalence class in A/p represented by a polynomial $f(w)$ as $\overline{f}(w)$ and in B/p as $\widetilde{f}(w)$. We shall write an element of $\mathbb{Z}/k\mathbb{Z}$ represented by the integer n as $[n]_k$.

Our analysis of these bialgebras can be divided into two parts: an analysis of how the inclusion $A \to B$ behaves under quotienting and a quest for reasonable bases of A/p and B/p.

3.1 From A to B

Our main theorem of this section is:

Theorem 3.6 *The inclusion $A \to B$ induces a surjection $A/p \to B/p$ and is an isomorphism when restricted to the ideal $\overline{w}^{p-1}A/p$. Hence the map $A/p \to B/p$ has a left inverse which is a homomorphism of bialgebras. The same holds true when A is replaced by $\mathcal{A}$ and B by $\mathcal{B}$.*

We shall see that in A/p the ideal generated by $\overline{w}^{p-1}$ is the same as the ideal generated by $\overline{w}$. However we stick with $\overline{w}^{p-1}$ as that also works for $\mathcal{A}/p$.

To prove this we need some facts about the structure of A/p and B/p.

Proposition 3.7 *1. The ideal $pA \subseteq A$ is characterised by the condition:*

$$f(w) \in pA \text{ if and only if } f(\mathbb{Z}) \subseteq p\mathbb{Z}.$$

2. *The element $\overline{w}^{p-1} \in A/p$ is an idempotent and is group-like.*

3. *In B/p, $\widetilde{w}^{p-1} = \widetilde{1}$.*

4. *In both A/p and B/p every element satisfies $a^p = a$.*

Proof. 1. This is straightforward. As $A \subseteq \mathbb{Q}[w]$ we can always divide by p in $\mathbb{Q}[w]$. Clearly $f(w) \in pA$ if and only if $\frac{1}{p} f(w) \in A$. Now for any $c \in \mathbb{Q}$ and $g(w) \in \mathbb{Q}[w]$, $(cg)(\mathbb{Z}) \subseteq c(g(\mathbb{Z}))$. Hence $\frac{1}{p} f(\mathbb{Z}) \subseteq \mathbb{Z}$ if and only if $f(\mathbb{Z}) \subseteq p\mathbb{Z}$.

2. By Fermat's little theorem the polynomial $\frac{1}{p}(w^p - w)$ lies in A. Hence $w^p - w \in pA$ and so $\overline{w}^p = \overline{w}$. Thus:

$$(\overline{w}^{p-1})^2 = \overline{w}^{2(p-1)} = \overline{w}^p \cdot \overline{w}^{p-2} = \overline{w} \cdot \overline{w}^{p-2} = \overline{w}^{p-1}.$$

The coproduct is obvious from $\Delta w = w \otimes w$.

3. Since $\frac{1}{p}(w^p - w) \in A$, $\frac{1}{p}(w^{p-1} - 1) \in B$. Hence $\widetilde{w}^{p-1} = \widetilde{1}$.

4. Clearly A is closed under composition and thus, by Fermat's little theorem, if $f(w) \in A$ then $\frac{1}{p}(f(w)^p - f(w)) \in A$ and hence $f(w)^p - f(w) \in pA$ so $\overline{f}(w)^p = \overline{f}(w)$.

The same argument will work for B/p once we have established that if $f(w) \in B$ then $\frac{1}{p}(f(w)^p - f(w)) \in B$. For this we use the characterisation of B as those $f(w) \in \mathbb{Q}[w, w^{-1}]$ for which $f(n) \in \mathbb{Z}[\frac{1}{n}]$ for all $n \in \mathbb{Z} \setminus \{0\}$. From this we see that we need to show that, for any n, $\mathbb{Z}[\frac{1}{n}]$ is mapped to itself by the map $x \to \frac{1}{p}(x^p - x)$, considered as a map $\mathbb{Q} \to \mathbb{Q}$.

There are two cases to consider: either p divides n or it does not. It it does divide n then there is some r such that $n = pr$, whence for $a \in \mathbb{Z}[\frac{1}{n}]$:

$$\tfrac{1}{p}(a^p - a) = \tfrac{r}{n}(a^p - a) \in \mathbb{Z}[\frac{1}{n}].$$

If p does not divide n then $n^{p-1} \equiv 1 \mod p$ by Fermat's little theorem. Therefore for any $b \in \mathbb{Z}$ and $k \in \mathbb{N}$, $b^p - bn^{k(p-1)} \equiv 0 \mod p$. A general element of $\mathbb{Z}[\frac{1}{n}]$ is of the form b/n^k for some $b \in \mathbb{Z}$ and $k \in \mathbb{N}$, so:

$$\frac{1}{p}\left(\frac{b^p}{n^{pk}} - \frac{b}{n^k}\right) = \frac{1}{n^{pk}} \cdot \frac{1}{p}\left(b^p - bn^{k(p-1)}\right)$$

which lies in $\mathbb{Z}[\frac{1}{n}]$ as the second part is an integer.

□

The properties of $\overline{w}^{p-1}$ prove the following result:

Corollary 3.8 *There is a split short exact sequence of bialgebras:*

$$\overline{w}^{p-1}A/p \to A/p \to (A/p)/(\overline{w}^{p-1}A/p).$$

The splitting $A/p \to \overline{w}^{p-1}A/p$ is the bialgebra homomorphism given by multiplication by $\overline{w}^{p-1}$. The splitting $(A/p)/(\overline{w}^{p-1}A/p)$ has image $(1-\overline{w}^{p-1})A/p$ and is a homomorphism of algebras but not of bialgebras.

We can now prove theorem 3.6.

Proof of theorem 3.6. We shall start by proving that the map $A/p \to B/p$ is surjective. Let $f(w) \in B$. There is some $k \in \mathbb{N}$ such that $g(w) := w^k f(w) \in A$. As $w \in A$ we may increase k and so assume that $(p-1)$ divides k. Therefore $\widetilde{w}^k = \widetilde{1}$ in B/p and so $\widetilde{g}(w) = \widetilde{f}(w)$. Since $g(w) \in A$, every element of B/p can be represented by an element of A whence the map $A/p \to B/p$ is demonstrated to be surjective.

Since $\widetilde{w}^{p-1} = \widetilde{1}$ the kernel of the map $A/p \to B/p$ contains the ideal generated by $(\overline{1} - \overline{w}^{p-1})$. This is a non-trivial element in A/p. To complete the proof we need to show that this ideal contains the kernel of $A/p \to B/p$.

Let $f(w) \in A$ represent a non-zero element in the kernel. Thus $f(w) \notin pA$ but $f(w) \in pB$. Hence $\frac{1}{p}f(w) \in B$ and so there is some $k \in \mathbb{N}$, which we may assume to be divisible by $(p-1)$ as before, such that $w^k \frac{1}{p} f(w) \in A$. Hence $w^k f(w) \in pA$ and so $\overline{w}^k \overline{f}(w) = 0$. As k is a multiple of $(p-1)$ and $\overline{w}^{p-1}$ is an idempotent this means that $\overline{w}^{p-1}\overline{f}(w) = 0$. Hence $\overline{f}(w) = (\overline{1} - \overline{w}^{p-1})\overline{f}(w)$ so $\overline{f}(w)$ lies in the ideal generated by $(\overline{1} - \overline{w}^{p-1})$ as required.

This proof goes through essentially unchanged for $\mathcal{A}/p$ and $\mathcal{B}/p$ since we are only ever multiplying by powers of w^{p-1}. □

3.2 From Newton to Fermat, part A

In this section we shall consider various bases for A/p and $\mathcal{A}/p$. The ring A is free as a $\mathbb{Z}$–module with basis the Newton polynomials:

$$b_i(w) := \binom{w}{i} = \frac{w(w-1)\cdots(w-i+1)}{i!}.$$

As A is free this quotients to a basis for A/p. However whilst this basis is a very natural one it will not prove the most useful for our purposes. We therefore use it to devise a test for determining whether or not a given family of polynomials in A/p quotients to a basis for A/p. The family that we shall focus on is closely related to Fermat's little theorem.

Our test will use the p–adic valuation function, ν_p. The property that we shall use is that an integer k quotients to a non-zero element in $\mathbb{F}_p$ if and only if $\nu_p(k) = 0$. For more on the p–adic valuation function see the appendix.

Proposition 3.9 *Let $\{f_n(w)\}$ be a sequence of polynomials in A such that:*

1. *$f_n(w)$ is a polynomial of degree n,*
2. *the leading coefficient, a_n, of $f_n(w)$ satisfies $\nu_p(a_n) + \nu_p(n!) = 0$,*

then the linear span of $\{\overline{f_0}(w), \ldots, \overline{f_n}(w)\}$ is the same as that of $\{\overline{b_0}(w), \ldots, \overline{b_n}(w)\}$ and thus $\{\overline{f_n}(w)\}$ is a basis for A/p.

Proof. We shall prove that the linear space of $\{\overline{b_0}(w), \ldots, \overline{b_{n-1}}(w), \overline{f_n}(w)\}$ is the same as that of $\{\overline{b_0}(w), \ldots, \overline{b_n}(w)\}$. The main result will then follow by induction, with initial step provided by the case $n = 0$.

As the $\{b_i(w)\}$ form a basis for A as a $\mathbb{Z}$–module there are integers $\{c_i\}$, all but a finite number zero, such that:

$$f_n(w) = \sum c_i b_i(w).$$

This identity also holds in $\mathbb{Q}[w]$ and quotients down to A/p. Now $b_i(w)$ is a polynomial of degree i. Therefore if $k \in \mathbb{N}$ is the maximum of the (finite) set $\{i : c_i \neq 0\}$, $\sum c_i b_i(w)$ is a polynomial of degree k. As $f_n(w)$ is a polynomial of degree n we must have $k = n$. Thus:

$$f_n(w) = c_n b_n(w) + c_{n-1} b_{n-1}(w) + \cdots + c_0 b_0(w).$$

Equating the coefficients of w^n on both sides leads to:

$$a_n = \frac{c_n}{n!}.$$

Taking p–adic valuations (noting that all terms are non-zero) leads to:

$$\nu_p(c_n) = \nu_p(a_n) + \nu_p(n!) = 0.$$

Hence $[c_n]_p$ is invertible in $\mathbb{F}_p$. Thus the linear span of $\{\overline{b_0}(w), \ldots, \overline{b_n}(w)\}$ is the same as that of $\{\overline{b_0}(w), \ldots, \overline{b_{n-1}}(w), \overline{f_n}(w)\}$ as required. □

Corollary 3.10 *Let $b_{(i)}(w) = b_{p^i}(w)$. Then A/p is generated by the family $\{\overline{b_{(i)}}(w)\}$ subject only to the relations $\overline{b_{(i)}}(w)^p = \overline{b_{(i)}}(w)$.*

Proof. The $\overline{b_{(i)}}(w)$ satisfy those relations because, by proposition 3.7, all elements of A/p satisfy those relations.

We recall from the appendix some of the properties of the p–expansions of natural numbers. Any $n \in \mathbb{N}$ has a unique expansion of the form $n = \sum_{j\geq 0} n_j p^j$ with $0 \leq n_j \leq p - 1$, all but a finite number of terms zero. The set $\{0, \ldots, p^m - 1\}$ corresponds to the expansions which are zero after at most the first m terms (the sequence starts at the zeroth term). Finally, from proposition A.3:

$$\nu_p(n!) = \sum_{j\geq 0} n_j \frac{p^j - 1}{p - 1}.$$

For $n \in \mathbb{N}$ with p–expansion $\sum n_j p^j$ let $f_n(w)$ denote the polynomial:

$$f_n(w) := \prod_{j\geq 0} b_{(j)}(w)^{n_j}.$$

This involves only the first m of the $b_{(j)}(w)$ if and only if $n < p^m$. Hence the linear span of $\{\overline{f_0}(w), \ldots, \overline{f_{p^m-1}}(w)\}$ is the same as the sub-algebra generated by $\{\overline{b_{(0)}}(w), \ldots, \overline{b_{(m-1)}}(w)\}$.

As $b_{(j)}(w)$ is a polynomial of degree p^j, $f_n(w)$ is a polynomial of degree $\sum np^j = n$. Its leading coefficient is:

$$\prod_{j\geq 0}\left(\frac{1}{p^j!}\right)^{n_j}.$$

The p–adic valuation of this is:

$$\nu_p\left(\prod_{j\geq 0}\left(\frac{1}{p^j!}\right)^{n_j}\right) = -\sum_{j\geq 0} n_j \nu_p(p^j!) = -\sum_{j\geq 0} n_j \frac{p^j-1}{p-1} = -\nu_p(n!).$$

Hence the family $\{f_n(w)\}$ quotients to a basis for A/p. Thus the $\{\overline{b_{(i)}}(w)\}$ generate A/p. There can be no additional relations as any other relation would contradict the linear independence of finite subfamilies of the $\{\overline{f_n}(w)\}$. □

As the relations came from the standard properties of all elements of A/p we can expand on this result to find a condition for a generating set:

Corollary 3.11 *Let $\{g_n(w)\}$ be a sequence of polynomials in A such that:*

1. *$g_n(w)$ is of degree p^n,*
2. *the leading coefficient of $g_n(w)$ has p–adic valuation $-\frac{p^n-1}{p-1}$,*

then the family $\{\overline{g_n}(w)\}$ generates A/p subject only to the relations $\overline{g_n}(w)^p = \overline{g_n}(w)$. Moreover, the subalgebra generated by $\{\overline{g_0}(w), \ldots, \overline{g_n}(w)\}$ is the same as that generated by $\{\overline{b_{(0)}}(w), \ldots, \overline{b_{(n)}}(w)\}$.

The basis that we shall use is closely related to Fermat's little theorem. It is constructed using the operation $f(w) \to \frac{1}{p}(f(w)^p - f(w))$ which we have already looked at a little. We examine this now in more detail.

Definition 3.12 *Let R be a commutative ring in which p is invertible. Define the operation θ on R by:*

$$\theta(r) = \tfrac{1}{p}(r^p - r).$$

The operation has many useful properties which we list here:

Proposition 3.13 1. *The operation θ preserves the following $\mathbb{Z}$–submodules of the given $\mathbb{Q}$–algebras:*

   (a) $\mathbb{Z}$ of $\mathbb{Q}$;

   (b) A and $w^d\mathbb{Q}[w^{p-1}]$, for $0 \leq d < p-1$, of $\mathbb{Q}[w]$;

   (c) B and $w^d\mathbb{Q}[w^{p-1}, w^{1-p}]$, for $0 \leq d < p-1$, of $\mathbb{Q}[w, w^{-1}]$.

2. *Let $f(w) \in \mathbb{Q}[w]$ be a polynomial of degree $n \geq 1$ and leading coefficient a. Then $\theta(f(w))$ has degree np and leading coefficient a^p/p.*

3. *Let $m \geq 1$ and let $k, l \in \mathbb{Z}$ be such that $k \equiv l \mod p^{m+1}$. Then $\theta(k) \equiv \theta(l) \mod p^m$.*

4. *Let $m \geq 1$ and let $k \in \mathbb{Z}$ be such that $k^{p-1} \equiv 1 \mod p^{m+1}$. Then $\theta(kl) \equiv k\theta(l) \mod p^m$.*

Proof. 1. (a) This is Fermat's little theorem.

(b) We have already shown this for A.

As $\mathbb{Q}[w^{p-1}]$ is a subring of $\mathbb{Q}[w]$, for $f(w) \in w^a\mathbb{Q}[w^{p-1}]$ and $g(w) \in w^b\mathbb{Q}[w^{p-1}]$ then $f(w)g(w) \in w^{a+b}\mathbb{Q}[w^{p-1}]$. In particular, if $f(w) \in w^d\mathbb{Q}[w^{p-1}]$ then $f(w)^p \in w^{pd}\mathbb{Q}[w^{p-1}] \subseteq w^d\mathbb{Q}[w^{p-1}]$. Hence $\theta(f(w)) \in w^d\mathbb{Q}[w^{p-1}]$.

(c) We have already shown this for B whilst the proof for $w^d\mathbb{Q}[w^{p-1}]$ readily adapts to $w^d\mathbb{Q}[w^{p-1}, w^{1-p}]$.

2. This is obvious.

3. Write $k = l + sp^{m+1}$ for some $s \in \mathbb{Z}$. Then:

$$\begin{aligned}
\theta(k) &= \theta(l + sp^{m+1}) \\
&= \tfrac{1}{p}\left((l + sp^{m+1})^p - l - sp^{m+1}\right) \\
&= \frac{1}{p}\left(\sum_{j=0}^{p}\binom{p}{j} l^{p-j}s^j p^{j(m+1)} - l - sp^{m+1}\right) \\
&= \tfrac{1}{p}(l^p - l) - sp^m + \sum_{j=1}^{p}\binom{p}{j} l^{p-j}s^j p^{j(m+1)-1} \\
&\equiv \theta(l) \mod p^m.
\end{aligned}$$

4. Firstly, as $k^{p-1} \equiv 1 \mod p^{m+1}$, $k^p - k = p^{m+1}r$ for some $r \in \mathbb{Z}$ and hence $\theta(k) \equiv 0 \mod p^m$. Then:

$$\begin{aligned}
\theta(kl) &= \tfrac{1}{p}(k^p l^p - kl) \\
&= \tfrac{1}{p}(k^p l^p - kl^p + kl^p - kl) \\
&= \theta(k)l^p + k\theta(l) \\
&\equiv k\theta(l) \mod p^m.
\end{aligned}$$

□

Definition 3.14 *Define the sequence of polynomials $\{g_i(w)\}$ by $g_i(w) = \theta^{\circ i}(w)$, with $g_0(w) = w$.*

Proposition 3.15 *The elements $\{\overline{g_i}(w)\}$ generate A/p subject only to the relations $\overline{g_i}(w)^p = \overline{g_i}(w)$.*

The polynomial:

$$\prod_{i \geq 0} g_i(w)^{n_i}$$

lies in $w^d\mathbb{Q}[w^{p-1}]$ where $d \in \{0, \ldots, p-2\}$ satisfies $d \equiv \sum n_i \mod (p-1)$.

Proof. By induction, $g_i(w)$ is a polynomial of degree p^i with leading coefficient p^{-n_i} where $n_i = p^{i-1} + p^{i-2} + \ldots + p + 1$. The first part of the result follows by corollary 3.10.

From proposition 3.13, each $g_i(w)$ lies in $w\mathbb{Q}[w^{p-1}]$, from which the second part follows. □

The family:

$$\prod_{j \geq 0} \overline{g_j}(w)^{n_j},$$

with $0 \leq n_j \leq p-1$ for all j and all but a finite number of the n_j being zero, therefore forms a basis for A/p.

We shall use three different ways of referring to elements of this basis. Firstly, there is the expression above which gives the actual element. Secondly, there is the sequence $(n_0, n_1, n_2, \dots)$ subject to the above constraints. Thirdly, we have already noted that there is a one-to-one correspondence between positive integers and such sequences (n_j) via $(n_j) \to \sum n_j p^j$ and so we get an enumeration of the basis elements (starting at 0 for the trivial product).

One useful feature of the correspondence between sequences and integers is that $n \equiv \sum n_j \mod (p-1)$. Therefore the nth basis element lies in $w^d \mathbb{Q}[w^{p-1}]$ where $d \in \{0, \dots, p-2\}$ is such that $d \equiv n \mod (p-1)$.

This exhibition of the basis shows that A/p decomposes according to a variant on the notion of the degree of a polynomial. In light of the identity $\overline{w}^p = \overline{w}$ the actual degree of a representative is not an invariant of the equivalence class but its reduction mod $(p-1)$ is. We have the following decomposition:

Corollary 3.16 *As a vector space, A/p decomposes as the direct sum of the $(p-1)$ factors $(A \cap w^d \mathbb{Q}[w^{p-1}])/p$ with $0 \leq d \leq p-1$. The factor corresponding to $d = 0$ is the sub-bialgebra $\mathcal{A}/p$ and has basis:*

$$\prod_{j \geq 0} \overline{g_j}(w)^{n_j}$$

such that $\sum n_j \equiv 0 \mod (p-1)$.

3.3 From Newton to Fermat, part *B*

Having dealt with A/p and $\mathcal{A}/p$ we now turn to B/p and $\mathcal{B}/p$. As the map $A/p \to B/p$ is a surjection, the generators of A/p also generate B/p subject to additional relations. These additional relations generate the kernel of the map $A/p \to B/p$ which we have already identified as the ideal generated by $\overline{1} - \overline{w}^{p-1}$. Thus as $g_0(w) = w$ the sole additional relation is $\widetilde{g_0}(w)^{p-1} = \widetilde{1}$. Hence:

Theorem 3.17 *The family $\{\widetilde{g_n}(w)\}$ generates B/p subject only to the relations:*

$$\widetilde{g_0}(w)^{p-1} = \widetilde{1} \text{ and } \widetilde{g_n}(w)^p = \widetilde{g_n}(w).$$

Therefore we have a basis of the form:

$$\prod_{j \geq 0} \widetilde{g_j}(w)^{m_j}$$

with $0 \leq m_j \leq p-1$ as before but with the additional constraint that $m_O \neq p-1$. Since $\widetilde{g_0}(w)^{p-1} = \widetilde{1}$ we could equally well assume that $m_0 \neq 0$. This makes it slightly simpler to describe the basis elements in terms of their enumeration since the condition $m_0 \neq 0$ translates to $\sum m_j p^j \not\equiv 0 \mod p$. The real reason for making this switch is:

Proposition 3.18 *With this assumption on the way of writing the basis elements, the isomorphism $B/p \to \overline{w}^{p-1}A/p$ is given by:*

$$\begin{gathered}\prod_{j\geq 0} \widetilde{g_j}(w)^{m_j} \to \prod_{j\geq 0} \overline{g_j}(w)^{m_j}, \\ (m_0, m_1, m_2, \dots) \to (m_0, m_1, m_2, \dots), \\ m \to m.\end{gathered}$$

Proof. Clearly under the map $A/p \to B/p$ the right-hand basis element in A/p is taken to the corresponding left-hand element of B/p. Therefore the inverse is as described if we can show that the right-hand element lies in $\overline{w}^{p-1}A/p$.

The projection $A/p \to \overline{w}^{p-1}A/p$ is given by multiplying by $\overline{w}^{p-1}$. As this is a power of an element of the generating set for A/p the effect of multiplying a generic basis element by it is determined by its effect on the $\overline{g_0}(w)$ part. For $0 \leq k \leq p-1$:

$$\overline{w}^{p-1} \cdot \overline{w}^k = \begin{cases} \overline{w}^k & \text{if } k \neq 0, \\ \overline{w}^{p-1} & \text{if } k = 0. \end{cases}$$

Hence:

$$\overline{w}^{p-1} \prod_{j\geq 0} \overline{g_j}(w)^{m_j} = \begin{cases} \prod_{j\geq 0} \overline{g_j}(w)^{m_j} & \text{if } m_0 \neq 0, \\ \overline{g_0}(w)^{p-1} \prod_{j\geq 1} \overline{g_j}(w)^{m_j} & \text{if } m_0 = 0. \end{cases}$$

Therefore the basis element corresponding to $m = \sum m_j p^j$ lies in $\overline{w}^{p-1}A/p$ if and only if p does not divide m. The ones in the statement of the proposition are precisely those elements. □

Corollary 3.19 *The subalgebra $\mathcal{B}/p$ has basis consisting of those elements corresponding to integers m such that $m \not\equiv 0 \mod p$ and $m \equiv 0 \mod (p-1)$.*

It is generated by the family $\{\widetilde{w}^{p-2}\widetilde{g_n}(w) : n \geq 1\}$ subject to the usual relations.

The idea here is that the two conditions $n \not\equiv 0 \mod p$ and $n \equiv 0 \mod (p-1)$ mean that n_0 is completely specified by $(n_1, n_2, \dots)$ and we have free choice for this shortened sequence.

Proof. The first part comes from the same consideration as for $\mathcal{A}/p$: that a generic basis element lies in the summand corresponding to $w^d\mathbb{Q}[w^{p-1}]$ where $d \equiv \sum m_j p^j \mod (p-1)$.

For the second part we start by noting that the given family does lie in $\mathcal{B}/p$. We just need to show that we can get an arbitrary basis element. Let $(m_0, m_1, \dots)$ represent such an element. Consider:

$$\prod_{j\geq 1} \left(\widetilde{w}^{p-2}\widetilde{g_j}(w)\right)^{m_j} = \widetilde{w}^{\sum m_j(p-2)} \prod_{j\geq 1} \widetilde{g_j}(w)^{m_j}.$$

Using $\widetilde{w}^{p-1} = \widetilde{1}$ we can reduce the first term to something of the form $\widetilde{w}^k$ with $1 \leq k \leq p-1$ and $k \equiv \sum_{j\geq 1} m_j(p-2) \mod (p-1)$. Since $(p-2) \equiv -1 \mod (p-1)$ and $\sum_{j\geq 1} m_j \equiv -m_0 \mod (p-1)$ we have $k \equiv m_0 \mod (p-1)$. As both lie in the range $\{1, \dots, p-1\}$ they must be the same and so we have the required basis element. □

4 Cops and Ops *or* Coops and Oops

In this section we consider the duals of A/p etc. We have mentioned several times that in the cohomology theories under consideration, operations are dual to co-operations. This duality is of *topological algebras* so it is worth taking a moment to examine the topologies involved.

Let E be a generalised cohomology theory. The topology on the associated homology is discrete whilst cohomology is given the *pro-finite* topology. This is the filtration topology with respect to the ideals:

$$F^a E^*(X) := \ker(E^*(X) \to E^*(X_a))$$

where X_a runs through the finite subcomplexes of X.

The topology on $DE_*(X)$ is the *dual-finite* topology. This makes sense for the dual, $DM := \hom_R(M, R)$, of an arbitrary module M over a ring R. It is defined to be the filtration topology with respect to the ideals:

$$F^L DM := \ker(DM \to DL)$$

where L runs through the family of finitely generated submodules of M. We make two simple observations about this topology: firstly, any cofinal family of such submodules will generate the same topology and secondly, if the ring is a field, finitely generated submodules are finite dimensional subspaces.

There is a natural E^*–bilinear pairing $E^*(X) \times E_*(X) \to E^*$ which we use to define an E^*–linear map $d : E^*(X) \to DE_*(X)$. We shall examine this map in detail later, for now we just need the following result:

Proposition 4.1 ([Boa95, Theorem 4.14]) *If $E_*(X)$ is a free E^*–module then $d : E^*(X) \to DE_*(X)$ is a homeomorphism with respect to the above topologies.*

In our cases, the ring E^* is periodic of the form $R[x, x^{-1}]$ with R in degree 0 and $\deg x = -k$ for some $k > 0$. In this case we can simplify the duality somewhat:

Lemma 4.2 *Let M be a graded module over $R[x, x^{-1}]$, DM its dual. Then there is a natural homeomorphism $(DM)^n \cong D(M^{-n})$.*

As R is the degree zero component of $R[x, x^{-1}]$, the $R[x, x^{-1}]$–module structure on M induces an R–module structure on each component, M^k. This structure of an R–module defines the dual-finite topology on $D(M^{-n})$.

Proof. The R–module $(DM)^n$ consists of degree n maps $M \to R[x, x^{-1}]$ which are $R[x, x^{-1}]$–linear. Therefore we have a sequence of maps (f_m) with $f_m : M^{m-n} \to R[x, x^{-1}]^m$ which are R–linear and which satisfy $x^{-1} f_m x = f_{m+k}$.

Now $R[x, x^{-1}]^m$ is zero unless k divides m, in which case it is (isomorphic to) R. Thus f_m is the zero map unless k divides m so the sequence of maps is really (f_{lk}). Then $x^{-l} f_0 x^l = f_{lk}$ so this sequence is completely determined by $f_0 : M^{-n} \to R$. The map $(f_m) \to f_0$ is clearly a bijection and the statement about the topologies is obvious. □

As Kp^* and $K(1)^*$ are graded fields, any module over them is free and so we always have the duality. Hence:

Corollary 4.3 1. *Additive operations from degree zero to degree zero in Kp are isomorphic to:* $\hom_{\mathbb{F}_p}(A/p)$.

2. *Stable operations of degree zero in Kp are isomorphic to:* $\hom_{\mathbb{F}_p}(B/p)$.

3. *Additive operations from degree zero to degree zero in* $K(1)$ *are isomorphic to:* $\hom_{\mathbb{F}_p}(\mathcal{A}/p)$.

4. *Stable operations of degree zero in* $K(1)$ *are isomorphic to:* $\hom_{\mathbb{F}_p}(\mathcal{B}/p)$.

In each case the topology is as: $\varprojlim_V \hom_{\mathbb{F}_p}(V, \mathbb{F}_p)$ *where* V *runs over a cofinal family of finite dimensional subspaces.*

4.1 Identifying the Duals

In the light of corollary 4.3, the first step to analysing the various spaces of operations is finding a suitable family of finite dimensional subspaces of the various spaces of numerical polynomials.

Definition 4.4 *Let* $F^n A/p$ *be the subalgebra of* A/p *generated by* $\{\overline{g_0}(w), \dots, \overline{g_{n-1}}(w)\}$.

Proposition 4.5 1. $F^n A/p$ *is a finite dimensional subspace of* A/p *of dimension* p^n.

2. $\bigcup F^n A/p = A/p$.

3. *There is a commutative diagram:*

$$\begin{array}{ccc} F^n A/p & \xrightarrow{\cong} & \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p) \\ \downarrow & & \downarrow \\ F^{n+1} A/p & \xrightarrow{\cong} & \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p), \end{array}$$

where the horizontal maps send $\overline{f}(w) \in F^n A/p$ *to the map* $[k]_{p^n} \to \overline{f}([k]_{p^n}) := [f(k)]_p$*, the left-hand vertical map is the inclusion, and the right-hand vertical map is induced by the projection* $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$*. All the maps are of bialgebras.*

Proof. 1. The subspace $F^n A/p$ has basis:

$$\prod_{j=0}^{n-1} \overline{g_j}(w)^{n_j}$$

with $0 \le n_j \le p-1$ for all j. This is a finite set of size p^n.

2. The statement that $\{\prod \overline{g_j}(w)^{n_j}\}$ forms a basis for A/p means that every element of A/p is a finite linear combination of elements in this family. Therefore every element of A/p lies in the subalgebra generated by a finite number of the $\overline{g_j}(w)$, hence in some $F^n A/p$.

3. We start by showing that the horizontal maps are well-defined. There are two choices made in its definition: the representing polynomial and the representing integer.

   We start with the polynomial. Let $g(w)$ also represent $\overline{f}(w)$. Then $g(w) - f(w) \in pA$ so $g(k) - f(k) \in p\mathbb{Z}$. Hence $[g(k)]_p = [f(k)]_p$.

   For the integer, as $\overline{f}(w) \in F^n A/p$ we may choose as our representing polynomial a linear combination of products of $\{g_0(w), \dots, g_{n-1}(w)\}$. Hence

$[f(k)]_p$ is a linear combination of products of $\{[g_0(k)]_p, \ldots, [g_{n-1}(k)]_p\}$. By definition, this set is $\{[\theta^{\circ i}(k)]_p : 0 \le i \le n-1\}$. From proposition 3.13, if $k \equiv l \mod p^n$ then $\theta^{\circ i}(k) \equiv \theta^{\circ i}(l) \mod p$ for $0 \le i \le n-1$. Hence $[f(k)]_p$ depends only on the coset $k + p^n\mathbb{Z}$ and not on the choice of lift.

We now show that the horizontal maps are isomorphisms of $\mathbb{F}_p$–vector spaces. As both sides have the same dimension it is sufficient to show injectivity. Let $f(w) \in A$ represent an element of $F^n A/p$ which goes to the zero map. For $k \in \mathbb{Z}$ we thus have $[f(k)]_p = \overline{f}([k]_{p^n}) = 0$. Hence $f(k) \in p\mathbb{Z}$ for all $k \in \mathbb{Z}$ and thus $f(k) \in pA$. Therefore the horizontal maps are injective and so are isomorphisms.

To show that the diagram is commutative, consider the projection map $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$. Let $x \in \mathbb{Z}/p^{n+1}$ map to $y \in \mathbb{Z}/p^n$. Now we can choose $k \in \mathbb{Z}$ such that $[k]_{p^{n+1}} = x$ and $[k]_{p^n} = y$. Therefore for $\overline{f}(w) \in F^n A/p$, viewing it as an element of $F^n A/p$ we get $\overline{f}(y) = [f(k)]_p$ and viewing it as an element of $F^{n+1}A/p$ we get $\overline{f}(x) = [f(k)]_p$. Hence the diagram commutes.

That all the maps are algebra maps is obvious. The bialgebra structure on each side also corresponds since on the right it is such that $(\Delta f)(x \otimes y) = f(xy)$ and on the left it is determined by $\Delta w = w \otimes w$. A quick check shows that these correspond under the given map. □

We can repeat this for $\mathcal{A}/p$, B/p, and $\mathcal{B}/p$. In the latter two cases we identify B/p with $\overline{w}^{p-1}A/p$ and consider each as a subspace of A/p. Thus we get induced filtrations by intersection:

Definition 4.6 *Let $F^n\mathcal{A}/p$, F^nB/p, and $F^n\mathcal{B}/p$ be defined as, respectively, $\mathcal{A}/p \cap F^nA/p$, $B/p \cap F^nA/p$, and $\mathcal{B}/p \cap F^nA/p$.*

The following basic properties are simple to establish:

Lemma 4.7 *1. $F^n\mathcal{A}/p$, F^nB/p, and $F^n\mathcal{B}/p$ are all finite dimensional with dimensions, respectively, p^{n-1}, $p^{n-1}(p-1)$, and p^{n-1}.*

2. F^nB/p is the subalgebra of B/p generated by $\{\widetilde{g_0}(w), \ldots, \widetilde{g_{n-1}}(w)\}$ and $F^n\mathcal{B}/p$ is the subalgebra of $\mathcal{B}/p$ generated by $\{\widetilde{w}^{p-2}\widetilde{g_1}(w), \ldots, \widetilde{w}^{p-2}\widetilde{g_{n-1}}(w)\}$.

3. $\bigcup F^n\mathcal{A}/p = \mathcal{A}/p$, $\bigcup F^nB/p = B/p$, and $\bigcup F^n\mathcal{B}/p = \mathcal{B}/p$.

What we wish to establish is how these subspaces of F^nA/p are reflected in $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$. More precisely, we have compatible commutative diagrams of inclusions and projections:

$$\begin{array}{ccc} F^n\mathcal{A}/p & \longrightarrow & F^nA/p \\ \uparrow & & \uparrow \\ F^n\mathcal{B}/p & \longrightarrow & F^nB/p \end{array} \qquad \begin{array}{ccc} F^n\mathcal{A}/p & \longleftarrow & F^nA/p \\ \downarrow & & \downarrow \\ F^n\mathcal{B}/p & \longleftarrow & F^nB/p \end{array} \tag{4.1}$$

which we wish to translate into the language of $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$.

We need to identify our candidate spaces as subspaces of $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$. The first is $\mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)$ identified with the subspace of $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ of maps with support in $(\mathbb{Z}/p^n)^\times$. That is, given a map $f : (\mathbb{Z}/p^n)^\times \to \mathbb{F}_p$ we extend it to a map $\mathbb{Z}/p^n \to \mathbb{F}_p$ by defining it to be zero on the non-units in $\mathbb{Z}/p^n$. The projection onto $\mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)$ is thus defined by restriction.

For the second, we define an action of ${\mathbb{F}_p}^\times$ on $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ as follows: we identify ${\mathbb{F}_p}^\times$ with the unique cyclic subgroup of $(\mathbb{Z}/p^n)^\times$ of order $(p-1)$ (there is a canonical identification induced by the quotient map $\mathbb{Z}/p^n \to \mathbb{F}_p$). This acts on $\mathbb{Z}/p^n$ and hence acts linearly on $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ by precomposition. We define $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}$ to be the fixed point subspace of this action. As we are in characteristic p and ${\mathbb{F}_p}^\times$ is a finite group of order $(p-1)$, there is a projection onto this subspace given by:

$$f \to -\sum_{\alpha \in {\mathbb{F}_p}^\times} f^\alpha.$$

This also defines the third candidate, $\mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}$, since the action of ${\mathbb{F}_p}^\times$ on $\mathbb{Z}/p^n$ respects the description of $\mathbb{Z}/p^n$ as the disjoint union of units and non-units.

We therefore have diagrams of inclusions and projections:

$$\begin{array}{ccc}
\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)^{{\mathbb{F}_p}^\times} & \longrightarrow & \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p) \\
\uparrow & & \uparrow \\
\mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)^{{\mathbb{F}_p}^\times} & \longrightarrow & \mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p) \\
\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)^{{\mathbb{F}_p}^\times} & \longleftarrow & \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p) \\
\downarrow & & \downarrow \\
\mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)^{{\mathbb{F}_p}^\times} & \longleftarrow & \mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)
\end{array} \tag{4.2}$$

Proposition 4.8 *The isomorphism* $F^n A/p \cong \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ *induces isomorphisms:*

$$\begin{aligned}
F^n B/p &\cong \mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p), \\
F^n \mathcal{A}/p &\cong \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}, \\
F^n \mathcal{B}/p &\cong \mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)^{{\mathbb{F}_p}^\times},
\end{aligned}$$

and the diagrams in (4.1) *are mapped to the diagrams in* (4.2).

Proof. We start with $F^n B/p$. We are thinking of B/p as $\overline{w}^{p-1} A/p$ so we are really considering $F^n A/p \cap \overline{w}^{p-1} A/p$. Now the complement to $(\mathbb{Z}/p^n)^\times$ in $\mathbb{Z}/p^n$ is the image under the quotient map $\mathbb{Z} \to \mathbb{Z}/p^n$ of $p\mathbb{Z}$. Therefore it is sufficient to show that $f(w) \in A$ represents an element of $\overline{w}^{p-1} A/p$ if and only if it maps $p\mathbb{Z}$ into $p\mathbb{Z}$; equivalently, that $[f(pk)]_p = 0$ for all $k \in A$.

As $\overline{w}^{p-1}$ is an idempotent, $f(w)$ represents an element of $\overline{w}^{p-1} A/p$ if and only if $w^{p-1} f(w)$ represents the same element. That is, if and only if $(1 - w^{p-1}) f(w) \in pA$. Let $g(w) = (1 - w^{p-1}) f(w)$. We now wish to show that $g(w) \in pA$ if and only if $f(p\mathbb{Z}) \subseteq p\mathbb{Z}$.

Let $k \in \mathbb{Z}$ be not divisible by p. Then $k^{p-1} \equiv 1 \mod p$ so p divides $1 - k^{p-1}$. Hence $g(k)$ is divisible by p. This does not use any conditions on $f(w)$ (beyond it being in A) and so $g(\mathbb{Z}) \subseteq p\mathbb{Z}$ if and only if $g(p\mathbb{Z}) \subseteq p\mathbb{Z}$. For any k, $1 - p^{p-1} k^{p-1}$

is not divisible by p so as p is prime, $g(pk)$ is divisible by p if and only if $f(pk)$ is divisible by p. Hence $g(p\mathbb{Z}) \subseteq p\mathbb{Z}$ if and only if $f(p\mathbb{Z}) \subseteq p\mathbb{Z}$.

Thus $\overline{f}(w) \in \overline{w}^{p-1}A/p$ if and only if $f(p\mathbb{Z}) \subseteq p\mathbb{Z}$. Hence F^nB/p corresponds to the maps which have support in $(\mathbb{Z}/p^n)^\times$; which we have identified with $\mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)$.

We now consider $F^n\mathcal{A}/p$. The result for $F^n\mathcal{B}/p$ will follow as $\mathcal{B}/p = \mathcal{A}/p \cap B/p$ when thought of as subspaces of A/p and the analogous statement holds in $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$.

Consider the action of ${\mathbb{F}_p}^\times$ on a generator $\overline{g}_j(w)$ of F^nA/p. As this lies in F^nA/p, we have $j < n$. The map on $\mathbb{Z}/p^n$ corresponding to $\overline{g}_j(w)$ is $[l]_{p^n} \to [g_j(l)]_p$. Now the subgroup of $(\mathbb{Z}/p^n)^\times$ identified with ${\mathbb{F}_p}^\times$ consists of those elements which satisfy $\alpha^{p-1} = 1$. Therefore they are represented by integers k which satisfy $k^{p-1} \equiv 1 \mod p^n$. As $g_j(w) = \theta^{\circ j}(w)$, proposition 3.13 implies that $[g_j(kl)]_p = [kg_j(l)]_p$. Therefore ${\mathbb{F}_p}^\times$ acts on $\overline{g}_j(w)$ by scalar multiplication. On the basis element corresponding to the sequence $(m_0, m_1, \dots, m_{n-1}, 0, \dots)$ the action is thus scalar multiplication by $\alpha^{\sum m_j}$.

Hence the decomposition of A/p into the subspaces $(A \cap w^d\mathbb{Q}[w^{p-1}])/p$ corresponds precisely to the decomposition of $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ into the distinct representations of ${\mathbb{F}_p}^\times$. In particular, $\mathcal{A}/p$ corresponds to the fixed point subspace and the projection $A/p \to \mathcal{A}/p$ is the standard averaging map which kills the other representations. □

Corollary 4.9 *There are isomorphisms:*

$$\begin{aligned} A/p &\cong \varinjlim \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p), \\ B/p &\cong \varinjlim \mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p), \\ \mathcal{A}/p &\cong \varinjlim \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}, \\ \mathcal{B}/p &\cong \varinjlim \mathrm{Map}((\mathbb{Z}/p^n)^\times, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}.\mathbb{F}_p), \end{aligned}$$

where the connecting maps on the right are induced by the projection $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$.

At this point we wish to dualise to deduce some statement about the various spaces of operations. Our discussion now proceeds along two parallel but disconnected paths. One leads to a presentation of the operations as power series in a small number of generators whilst the other to a more elegant formulation in terms of p–adic integers.

4.2 Operations as Power Series

The dual of $\mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ is $\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$ with pairing given by evaluation. The connecting maps on the duals are again induced by projections $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$. Using the definition of the projective limit, we see that:

Lemma 4.10 *There are homeomorphisms of topological $\mathbb{F}_p$–algebras:*

$$\begin{aligned} \hom_{\mathbb{F}_p}(A/p, \mathbb{F}_p) &\cong \varprojlim \mathbb{F}_p\langle \mathbb{Z}/p^n \rangle, \\ \hom_{\mathbb{F}_p}(B/p, \mathbb{F}_p) &\cong \varprojlim \mathbb{F}_p\langle (\mathbb{Z}/p^n)^\times \rangle, \\ \hom_{\mathbb{F}_p}(\mathcal{A}/p, \mathbb{F}_p) &\cong \varprojlim \mathbb{F}_p\langle \mathbb{Z}/p^n \rangle^{{\mathbb{F}_p}^\times}, \\ \hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p) &\cong \varprojlim \mathbb{F}_p\langle (\mathbb{Z}/p^n)^\times \rangle^{{\mathbb{F}_p}^\times}. \end{aligned}$$

Thus to give a linear functional $A/p \to \mathbb{F}_p$ is equivalent to giving a sequence (α_n) where $\alpha_n \in \mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$ and $\alpha_{n+1} \to \alpha_n$ under the map $\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle \to \mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$. The isomorphism $F^n A/p \cong \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$ shows that the isomorphisms in lemma 4.10 are as follows: given a sequence $\alpha := (\alpha_n)$, we choose a sequence of integers (a_n) such that $\alpha_n = [a_n]_{p^n}$, then:

$$\alpha(\overline{f}(w)) = \lim [f(a_n)]_p = [f(a_N)]_p,$$

for N sufficiently large.

We introduce some notation. To distinguish between an element $\alpha \in \mathbb{Z}/p^n$ and the corresponding basis element of $\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$ we write the latter as Ψ^α. If we have given a label, say β, to an admissible sequence in the $\mathbb{Z}/p^n$, we give the corresponding sequence in $\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$ the label Ψ^β. In particular, an integer defines a unique sequence of elements in $\mathbb{Z}/p^n$ via $k \to ([k]_{p^n})$, we use the label k to refer to this sequence and so write Ψ^k for the sequence of corresponding elements in $\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$. Thus:

$$\Psi^k(\overline{f}(w)) = [f(k)]_p.$$

The algebra structure on the dual is determined by the coalgebra structure on the original space. Since this was itself determined by the multiplications on each $\mathbb{Z}/p^n$, the correct algebra structure on each $\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle$ is given by the *multiplication* in $\mathbb{Z}/p^n$.

Theorem 4.11 *Let $q \in \mathbb{Z}$ be primitive modulo p^2. Let $\tilde{q} = q^{p-1}$.*
There is an isomorphism of topological algebras:

$$\hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p) \cong \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1]\!].$$

Proof. There is a split short exact sequence of cyclic groups:

$${\mathbb{F}_p}^\times \to (\mathbb{Z}/p^n)^\times \to (\mathbb{Z}/p^n)^\times/{\mathbb{F}_p}^\times$$

which is natural in n in that the quotient $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ induces a map of short exact sequences. The splitting map defines an isomorphism:

$$\mathbb{F}_p\langle (\mathbb{Z}/p^n)^\times/{\mathbb{F}_p}^\times \rangle \to \mathbb{F}_p\langle (\mathbb{Z}/p^n)^\times \rangle^{{\mathbb{F}_p}^\times}.$$

Now $(\mathbb{Z}/p^n)^\times/{\mathbb{F}_p}^\times$ is cyclic of order p^{n-1}. Hence if t_n is a generator,

$$\mathbb{F}_p\langle (\mathbb{Z}/p^n)^\times \rangle^{{\mathbb{F}_p}^\times} \cong \mathbb{F}_p[t_n]/({t_n}^{p^{n-1}} - 1).$$

In characteristic p the pth power map is additive and so $(t_n - 1)^{p^{n-1}} = {t_n}^{p^{n-1}} - 1$. Thus we can rewrite this as:

$$\mathbb{F}_p\langle (\mathbb{Z}/p^n)^\times \rangle^{{\mathbb{F}_p}^\times} \cong \mathbb{F}_p[t_n - 1]/(t_n - 1)^{p^{n-1}}.$$

Let $t = (t_n)$ be a sequence of generators of $(\mathbb{Z}/p^n)^\times/{\mathbb{F}_p}^\times$ such that $t_{n+1} \to t_n$ under the projection:

$$(\mathbb{Z}/p^{n+1})^\times/{\mathbb{F}_p}^\times \to (\mathbb{Z}/p^n)^\times/{\mathbb{F}_p}^\times.$$

Then since $\mathbb{F}_p[\![x]\!] = \varprojlim \mathbb{F}_p[x]/x^k$:

$$\varprojlim \mathbb{F}_p\langle(\mathbb{Z}/p^n)^\times\rangle^{\mathbb{F}_p{}^\times} \cong \mathbb{F}_p[\![t-1]\!].$$

Hence there is an isomorphism of topological algebras:

$$\hom_{\mathbb{F}_p}(\mathcal{B}, \mathbb{F}_p) \cong \mathbb{F}_p[\![t-1]\!].$$

It remains to find a suitable family t.

As q is primitive modulo p^2 it is primitive modulo p^n for all n. Hence $[q]_{p^n}$ generates $(\mathbb{Z}/p^n)^\times$, which is cyclic of order $p^{n-1}(p-1)$. The subgroup corresponding to $(\mathbb{Z}/p^n)^\times/\mathbb{F}_p{}^\times$ is the unique p^{n-1} cyclic subgroup of $(\mathbb{Z}/p^n)^\times$ and hence is generated by $[q^{p-1}]_{p^n}$. Thus the sequence $([q^{p-1}]_{p^n})$ is a suitable family of generators. This corresponds to the element $\Psi^{\tilde{q}}$ and hence:

$$\hom_{\mathbb{F}_p}(\mathcal{B}, \mathbb{F}_p) \cong \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1]\!]$$

as required. □

Theorem 4.12 *Let $a = (a_n)$ be an admissible sequence of integers such that $[a_n]_{p^n}$ generates the $(p-1)$th roots of unity in $\mathbb{Z}/p^n$.*

There is an isomorphism of topological algebras:

$$\begin{aligned}\hom_{\mathbb{F}_p}(B/p, \mathbb{F}_p) &\cong \mathbb{F}_p\langle\mathbb{F}_p{}^\times\rangle \otimes \hom_{\mathbb{F}_p}(\mathcal{B}, \mathbb{F}_p)\\ &\cong \mathbb{F}_p[\![\Psi^a, \Psi^{\tilde{q}} - 1]\!]/((\Psi^a)^{p-1} - 1)\\ &\cong \left(\mathbb{F}_p[\Psi^a]/((\Psi^a)^{p-1} - 1)\right) \otimes \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1]\!].\end{aligned}$$

Proof. The split short exact sequence that we have already considered induces an isomorphism for each n, natural in n:

$$(\mathbb{Z}/p^n)^\times \cong \mathbb{F}_p{}^\times \times (\mathbb{Z}/p^n)^\times/\mathbb{F}_p{}^\times.$$

Hence:

$$\mathbb{F}_p\langle(\mathbb{Z}/p^n)^\times\rangle \cong \mathbb{F}_p\langle\mathbb{F}_p{}^\times\rangle \otimes \mathbb{F}_p\langle(\mathbb{Z}/p^n)^\times\rangle^{\mathbb{F}_p{}^\times}.$$

As n increases, the first factor does not change whilst the second factor becomes $\hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p)$.

The second and third parts merely involve rewriting the $\mathbb{F}_p\langle\mathbb{F}_p{}^\times\rangle$ factors. As $\mathbb{F}_p{}^\times$ is cyclic of order $(p-1)$ there is an isomorphism:

$$\mathbb{F}_p\langle\mathbb{F}_p{}^\times\rangle \cong \mathbb{F}_p[s]/(s^{p-1} - 1).$$

As the projection $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ takes the $(p-1)$th roots of unity in $\mathbb{Z}/p^{n+1}$ onto those in $\mathbb{Z}/p^n$, it is possible to choose a sequence $a = (a_n)$ satisfying the requirements. Hence we may put $s = \Psi^a$ in the above. □

We can actually find a single generator for $\hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p)$: Ψ^q. This is because $[q]_{p^n}$ generates $(\mathbb{Z}/p^n)^\times$ for all n. However, it satisfies the relation: $q^{p^{n-1}(p-1)} = 1$ and so the difficulty in explaining the limit outweighs the simplicity in having only one generator.

Theorem 4.13 *There are isomorphisms of topological algebras:*

$$
\begin{aligned}
\hom_{\mathbb{F}_p}(\mathcal{A}/p, \mathbb{F}_p) &\cong \mathbb{F}_p \oplus \hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p)[\![\Psi^p]\!] \\
&\cong \mathbb{F}_p \oplus \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1, \Psi^p]\!]. \\
\hom_{\mathbb{F}_p}(A/p, \mathbb{F}_p) &\cong \mathbb{F}_p \oplus \hom_{\mathbb{F}_p}(B/p, \mathbb{F}_p)[\![\Psi^p]\!] \\
&\cong \mathbb{F}_p \oplus \mathbb{F}_p[\![\Psi^a, \Psi^{\tilde{q}} - 1, \Psi^p]\!]/((\Psi^a)^{p-1} - 1) \\
&\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p[\Psi^a]/((\Psi^a)^{p-1} - 1)\Big) \otimes \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1, \Psi^p]\!].
\end{aligned}
$$

The lone $\mathbb{F}_p$ corresponds to Ψ^0. Multiplication by Ψ^0 is the same as the projection onto the $\mathbb{F}_p$ factor.

Proof. These follow from the observation that as monoids:

$$
\mathbb{Z}/p^n = \{0\} \amalg \coprod_{j=0}^{n-1} p^j (\mathbb{Z}/p^{n-j})^\times,
$$

and that this is compatible both with the projections $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ and the ${\mathbb{F}_p}^\times$–action. Hence:

$$
\begin{aligned}
\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle^{\mathbb{F}_p^\times} &\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p[\Psi^{\tilde{q}} - 1, \Psi^p]/(\Psi^{pj}(\Psi^{\tilde{q}} - 1)^{p^{n-j-1}})\Big) \\
\mathbb{F}_p\langle \mathbb{Z}/p^n \rangle &\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p[\Psi^a, \Psi^{\tilde{q}} - 1, \Psi^p]/(\Psi^{pj}(\Psi^{\tilde{q}} - 1)^{p^{n-j-1}}, (\Psi^a)^{p-1} - 1)\Big).
\end{aligned}
$$

Taking the projective limits yields the desired results. □

We summarise what we view as the simplest form of these:

$$
\begin{aligned}
\hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p) &\cong \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1]\!], \\
\hom_{\mathbb{F}_p}(B/p, \mathbb{F}_p) &\cong \mathbb{F}_p\langle {\mathbb{F}_p}^\times \rangle \otimes \mathbb{F}_p[\![\Psi^{\tilde{q}} - 1]\!], \\
\hom_{\mathbb{F}_p}(\mathcal{A}/p, \mathbb{F}_p) &\cong \mathbb{F}_p \oplus \mathbb{F}_p[\![\Psi^p, \Psi^{\tilde{q}} - 1]\!], \\
\hom_{\mathbb{F}_p}(A/p, \mathbb{F}_p) &\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p\langle {\mathbb{F}_p}^\times \rangle \otimes \mathbb{F}_p[\![\Psi^p, \Psi^{\tilde{q}} - 1]\!]\Big).
\end{aligned}
$$

Each of these isomorphisms is of topological algebras, with the topology on the right-hand side being the standard one.

4.3 Operations as *p*–adic Integers

In this section we look for a description of the various spaces of operations involving the *p*–adic integers. That such a description exists is suggested by all the inverse limits of cyclic groups in the previous section. The definition of the *p*–adic integers and various standard properties are contained in the appendix. Our main technical result is:

Proposition 4.14 $C^0(\mathbb{Z}_p, \mathbb{F}_p) \cong \varinjlim \mathrm{Map}(\mathbb{Z}/p^n, \mathbb{F}_p)$.

Here, as elsewhere, $\mathbb{F}_p$ and $\mathbb{Z}/p^n$ have the discrete topology.

Proof. We shall prove one direction of this by showing that for a continuous map, $f : \mathbb{Z}_p \to \mathbb{F}_p$, there is some $n \in \mathbb{N}$ such that f factors through the projection $\mathbb{Z}_p \to \mathbb{Z}/p^n$.

For each $\alpha \in \mathbb{Z}_p$ there is some $n_\alpha \in \mathbb{N}$ such that f is constant on $\alpha + p^{n_\alpha}\mathbb{Z}_p$. This family is an open cover of $\mathbb{Z}_p$ and so has a finite subcover. Hence there is some $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$ such that:

$$\mathbb{Z}_p = \bigcup_{j=1}^{m} \alpha_j + p^{n_{\alpha_j}}\mathbb{Z}_p.$$

and f is constant on each subset in this decomposition.

Let $n = \max\{n_{\alpha_j}\}$ and consider the family $\{k + p^n\mathbb{Z}_p : 0 \leq k \leq p^n - 1\}$. For each such k, $k \in \alpha_j + p^{n_{\alpha_j}}\mathbb{Z}_p$ for some j. Since $n \geq n_{\alpha_j}$, this implies that $k + p^n\mathbb{Z}_p \subseteq \alpha_j + p^{n_{\alpha_j}}\mathbb{Z}_p$. Hence f is constant on $k + p^n\mathbb{Z}_p$. Thus f is constant on all the cosets of $p^n\mathbb{Z}_p$ and hence factors through the projection $\mathbb{Z}_p \to \mathbb{Z}/p^n$.

As we have equipped $\mathbb{Z}/p^n$ with the discrete topology there are no continuity conditions to be checked on the induced map $\mathbb{Z}/p^n \to \mathbb{F}_p$.

For the reverse, we just note that the projection $\mathbb{Z}_p \to \mathbb{Z}/p^n$ is continuous as the inverse image of a point is of the form $k + p^n\mathbb{Z}_p$, i.e. an open ball. Hence any map $\mathbb{Z}/p^n \to \mathbb{F}_p$ extends to a continuous map $\mathbb{Z}_p \to \mathbb{F}_p$. □

As the projections $\mathbb{Z}_p \to \mathbb{Z}/p^n$ take units to units and non-units to non-units, and are ${\mathbb{F}_p}^\times$–equivariant we get analogous results on ${\mathbb{Z}_p}^\times$ and for the ${\mathbb{F}_p}^\times$–fixed point subspaces. Hence:

Corollary 4.15 *There are isomorphisms of bialgebras:*

$$\begin{aligned} A/p &\cong C^0(\mathbb{Z}_p, \mathbb{F}_p), \\ B/p &\cong C^0({\mathbb{Z}_p}^\times, \mathbb{F}_p), \\ \mathcal{A}/p &\cong C^0(\mathbb{Z}_p, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}, \\ \mathcal{B}/p &\cong C^0({\mathbb{Z}_p}^\times, \mathbb{F}_p)^{{\mathbb{F}_p}^\times}. \end{aligned}$$

We turn now to the duals. The dual of $\operatorname{Map}(X, \mathbb{F}_p)$ is $\mathbb{F}_p\langle X\rangle$ so we would expect the dual of $C^0(X, \mathbb{F}_p)$ to be some completion of $\mathbb{F}_p\langle X\rangle$. To identify the correct completion, we make a definition:

Definition 4.16 *Let M be an abelian monoid and suppose that the map $M \to \varprojlim_N M/N$ is injective where N runs over the directed family of submonoids of finite index. Define the* pro-finite completion *of the monoid ring of M over $\mathbb{F}_p$, $\mathbb{F}_p\langle\langle M\rangle\rangle$, to be the projective limit of the family $\mathbb{F}_p\langle M/N\rangle$.*

This is the completion of $\mathbb{F}_p\langle M\rangle$ with respect to the filtration given by the finite index submonoids. As an example, consider $\mathbb{N}_0$ (i.e. including zero) with the operation of addition. The monoid ring is isomorphic to $\mathbb{F}_p[t]$ whereas the pro-finite completion is isomorphic to $\mathbb{F}_p[[t]]$.

Theorem 4.17 *There are isomorphisms of topological algebras:*

$$\begin{aligned}
\hom_{\mathbb{F}_p}(\mathcal{B}/p, \mathbb{F}_p) &\cong \mathbb{F}_p\langle\!\langle \mathbb{Z}_p{}^\times \rangle\!\rangle^{\mathbb{F}_p{}^\times}, \\
&\cong \mathbb{F}_p\langle\!\langle 1 + p\mathbb{Z}_p \rangle\!\rangle, \\
\hom_{\mathbb{F}_p}(B/p, \mathbb{F}_p) &\cong \mathbb{F}_p\langle\!\langle \mathbb{Z}_p{}^\times \rangle\!\rangle, \\
&\cong \mathbb{F}_p\langle \mathbb{F}_p{}^\times \rangle \otimes \mathbb{F}_p\langle\!\langle 1 + p\mathbb{Z}_p \rangle\!\rangle, \\
\hom_{\mathbb{F}_p}(\mathcal{A}/p, \mathbb{F}_p) &\cong \mathbb{F}_p\langle\!\langle \mathbb{Z}_p \rangle\!\rangle^{\mathbb{F}_p{}^\times}, \\
&\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p\langle\!\langle (1 + p\mathbb{Z}_p) \rangle\!\rangle \widehat{\otimes} \mathbb{F}_p\langle\!\langle \mathbb{N}_0 \rangle\!\rangle \Big), \\
\hom_{\mathbb{F}_p}(A/p, \mathbb{F}_p) &\cong \mathbb{F}_p\langle\!\langle \mathbb{Z}_p \rangle\!\rangle, \\
&\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p\langle\!\langle \mathbb{Z}_p{}^\times \rangle\!\rangle \widehat{\otimes} \mathbb{F}_p\langle\!\langle \mathbb{N}_0 \rangle\!\rangle \Big), \\
&\cong \mathbb{F}_p \oplus \Big(\mathbb{F}_p\langle \mathbb{F}_p{}^\times \rangle \otimes \mathbb{F}_p\langle\!\langle (1 + p\mathbb{Z}_p) \rangle\!\rangle \widehat{\otimes} \mathbb{F}_p\langle\!\langle \mathbb{N}_0 \rangle\!\rangle \Big).
\end{aligned}$$

Here, $\widehat{\otimes}$ denotes the obvious completed tensor product. It is not needed with the factor $\mathbb{F}_p\langle \mathbb{F}_p{}^\times \rangle$ as this is finite dimensional (which also accounts for the lack of completion on this factor).

Proof. The first line in each follows from the fact that in each case the pro-finite completion agrees with the projective topology induced by the projections onto the corresponding piece of $\mathbb{Z}/p^n$.

The second line for $\mathcal{B}/p$ comes from the fact that the short exact sequence:

$$\mathbb{F}_p{}^\times \to \mathbb{Z}_p{}^\times \to \mathbb{Z}_p{}^\times/\mathbb{F}_p{}^\times$$

is split via a map $\mathbb{Z}_p{}^\times/\mathbb{F}_p{}^\times \to 1 + p\mathbb{Z}_p$.

For A/p, we have the following description of $\mathbb{Z}_p$:

$$\begin{aligned}
\mathbb{Z}_p &= \{0\} \amalg \coprod_{j \geq 0} p^j \mathbb{Z}_p{}^\times \\
&\cong \{0\} \amalg (\mathbb{Z}_p{}^\times \times \mathbb{N}_0) \\
&\cong \{0\} \amalg (\mathbb{F}_p{}^\times \times (1 + p\mathbb{Z}_p) \times \mathbb{N}_0).
\end{aligned}$$

with the factor $\mathbb{N}_0$ acting via $j \to p^j$. This is compatible with the action of $F_p{}^\times$ which leads to the description for $\mathcal{A}/p$. □

5 Unstable Operations

In this final part we examine the algebra of unstable operations of degree zero. Our main tool here is knowledge of the co-operations and of the additive operations. Thus operations from ordinary K–theory appear only via additive operations. Although we still have a strong connection between unstable operations in ordinary K–theory and unstable operations in K–theory with mod p coefficients, to get a precise statement would involve investigating a highly non-trivial spectral sequence. This spectral sequence has already been computed for the dual world of co-operations, [Wil84], and so to avoid reinventing the wheel we shall use the structure of the co-operations together with what we already know about additive operations to investigate the unstable realm.

5.1 Co-operations in Degree Zero

Co-operations in a multiplicative generalised cohomology theory form a *enriched Hopf ring*. The essential ingredients of this structure are two multiplications and one comultiplication. The multiplications are induced by the structure maps which define the multiplication and the addition on the cohomology theory. Following [BJW95] we write these as, respectively, $\circ$–multiplication and $*$–multiplication. One can $\circ$–multiply any two co-operations but can only $*$–multiply two co-operations that lie in the homology of the same space.

The rules of how these multiplications interact mean that the most straightforward way to describe the co-operations is to give a list of elements from which all possible $\circ$–products are formed to give a family that $*$–generates the co-operations.

For $K(1)_*(\underline{K(1)}_*)$ these initial elements are:

$$
\begin{aligned}
b_{(i)} &\in K(1)_{2p^i}(\underline{K(1)}_2) \quad \text{for } i \geq 0, \\
a_{(0)} &\in K(1)_2(\underline{K(1)}_1), \\
e &\in K(1)_1(\underline{K(1)}_1), \\
[v_1] &\in K(1)_0(\underline{K(1)}_{-2(p-1)}), \\
[{v_1}^{-1}] &\in K(1)_0(\underline{K(1)}_{2(p-1)}).
\end{aligned}
$$

The relations that these elements satisfy are:

$$
\begin{aligned}
[1]^{*p} &= 1_0, \\
[v_1] \circ [{v_1}^{-1}] &= [1[, \\
e^{\circ 2} &= -b_{(0)}, \\
{b_{(i)}}^{\circ p} &= {v_1}^{p^i} b_{(i)} \circ [{v_1}^{-1}], \\
e \circ {b_{(0)}}^{p-1} &= v_1 e \circ [{v_1}^{-1}], \\
{a_{(0)}}^{*p} &= v_1 a_{(0)} - a_{(0)} \circ {b_{(0)}}^{p-1} \circ [v_1].
\end{aligned}
$$

Here, $[1]$ is the unit with respect to the $\circ$–multiplication and 1_k is the unit with respect to the $*$–multiplication in the homology of the kth space.

We wish to determine the part lying in $K(1)_0(\underline{K(1)}_0)$. As the $*$–multiplication does not change the spacial degree, we start by determining which $*$–generators lie in $K(1)_*(\underline{K(1)}_0)$. From the relations we find that the $*$–generators are of the form:

$$a_{(0)}{}^{\alpha} \circ e^{\epsilon} \circ b^{\circ J} \circ [v_1{}^k]$$

with $\alpha, \epsilon \in \{0,1\}$, $k \in \mathbb{Z}$, and $J = (j_0, j_1, \dots)$ is a multi-index with $0 \leq j_i < p$ and all but a finite number zero. If either α or ϵ is non-zero we could impose the additional constraint that $j_0 \neq p-1$ but choose, as previously, to impose the constraint that $j_0 \neq 0$. This generic $*$–generator lies in $K(1)_n(\underline{K(1)}_m)$ where:

$$\begin{aligned} n &= 2\alpha + \epsilon + \sum 2 j_i p^i, \\ m &= \alpha + \epsilon + \sum 2 j_i - 2k(p-1). \end{aligned}$$

Thus to get $m = 0$ we need $\alpha = \epsilon$ and $\sum j_i \equiv -\alpha \mod (p-1)$. Note that ϵ and k are completely determined by α and J, moreover if α is non-zero then J must also be non-zero (i.e. have a non-zero term).

The $*$–relations that these elements satisfy are simple to deduce:

Lemma 5.1 *For J with a non-zero term:*

$$\begin{aligned} (b^{\circ J} \circ [v_1{}^k])^{*p} &= 0, \\ (a_{(0)} \circ e \circ b^{\circ J} \circ [v_1{}^k])^{*2} &= 0. \end{aligned}$$

Proof. The second identity is the easier: the element has odd degree and therefore must square to zero as our characteristic is odd.

For the first identity we start by noticing that the $[v_1{}^k]$ is a red herring: as the coproduct of $[v_1{}^k]$ is $[v_1{}^k] \otimes [v_1{}^k]$, [BJW95, Equation (10.11)] implies that for any c, d:

$$(c \circ [v_1{}^k]) * (d \circ [v_1{}^k]) = (c * d) \circ [v_1{}^k].$$

Thus it is sufficient to show that $(b^{\circ J})^{*p} = 0$. This follows from [BJW95, Equation 15.13]:

$$(c \circ b^{\circ J})^{*p} = c^{*p} \circ b^{\circ s(J)}$$

where $s(J)$ is J shifted by 1: if $J = (j_0, j_1, \dots)$ then $s(J) = (0, j_0, j_1, \dots)$. We apply this in the case that $c = [1]$ for then $c^{*p} = [p] = 1_0$. Now $1_0 \circ b^{\circ J} = 0$ for J non-zero but $[1] \circ b^{\circ J} = b^{\circ J}$. Hence:

$$(b^{\circ J})^{*p} = ([1] \circ b^{\circ J})^{*p} = [1]^{*p} \circ b^{\circ s(J)} = 1_0 \circ b^{\circ s(J)} = 0,$$

as required. □

The expression $a \circ e \circ b^{\circ J}$ can be simplified. As $e = v_1{}^{-1} e \circ b_{(0)}{}^{p-1} \circ [v_1]$, we can write simplify as follows:

$$\begin{aligned} a \circ e \circ b^{\circ J} &= v_1{}^{-1} a \circ e \circ b_{(0}{}^{p-1} \circ [v_1] \circ b^{\circ J} \\ &= (v_1{}^{-1} a \circ e \circ b_{(0)}{}^{p-2} \circ [v_1]) \circ (b_1 \circ b^{\circ J}) \\ &= a \circ b^{\circ J'}; \end{aligned}$$

where $a = v_1{}^{-1} a \circ e \circ b_{(0)}{}^{p-2} \circ [v_1]$ and J' is obtained from J by adding a factor of b_1. This has the effect of replacing j_0 by $j_O + 1$ if $j_O \neq p-1$ and by 1 if $j_0 = p-1$.

Note that we have kept the condition that $j'_0 \neq 0$ and now we have $\sum j'_i \equiv 0 \mod (p-1)$.

We have $*$–generators of $K(1)_*(\underline{K(1)}_0)$ indexed by particular sequences, J. For notational reasons it will be useful to reindex these. Recall that there is a one-to-one correspondence between finite sequences $J = (j_0, j_1, \dots)$ with $0 \leq j_i < p$ and natural numbers via $J \to \sum j_i p^i$.

Our restrictions on J translate to restrictions on the natural number, n. Since $n \equiv \sum j_i \mod (p-1)$ the restriction $\sum j_i \equiv 0 \mod (p-1)$ translates to $n \equiv 0 \mod (p-1)$. Then as $n \equiv j_0 \mod p$, the condition $j_0 \neq 0$ translates to $n \not\equiv 0 \mod p$.

Definition 5.2 *For $n \equiv 0 \mod (p-1)$ we define the elements:*

$$\begin{aligned} \beta_n &= {v_1}^{l_n} b^{\circ J_n} \circ [{v_1}^{k_n}] && \in K(1)_0(\underline{K(1)}_0), \\ \alpha_n &= {v_1}^{l_n} a \circ b^{\circ J_n} \circ [{v_1}^{k_n}] && \in K(1)_1(\underline{K(1)}_0); \end{aligned}$$

where l_n and k_n are the integers, uniquely determined by n, which make the degrees as specified. In the second case we make the additional constraint that $n \not\equiv 0 \mod p$.

Proposition 5.3 *The Hopf ring $K(1)_*(\underline{K(1)}_0)$ is the tensor product of a truncated polynomial algebra on the β_n, subject to ${\beta_n}^{*p} = 0$, and an exterior algebra on the α_n.*

From this we extract the degree zero part. As the β_n are in degree zero the truncated polynomial algebra remains, albeit over $\mathbb{F}_p$ rather than $K(1)^*$. The exterior generators lie in degree -1 so we need to take the degree zero part.

Using the periodicity of $K(1)^*$ we find that this slice has the following description: Let $\mathcal{I}$ be the family of sets $I \subseteq \mathbb{N}$ such that:

1. each element $i \in I$ satisfies:
   (a) $i \equiv -1 \mod (p-1)$,
   (b) $i \not\equiv 0 \mod p$;
2. $|I|$ is divisible by $2(p-1)$

and define:

$$\alpha_I := {v_1}^{-r_I} \bigstar_{i \in I} \alpha_i$$

where $2(p-1)r_I = |I|$ and the $*$–product is understood to be taken in increasing order of indices. These elements form a basis for the part of $K(1)_0(\underline{K(1)}_0)$ coming from the exterior algebra. The rules for multiplying these elements are:

$$\alpha_I * \alpha_J = \begin{cases} (-1)^{IJ} \alpha_{I \cup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

where $(-1)^{IJ}$ is the sign of the permutation that puts $I \cup J$ into the correct order.

5.2 What the Additives See

Now that we have the description of the co-operations the next stage is to see what the additive operations generate. The additive operations themselves obviously see that part of the co-operations which quotients to the additive co-operations. To determine what products of additive operations see we need to combine this quotient map with the coproduct on unstable co-operations.

The quotient map to additive co-operations essentially kills all $*$–products. Specifically:

$$q(c * d) = \epsilon(c)q(d) + \epsilon(d)q(c)$$

where ϵ is the augmentation. All the non-trivial elements that we have been working with lie in the kernel of the augmentation so the quotient of a non-trivial $*$–product of $*$–generators is zero. The other use that we shall make of the augmentation is the identity $c \circ 1_k = \epsilon(c)1_{k+m}$.

The coproduct obeys the following rules:

$$\begin{aligned}
\psi(c * d) &= \psi(c) * \psi(d), \\
\psi(c \circ d) &= \psi(c) \circ \psi(d), \\
\psi(v_1 c) &= v_1 \psi(c), \\
\psi([v_1]) &= [v_1] \otimes [v_1], \\
\psi(a) &= 1_1 \otimes a + a \otimes 1_1, \\
\psi(e) &= 1_1 \otimes e + e \otimes 1_1, \\
\psi(b_k) &= \sum_{i+j=k} b_i \otimes b_j.
\end{aligned}$$

Note that in the last statement we are using the b_i rather than the $b_{(i)}$. The relationship is $b_{(i)} = b_{p^i}$.

We shall start our analysis with the polynomial part of $K(1)_0(\underline{K(1)}_0)$. The formula for $\psi^l \beta_n$ can get a little complicated, but things are simplified considerably when we quotient down to additive co-operations.

Proposition 5.4 *Let $n_1, \dots, n_k$ be a sequence of non-necessarily-distinct non-zero natural numbers divisible by $p-1$. Then:*

$$q\psi^l(\beta_{n_1} * \cdots * \beta_{n_k}) = \begin{cases} 0 & \text{if } l < k, \\ \sum_{\sigma \in S_k} q\beta_{n_{\sigma 1}} \otimes \cdots \otimes q\beta_{n_{\sigma k}} & \text{if } l = k. \end{cases}$$

The analogous result for $l > k$ is non-trivial and not overly complicated but we shall not need it save in one special case. The requirement that each n_i be non-zero merely ensures that the β_{n_i} are non-trivial.

Proof. We start by considering a product, B, of a sequence of elements of the form $d_{ij} := b_i \circ [v_1{}^j]$ chained together by $\circ$– and $*$–multiplications (where we give the $\circ$–multiplication higher binding than the $*$–one). We shall use two ways of referring to this product. The coproduct treats both multiplications in the same way so when working out $\psi^l B$ we don't need to know which multiplication is which. Therefore we shall write:

$$B := d_{i_1 j_1} d_{i_2 j_2} \cdots d_{i_k j_k}.$$

On the other hand, the quotient maps does treat the two multiplications differently. In this case we shall be most focused on the $*$–product and so shall write:

$$B := c_1 * c_2 * \cdots * c_m$$

where each c_j is a $\circ$–product of d_{ij}s.

The formulæ for ψb_n and for $\psi[v_1]$ show that:

$$\begin{aligned}\psi d_{ij} &= \Big(\sum_{r+s=i} b_r \otimes b_s\Big) \circ [v_1{}^j] \otimes [v_1{}^j] \\ &= \sum_{r+s=i} (b_r \circ [v_1{}^j]) \otimes (b_s \circ [v_1{}^j]) \\ &= \sum_{r+s=i} d_{rj} \otimes d_{sj}.\end{aligned}$$

Iterating this formula shows that:

$$\psi^l d_{ij} = \sum_{r_1+\cdots+r_l=i} d_{r_1 j} \otimes \cdots \otimes d_{r_l j}.$$

Hence:

$$\psi^l(B) = \sum \bigotimes_{s=1}^{l} d_{r_{1s}} j_1 d_{r_{2s}} j_2 \cdots d_{r_{ks}} j_k$$

where $\sum_t r_{ts} = i_s$ and the sum is over all (ordered) families of such decompositions.

When applying the quotient map, q, a term in the tensor product will only survive if there are no non-trivial $*$–multiplications. Therefore if we group the $\circ$–multiplications together we can only have one non-trivial such grouping. A typical piece of the summation is:

$$(c_{11} * c_{12} * \cdots * c_{1k}) \otimes \cdots \otimes (c_{l1} * c_{l2} * \cdots * c_{lk}).$$

For this term to survive the quotient map then for each $1 \le j \le k$ only one of the c_{ij}s can be non-trivial. However we also have constraints on the c_{ij} since they must "add up" to the corresponding term in B. Thus for a given $1 \le i \le l$ at least one of the c_{ij} must be non-trivial.

If $l < k$ there is no way to satisfy both of these conditions. If $l = k$ we can satisfy these conditions but are forced to have exactly one of the c_{ij} non-trivial for a given i. Thus we get one possibility for each permutation in S_k. As the c_{ij} "add up" to c_i, the non-zero term must be c_i. Hence:

$$q\psi^k(c_1 * \cdots * c_k) = \sum_{\sigma \in S_k} c_{\sigma 1} \otimes \cdots \otimes c_{\sigma k}.$$

Finally, we note that each β_j is of the form $v_1 c_j$ for c_j as above. Therefore as the coproduct and quotient maps are $K(1)^*$–linear we have the desired result. □

Using this formula we can describe the sub-algebra of $K(1)^0(\underline{K(1)}_0)$ generated by the additive operations of degree zero.

Theorem 5.5 *The dual of the subalgebra of $K(1)_0(\underline{K(1)}_0)$ generated by the β_n is generated by the additive operations of degree zero, $PK(1)^0(\underline{K(1)}_0)$. The only relations are that if ϕ is an additive operation then so is ϕ^p.*

Alternatively, this subalgebra is generated by the stable operations with no relations.

What this means is that given a basis $\{\phi_j\}$ for the additive operations then the subalgebra of the unstable operations generated by the additive operations is the power series ring on the $\{\phi_j\}$ quotiented by relations of the form ${\phi_j}^p = \sum c_i \phi_i$. The alternative description is that this subalgebra is the power series ring on a basis for the stable operations with no relations.

Proof. The elements $q\beta_n$ are a basis for $Q(K(1))_0^0$. Therefore for any n we have additive operations $\phi_1, \dots, \phi_n$ such that $\phi_i(\beta_j) = \delta_{ij}$ for $1 \le i, j \le n$. Consider the subalgebra of $K(1)_0(\underline{K(1)}_0)$ generated by $\{\beta_1, \dots, \beta_n\}$. Then:

$$(\phi_{i_1} \cdots \phi_{i_l})(\beta_{n_1} * \cdots * \beta_{n_k}) = (\phi_{i_1} \otimes \cdots \otimes \phi_{i_l}) q \psi^k (\beta_{n_1} * \cdots * \beta_{n_k}).$$

If $l < k$ this is zero and if $l = k$ we get:

$$\begin{aligned}(\phi_{i_1} \cdots \phi_{i_l})(\beta_{n_1} * \cdots * \beta_{n_k}) &= \sum_{\sigma \in S_k} \phi_{i_1}(\beta_{n_{\sigma 1}}) \cdots \phi_{i_k}(\beta_{n_{\sigma k}}) \\ &= \sum_{\sigma \in S_k} \delta_{i_1 n_{\sigma 1}} \cdots \delta_{i_k n_{\sigma k}}.\end{aligned}$$

Thus $\phi_{i_1} \cdots \phi_{i_l}$ does not see any $*$–product of length greater than l and of those of length l it can only see $\beta_{i_1} * \cdots * \beta_{i_l}$. On this element its value is $l_1! \cdots l_m!$ where $l_1, \dots, l_m$ are the number of repetitions in the set $\{i_1, \cdots, i_l\}$. Since ${\beta_i}^{*p} = 0$ a non-trivial product can only contain repetitions of length less than p and so $l_1! \cdots l_m!$ is invertible in $\mathbb{F}_p$.

From this we deduce that the products of the ϕ_j span the dual of the subalgebra generated by $\{\beta_1, \dots, \beta_n\}$ and a basis is given by those products in which each ϕ_j occurs no more than $(p-1)$ times. Hence the dual of the subalgebra of $K(1)_0(\underline{K(1)}_0)$ generated by the β_j is generated by the additive operations subject to some relations expressing ϕ^p as an additive operation.

The map $\phi \to \phi^p$ is the *Frobenius* map and is linear as we are in characteristic p. To work out what it is we consider the pth iterate of the coproduct map. Using the notation of proposition 5.4 we find that if $B = d_{1_1 j_1} \cdots d_{i_k j_k}$ then:

$$\psi^p(B) = \sum \bigotimes_{s=1}^{p} d_{r_{1s} j_1} d_{r_{2s} j_2} \cdots d_{r_{ks} j_k}$$

with $\sum_t r_{st} = i_s$. As the division $(r_{s1}, \dots, r_{sp})$ of i_S is ordered, a given *unordered* division may occur more than once. When applying $\phi \otimes \cdots \otimes \phi$ to $\psi^p(B)$ any two terms that differ only by the ordering will contribute the same amount. Now the number of orderings on a fixed unordered division depends on the number of repetitions; specifically it is:

$$\frac{p!}{n_1! n_2! \cdots n_l!}$$

where $n_1, \dots, n_l$ are the number of repetitions. Since $n_1 + \cdots + n_l = p$ this number will be divisible by p unless $n_1 = p$. This forces i_s to be divisible by p. Hence $\phi^p q \psi^p(B) = 0$ unless each i_s is divisible by p whence, if $i_S = p m_S$:

$$\phi^p(B) = \left(\phi(d_{m_1 j_1} d_{m_2 j_2} \cdots d_{m_k j_k})\right)^p = \phi(d_{m_1 j_1} \cdots d_{m_k j_k}).$$

The last step is from Fermat's little theorem as we are in $\mathbb{F}_p$ by this point.

In particular, $\phi^p(B) = 0$ if B contains a non-trivial $*$–product. Hence ϕ^p factors through the quotient map q and so is additive. To determine which additive operation it is we compute $\phi^p(q\beta_n)$. The relevant part of β_n is the $b^{\circ J}$ piece. This is ${b_1}^{\circ j_0} \circ {b_p}^{\circ j_1} \circ \cdots$. Each of these terms is divisible by p except the first and so we get a zero contribution if $j_0 \neq 0$, otherwise the exponents shift down one. That is:

$$\phi^p(\beta_n) = \begin{cases} \phi(\beta_m) & \text{if } n = pm \\ 0 & \text{otherwise.} \end{cases}$$

In particular consider the Adams' operation Ψ^k. The value of this on β_n is the mod p reduction of:

$$\prod_{i \geq 0} \binom{k}{p^i}^{n_i}$$

where $n = \sum n_i p^i$. Thus $(\Psi^k)^p(\beta_{mp})$ is the mod p reduction of:

$$\prod_{i \geq 0} \binom{k}{p^i}^{m_{i+1}}.$$

From the appendix we see that $\binom{r}{s} \equiv \binom{rp}{sp} \mod p$ and so this is equivalent to:

$$\prod_{i \geq 0} \binom{kp}{p^{i+1}}^{m_{i+1}}$$

which is $\Psi^{kp}(\beta_{mp})$. Now if n is not divisible by p, $(\Psi^k)^p(\beta_n) = 0$ and $\Psi^{kp}(\beta_n)$ is the reduction of:

$$\prod_{i \geq 0} \binom{kp}{p^i}^{n_i}.$$

The first term of this is $\binom{kp}{1}$ to a non-zero power and thus is divisible by p. Hence $\Psi^{kp}(\beta_n) = 0$ and so:

$$(\Psi^k)^p = \Psi^{kp}.$$

We can use this in two ways: either we can say that the operations we are looking at are generated by *all* the Adams' operations subject to the above identity, or we can use the above identity to through out all the Adams' operations corresponding to integers divisible by p. This leaves precisely the stable operations, and Ψ^0. □

Recall that the stable operations were generated under composition by $\Psi^{\tilde{q}} - 1$ where q is primitive modulo p^2 and $\tilde{q} = q^{p-1}$. Therefore the operations that we are considering are:

$$\mathbb{F}_p \oplus \mathbb{F}_p[\![(\Psi^{\tilde{q}} - 1)^{\circ k}]\!]$$

where, for operations, $\circ$ denotes composition. It would be nice to write this in a similar sort of language to the co-operations, namely to say that the operations have two multiplications and that they are generated by $*$–products of $\circ$–products of the generator $\Psi^{\tilde{q}} - 1$. This is almost true. Where it breaks down is that composition is not a multiplication as it does not distribute over addition of all operations.

5.3 The Exterior Slice

The other part of the operations is the exterior algebra generated by the α_n. To work out the iterated coproduct of the α_n we first consider the coproduct of a.

Lemma 5.6 *The element a is primitive. That is:*

$$\psi(a) = a \otimes 1_1 + 1_1 \otimes a.$$

Proof. By definition, $a = {v_1}^{-1} a_{(0)} \circ e \circ {b_{(0)}}^{p-2} \circ [v_1]$. The coproduct distributes over the $\circ$–multiplication so we consider each term in turn. Each piece is primitive (recall that $b_{(0)} = b_1$) save for $[v_1]$ which is group-like. Moreover, each piece annihilates on contact with 1_k, again save $[v_1]$ which satisfies $[v_1] \circ 1_k = 1_{1-2(p-1)}$. Thus in the product we are left with:

$$\begin{aligned}\psi(a) &= {v_1}^{-1}\Big((a_{(0)} \circ e \circ {b_{(0)}}^{p-2}) \otimes 1_1 + 1_1 \otimes (a_{(0)} \circ e \circ {b_{(0)}}^{p-2})\Big) \circ ([v_1] \otimes [v_1]) \\ &= a \otimes 1_1 + 1_1 \otimes a\end{aligned}$$

□

Now when we consider something like $\psi^l(a \circ b_k)$, we are going to be taking the $\circ$–product of this with:

$$\sum_{i+j=k} b_i \otimes b_k.$$

Any non-trivial b_i meeting 1_1 will annihilate. Moreover, any a meeting 1_1 will annihilate. The only surviving terms will be of the form:

$$(a \circ b_k) \otimes 1_4 + 1_4 \otimes (a \circ b_k).$$

Hence each α_n is primitive.

Lemma 5.7 *Let $n_1, \dots, n_k$ be distinct natural numbers all divisible by $(p-1)$. Then:*

$$q\psi^l(\alpha_{n_1} * \cdots * \alpha_{n_k}) = \begin{cases} \sum_{\sigma \in S_k} |\sigma|\, \alpha_{n_{\sigma 1}} \otimes \cdots \otimes \alpha_{n_{\sigma k}} & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since 1_n is the $*$–unit, applying the iterated coproduct to the expression $\alpha_{n_1} * \cdots * \alpha_{n_k}$ produces the sum of all the ways to put each α_n into each slot. When we apply the quotient map,q, we kill any term where more than one α_n occurs in a given slot, as q kills $*$–products, and we kill any term with a vacant slot, as q kills the $*$–units. Hence we get zero unless $k = l$ whence we get the expression as given. As all the α_n are odd, the shuffling introduces signs. □

Corollary 5.8 *The exterior part of $K(1)^*(\underline{K(1)}_0)$ is generated by the additive operations in $PK(1)^1(\underline{K(1)}_0)$.*

Proof. Looking at the presentation of the additive co-operations, we see that a basis for $Q(K(1))^0_1$ is the family $q\alpha_n$. Therefore for a given admissible $n \in \mathbb{N}$ there is a family of additive operations ϕ_j such that $\phi_j(\alpha_i) = \delta_{ij}$ for $i, j \leq n$ satisfying the restrictions. As these have degree one they are exterior generators. Using the coproduct formula we can see that the family of products of the ϕ_j is dual to the family of products of the α_i. □

5.4 A Distinctly Odd Operation

We have introduced to our discussion the $\mathbb{F}_p$–vector space $PK(1)^1(\underline{K(1)}_0)$ which we have not previously considered. Therefore to complete our understanding of unstable operations in degree zero we should consider this space. We begin with the perhaps surprising result that every element of $PK(1)^1(\underline{K(1)}_0)$ is a component of a stable operation. This depends on the work of [SW05].

Proposition 5.9 *Every additive operation of odd total degree is a component of a unique stable operation.*

Proof. From the presentation of the generators of $Q(K(1))^*_*$ we have the generic basis element:

$$a_{(0)}{}^{\alpha} e^{\epsilon} b^J w_1{}^k$$

with $\alpha, \epsilon \in \{0, 1\}$, $k \in \mathbb{Z}$, and J a multi-index as usual. For this to be of odd total degree we must have $\alpha = 1$ since $a_{(0)}$ is the only element of odd total degree.

In [SW05] it is shown that there is an idempotent $s \in Q(K(1))^0_0$ such that $sQ(K(1))^k_i \to K(1)_{i-k}(K(1), o)$ is an isomorphism. This idempotent is $v_1{}^{-1} b_1{}^{p-1} w_1$. Now the element $a_{(0)}$ satisfies the identity:

$$a_{(0)} b_1{}^{p-1} = v_1 a_{(0)} w_1{}^{-1}$$

from which we can easily see that $sa_{(0)} = a_{(0)}$. Hence any basis element of odd total degree lies in the ideal generated by s and thus the stabilisation map $Q(K(1))^k_i \to K(1)_{i-k}(K(1), o)$ is an isomorphism for $(i - k)$ odd.

The dual of this statement is that the natural map $K(1)^{i-k}(K(1), o) \to PK(1)^i(\underline{K(1)}_k)$ is surjective for $(i - k)$ odd and hence every additive operation of odd degree is the component of a stable operation. This stable operation is unique as for $K(1)$–theory stable operations inject into additive ones. □

Therefore we are really studying $K(1)^1(K(1), o)$. To figure out this space we return briefly to K–theory with mod p coefficients. Recall that we showed that $Kp^0(Kp, o)$ was isomorphic to $K^0(K, o)/p$. We can extend this result to give $Kp^*(Kp, o)$ in terms of $K^0(K, o)/p$:

Proposition 5.10 *There is an isomorphism:*

$$Kp^*(Kp, o) \cong K^*(K, o)/p \otimes \Lambda(\alpha_1)$$

for some central $\alpha_1 \in Kp^1(Kp, o)$.

Proof. For the same reason as for $K(1)^*(-)$, $Kp^*(-)$ has a Künneth formula. Therefore using $Kp = K \wedge M\mathbb{F}_p$ we see that:

$$Kp^*(Kp, o) \cong Kp^*(K, o) \otimes Kp^*(M\mathbb{F}_p).$$

We know that the first term is $K^*(K, o)/p$. The second term we compute from the cofibre sequence $S \xrightarrow{p} S \to M\mathbb{F}_p$. As Kp^* has characteristic p we see that there are short exact sequences:

$$Kp^{i-1}(S) \to Kp^i(M\mathbb{F}_p) \to Kp^i(S)$$

and thus $Kp^*(M\mathbb{F}_p) \cong \Lambda(\eta_1)$ where $\eta_1 \in Kp^1(M\mathbb{F}_p)$ is the image of the unit in $Kp^0(S)$.

This element η_1 defines an element $\alpha_1 \in Kp^1(Kp, o)$ via $\alpha_1 = \eta_1 \wedge 1_K$. From this definition it is clear that α_1 commutes with all operations in K–theory with mod p coefficients that come from ordinary K–theory. As we now know this to be all of them, α_1 is central. □

As α_1 is central it commutes in particular with the Adams' idempotents and thus preserves the splitting of K–theory with mod p coefficients into the first Morava K–theory. Hence we have an operation $\alpha_1 \in K(1)^1(K(1), o)$ such that multiplication by α_1 is an isomorphism $K(1)^0(K(1), o) \to K(1)^1(K(1), o)$.

We also have an isomorphism $K(1)_0(K(1), o) \to K(1)_1(K(1), o)$ given by multiplication by the stabilisation of the co-operation a (note that stably $a = a_{(0)}$). Dualising yields an isomorphism $D(a) : K(1)^1(K(1), o) \to K(1)^0(K(1), o)$. To compare the two we need to compute $\alpha_1\phi(ac)$ for $\phi \in K(1)^0(K(1), o)$ and $c \in K(1)_0(K(1), o)$.

The version of α_1 for K–theory with mod p coefficients can also be defined directly using the map $M\mathbb{F}_p \to S$ of degree one coming from the cofibre sequence $S \xrightarrow{p} S \to M\mathbb{F}_p$. This definition expresses α_1 as the composition:

$$Kp \simeq M\mathbb{F}_p \wedge K \to S \wedge K \to M\mathbb{F}_p \wedge K \simeq Kp.$$

Lemma 5.11 *The two definitions of α_1 agree.*

Proof. The original definition uses $\eta_1 : M\mathbb{F}_p \to Kp$ and writes α_1 as:

$$Kp \simeq M\mathbb{F}_p \wedge K \xrightarrow{\eta_1 \wedge 1} Kp \wedge K \to Kp \wedge Kp \to Kp.$$

The map η_1 is defined so that the following diagram commutes:

$$\begin{array}{ccc} M\mathbb{F}_p & \xrightarrow{\eta_1} & Kp \\ \downarrow & & \downarrow{=} \\ S & \xrightarrow{\eta} & Kp \end{array}$$

This allows us to compare the two definitions of α_1 since we now have the commutative diagram:

$$\begin{array}{ccccccc} M\mathbb{F}_p \wedge K & \xrightarrow{\eta_1 \wedge 1} & Kp \wedge K & \longrightarrow & Kp \wedge Kp & \longrightarrow & Kp \\ \downarrow & & {=}\uparrow & & & & \\ S \wedge K & \xrightarrow{\eta \wedge 1} & Kp \wedge K & & & & \end{array}$$

Taking the lower route we find that the map $S \wedge K \to Kp$ is just the projection $K \to Kp$ because η is the unit for the multiplication map. Hence the lower route results in the new definition of α_1 and the two are thus equivalent. □

The advantage of this definition is that we see clearly that α_1 comes from the Moore spectrum piece of Kp and not from K. Therefore its properties can be deduced from those of the Moore spectrum.

Lemma 5.12 $\alpha_1\mu = \mu(\alpha_1 \wedge 1 + 1 \wedge \alpha_1)$

Proof. As the above remark indicates, it is sufficient to study the map $\eta_1\mu$: $M\mathbb{F}_p \wedge M\mathbb{F}_p \to M\mathbb{F}_p$ of degree one. Repeated analysis of the cofibre sequence $S \to S \to M\mathbb{F}_p$, using the fact that p is odd, leads to the short exact sequence:

$$\{M\mathbb{F}_p, M\mathbb{F}_p\}^0 \xrightarrow{\xi} \{M\mathbb{F}_p \wedge M\mathbb{F}_p, M\mathbb{F}_p\}^1 \xrightarrow{\zeta} \{M\mathbb{F}_p, M\mathbb{F}_p\}^1.$$

Both the outer spaces are F_p–vector spaces of dimension one. The generator for the first is the identity map and for the second is the map η_1. To describe the maps ξ and ζ we need to give some names to the various maps that we are using: let $\pi : S \to M\mathbb{F}_p$ and $\delta : M\mathbb{F}_p \to S$ be the maps in the cofibre sequence, so $\eta_1 = \pi\delta$. In terms of these, ξ and ζ are:

$$\xi(\gamma) = M\mathbb{F}_p \wedge M\mathbb{F}_p \xrightarrow{\delta\wedge 1} S \wedge M\mathbb{F}_p \simeq M\mathbb{F}_p \xrightarrow{\gamma} M\mathbb{F}_p,$$
$$\zeta(\gamma) = S \wedge M\mathbb{F}_p \xrightarrow{\pi\wedge 1} M\mathbb{F}_p \wedge M\mathbb{F}_p \xrightarrow{\gamma} M\mathbb{F}_p.$$

Clearly $\xi(1) = \delta \wedge 1$ (identifying $S \wedge M\mathbb{F}_p$ with $M\mathbb{F}_p$). We also have the map $1 \wedge \delta$. Applying ζ yields:

$$\zeta(1 \wedge \delta) = S \wedge M\mathbb{F}_p \xrightarrow{\pi\wedge 1} M\mathbb{F}_p \wedge M\mathbb{F}_p \xrightarrow{1\wedge\delta} M\mathbb{F}_p \wedge S \simeq M\mathbb{F}_p$$

As π has degree zero, this is the same as the map:

$$S \wedge M\mathbb{F}_p \xrightarrow{1\wedge\delta} S \wedge S \xrightarrow{\pi\wedge 1} M\mathbb{F}_p \wedge S \simeq M\mathbb{F}_p$$

which is $\pi\delta$; i.e. η_1.

Hence $\{M\mathbb{F}_p \wedge M\mathbb{F}_p, M\mathbb{F}_p\}$ is the two dimensional $\mathbb{F}_p$–vector space with basis $\{1 \wedge \delta, \delta \wedge 1\}$. Now $\pi : S \to M\mathbb{F}_p$ is also the unit for the multiplication on $M\mathbb{F}_p$, hence the equivalence $S \wedge M\mathbb{F}_p \simeq M\mathbb{F}_p$ is the composition $\mu(\pi \wedge 1)$ and also $\mu(1 \wedge \pi)$. Thus $1 \wedge \delta$ is the same as $\mu(1 \wedge \eta_1)$ and similarly for the other basis element.

Therefore there are coefficients $u, v \in \mathbb{F}_p$ such that $\eta_1\mu = u\mu(1\wedge\eta_1)+v\mu(\eta_1\wedge 1)$. By symmetry we must have $u = v$. To determine the coefficient we look at $\zeta(\eta_1\mu)$. This unravels as follows:

$$\zeta(\eta_1\mu) = \eta_1\mu(\pi \wedge 1) = \eta_1 = \zeta(\mu(1 \wedge \eta_1))$$

and thus $u = 1$ so $\eta_1\mu = \mu(1 \wedge \eta_1 + \eta_1 \wedge 1)$. □

Corollary 5.13 *Let $c \in K(1)_0(K(1), o)$. Then $\alpha_1(ac) = \epsilon(c)$.*

Proof. We know that $\alpha_1(d) = \epsilon\alpha_{1*}(d)$. Using the previous lemma we see that:

$$\alpha_{1*}(ac) = (\alpha_{1*}a)c + (\alpha_{1*}c)a.$$

As $\epsilon(a) = 0$, applying ϵ produces:

$$\epsilon(\alpha_{1*}a)\epsilon(c) = \alpha_1(a)\epsilon(c).$$

From [Wil84], the element a is in some sense universal. It originates in $H_2(K(\mathbb{Z}/p, 1), o)$ which is isomorphic to $\mathbb{F}_p$ with a canonical generator. Stabilisation defines an isomorphism $H_2(K(\mathbb{Z}/p, 1), o) \to H_1(H \wedge M\mathbb{F}_p)$ to which we apply $(1 \wedge \eta_1)_*$ and end up in $H_0(H \wedge M\mathbb{F}_p, o) \cong \mathbb{F}_p$. This isomorphism takes the canonical generator of $H_1(H \wedge M\mathbb{F}_p, o)$ to the unit in $H_0(H \wedge M\mathbb{F}_p, o)$. Hence $\alpha_1(a) = 1$. □

Corollary 5.14 *$\alpha_1\phi(ac) = \phi(c)$, or $-D(a)\alpha_1 = 1$.*

Proof. Write the coproduct of c as $\sum_i c_i' \otimes c_i''$. Expanding the left-hand side produces:

$$\begin{aligned}
\alpha_1\phi(ac) &= (\alpha_1 \otimes \phi)(a \otimes 1 + 1 \otimes a)(\sum_i c_i' \otimes c_i'') \\
&= (\alpha_1 \otimes \phi)\sum_i (ac_i' \otimes c_i'' + (-1)^{|c_i'|} c_i' \otimes ac_i'') \\
&= \sum_i \alpha_1(ac_i')\phi(c_i'') \\
&= \sum_i \epsilon(c_i')\phi(c_i'') \\
&= \phi\Big(\sum_i \epsilon(c_i')c_i''\Big) \\
&= \phi(c).
\end{aligned}$$

□

We conclude with our final description of the unstable operations. As the stable operations are concentrated in degrees zero and one we have the following result:

Theorem 5.15 *The unstable operations $K(1)^*(\underline{K(1)}_0, o)$ is the free completed graded commutative ring on the additive group of all stable operations.*

Remark 5.16 We could continue this analysis in two essentially equivalent ways: firstly we could determine the additive and unstable operations from degrees other than zero, secondly we could determine the unstable operations in K–theory with mod p coefficients. These would not be difficult to do, at least for even source degrees.

A p–Adically Minded

In this appendix we gather together the standard definitions and results that we need corresponding to the p–adic valuation and p–adic integers.

We start with the observation that any natural number has a unique expansion of the form $\sum_{j\geq 0} n_j p^j$ with $0 \leq n_j \leq p-1$ and, obviously, all but a finite number of terms zero. This is the p–version of the decimal expansion. The analogue of the test for divisibility by 9 is the statement that: $n \equiv \sum n_j \mod (p-1)$.

We turn to the *p–adic valuation*:

Definition A.1 *The* p–adic valuation, $\nu_p : \mathbb{Q}\setminus\{0\} \to \mathbb{Z}$ *is the unique function which satisfies:*

$$x = p^{\nu_p(x)} \frac{a}{b}$$

with $a, b \in \mathbb{Z}$ *coprime to* p.

It has various simple properties:

Lemma A.2 *The p–adic valuation satisfies:*

1. $\nu_p(xy) = \nu_p(x) + \nu_p(y)$,
2. $\nu_p(x^{-1}) = -\nu_p(x)$,
3. *for* $k \in \mathbb{Z}$, $[k]_p = 0$ *if and only if* $\nu_p(k) > 0$,

One property that we need is that the p–adic valuation of $n!$ can be computed from the p–expansion of n as $\sum_{j\geq 0} n_j p^j$.

Proposition A.3 $\nu_p(n!) = \sum_{j\geq 0} n_j \dfrac{p^j - 1}{p-1}$.

Proof. We need to compute the factors of p in $n!$, equivalently in each of $\{1, \ldots, n\}$. Now there are $n_1 + n_2 p + \cdots$ numbers less than or equal to n which are divisible by p so we start our count with this. Next, there are $n_2 + n_3 p + \cdots$ which are divisible by p^2. We have already counted these once so have accounted for one factor of p for each of these, thus we add $n_2 + n_3 p + \cdots$ to our count. We end up with:

$$\begin{aligned}
\sum_{j\geq 1} n_j + n_{j+1} p + n_{j+2} p^2 + \cdots &= \sum_{j\geq 1} \sum_{i \geq j} n_i p^{i-j} \\
&= \sum_{i\geq 1} n_i \sum_{j=0}^{i-1} p^j \\
&= \sum_{i\geq 1} n_i \frac{p^i - 1}{p-1}.
\end{aligned}$$

At each stage the sum is finite as only a finite number of the n_j are non-zero. We can include the zeroth term in the last summation since it is $n_0 \frac{p^0-1}{p-1} = 0$. □

There is a neat corollary of this which gives a condition for the p–divisibility of binomial coefficients. The condition relies on the fact that if $n = k + l$ then the p–expansion of n can be obtained from the p–expansions of k and l by *long addition*.

Corollary A.4 *For natural numbers $k_1, \ldots, k_r, n$ with $\sum k_i = n$ then the multinomial coefficient:*

$$\binom{n}{k_1, \ldots, k_r} := \frac{n!}{k_1! \cdots k_r!}$$

is not divisible by p if and only if in the long addition there were no "carries". In particular, if there is some m such that $n \geq p^m > k_j$ then $\binom{n}{k_1,\ldots,k_r}$ is divisible by p.

Proof. Continuing from the expansion of $\nu_p(n!)$ we see that:

$$\nu_p(n!) = \frac{1}{p-1}\Big(n - \sum_{j \geq 0} n_j\Big).$$

Hence:

$$\nu_p(\binom{n}{k_1, \ldots, k_r}) = \nu_p(n!) - \nu_p(k_1!) - \cdots - \nu_p(k_r!) = \frac{1}{p-1}\sum_{j\geq 0}(k_{1j} + \cdots + k_{rj} - n_j).$$

When doing the sum $\sum k_i = n$ by long addition, if we never "carry a one" then at each stage $\sum k_{ij} = n_j$ and so we get zero in the above, hence $\binom{n}{k_1,\ldots,k_r}$ is not divisible by p. However, if at the jth stage we have to do a carry (and assume, for simplicity, that we did not at the $(j-1)$th stage) then we see that $\sum k_{ij} = lp + n_j$ for some $l \geq 1$. This l "carries" to the next stage and we are considering $\sum k_{i(j+1} + l$. The overall effect is to subtract $(p-1)$ from the total $\sum k_{ij}$. Hence we get a non-zero p–adic valuation and so $\binom{n}{k_1,\ldots,k_r}$ is divisible by p.

If there is some m such that $n \geq p^m > k_j$ all j then the p–expansion of n goes on to the mth term whereas those for the k_j terminate at least by the $(m-1)$th term. Thus the last stage in the long addition must involve a "carry" and so $\binom{n}{k_1,\ldots,k_r}$ is divisible by p. □

Our main use for the p–adic valuation beyond a test for divisibility for factorials is its place in the construction of the p–adic integers.

Definition A.5 *The ring of* p–adic integers, *$\mathbb{Z}_p$, is the completion of $\mathbb{Z}$ with respect to the p–adic norm, $|\cdot|_p$, defined by:*

$$|k|_p = \begin{cases} 0 & \text{if } k = 0, \\ p^{-\nu_p(k)} & \text{otherwise.} \end{cases}$$

The standard properties of the p–adic integers are as follows:

1. The function $|\cdot|_p$ is a non-Archimedean norm. That is, it satisfies the identities:

   (a) $|k|_p = 0$ if and only if $k = 0$,

   (b) $|kl|_p = |k|_p \, |l|_p$,

   (c) $|k + l|_p \leq \max\{|k|_p, |l|_p\}$ with equality if $|k|_p \neq |l|_p$.

2. The ring structure on $\mathbb{Z}$ extends to define a ring structure on $\mathbb{Z}_p$.

3. Each p–adic integer is the limit of a unique series of the form $\sum_{j\geq 0} n_j p^j$ with $0 \leq n_j \leq p-1$ for all j. This series is called the *p–adic expansion* of the p–adic integer. Every such series converges yielding a one-to-one correspondence between p–adic integers and series $\sum n_j p^j$ of this form.

4. The inclusion $\mathbb{Z} \to \mathbb{Z}_p$ induces an isomorphism $\mathbb{Z}/p^n \to \mathbb{Z}_p/p^n$ for all p. In terms of p–adic expansion, the projection $\mathbb{Z}_p \to \mathbb{Z}/p^n$ is given by:

$$\sum n_j p^j \to \sum_{j=0}^{n-1} [n_j]_{p^n}.$$

5. The group of units, ${\mathbb{Z}_p}^\times$, consists of those p–adic integers whose expansion has non-zero leading term.

6. $\mathbb{Z}_p$ contains a unique cyclic subgroup of order $(p-1)$. Under the projection $\mathbb{Z}_p \to \mathbb{Z}/p^n$, this is taken to the unique cyclic subgroup of order $(p-1)$ in $(\mathbb{Z}/p^n)^\times$. In particular, the projection $\mathbb{Z}_p \to \mathbb{F}_p$ identifies it with ${\mathbb{F}_p}^\times$.

7. The open balls in $\mathbb{Z}_p$ are of the form $\alpha + p^n\mathbb{Z}_p$. These satisfy the following properties:

   (a) if $\beta \in \alpha + p^n\mathbb{Z}_p$ then $\beta + p^n\mathbb{Z}_p = \alpha + p^n\mathbb{Z}_p$,

   (b) let $m \leq n$, $\alpha, \beta \in \mathbb{Z}_p$, then $\alpha + p^m\mathbb{Z}_p$ and $\beta + p^n\mathbb{Z}_p$ are disjoint or $\beta + p^n\mathbb{Z}_p \subseteq \alpha + p^m\mathbb{Z}_p$ with equality if $n = m$,

   (c) every open ball is of the form $k + p^n\mathbb{Z}_p$ for some $k \in \{0, \ldots, p^n - 1\}$.

8. $\mathbb{Z}_p$ is compact.

We also have the following technical results regarding the action of ${\mathbb{F}_p}^\times$ on $\mathbb{Z}_p$ and ${\mathbb{Z}_p}^\times$.

Lemma A.6 *The projection $\mathbb{Z}_p \to \mathbb{Z}_p/{\mathbb{F}_p}^\times$ is split. The image of the splitting is:*

$$\{0\} \amalg \coprod_{n\geq 0} p^n(1 + p\mathbb{Z}_p).$$

On the units, there is an isomorphism of groups: ${\mathbb{Z}_p}^\times \cong {\mathbb{F}_p}^\times \times (1 + p\mathbb{Z}_p)$.

Proof. When considering the p–adic expansion of a product of non-zero p–adic integers, the first non-zero term is the product of the first non-zero terms. Therefore, as the $(p-1)$th roots of unity have distinct leading terms, the elements of the orbit of a non-zero $\alpha \in \mathbb{Z}_p$ are distinguishable by their first non-zero term. In particular, there is a unique element $\widehat{\alpha} \in \mathbb{Z}_p$ with first non-zero term 1 such that $\widehat{\alpha}^{p-1} = \alpha^{p-1}$. The splitting $\mathbb{Z}_p/{\mathbb{F}_p}^\times \to \mathbb{Z}_p$ is defined by:

$$\begin{aligned} 0 &\to 0, \\ {\mathbb{F}_p}^\times \cdot \alpha &\to \widehat{\alpha}. \end{aligned}$$

The non-zero elements in the image of this splitting are those with first non-zero term 1 in their p–adic expansion, which is the set given in the statement. The result on the units is clear since the first non-zero term of a unit must be the first term. □

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