Mechanics to Manifolds

I. An Introduction to Mechanics

Andrew Stacey

May 4, 1999

1 Introduction

This seminar is the first in a series of three which aim to explain some of the connections between Mechanics and Geometry. This first is very much focused on the Physics end. The aim of this talk is to be an introduction to Mechanics.

There are a few theories of Mechanics, and it can sometimes be hard to separate them in the literature and applications. There are three main principles behind them. These are the Newtonian[1], Relativistic and Quantum Theories. I have thus divided Mechanics into six areas:

1. Newtonian Mechanics.
   Applicable to systems on a large scale which are moving slowly[2].
   Newton published *Principia* in 1687.

2. Special Relativity.
   Not so much a theory of mechanics, but it has implications for mechanical theories. Applicable to systems on a large scale which are moving fast in the absence of a gravitational field.
   Einstein published his first papers on Special Relativity in 1905.
   Minkowski published *Space and Time* in 1908.

3. General Relativity.
   This includes gravitation into relativity.
   Einstein published his first papers on General Relativity in 1915.

4. Quantum Mechanics.
   Applicable to systems on a small scale which are moving slowly.
   Planck proposed that energy came in quanta in 1900.
   Einstein proposed his Quantum Theory of light in 1905.
   de Broglie proposed the principle of wave-particle duality in 1923.
   Schrödinger published his theory of wave mechanics in 1926.

---

[1] Often called Classical.

[2] "Large" and "small" are in comparison to the nanometre ($1 \times 10^{-9} m$). "Slow" and "fast" are in comparison to the megametre per second ($1 \times 10^{6} ms^{-1}$).

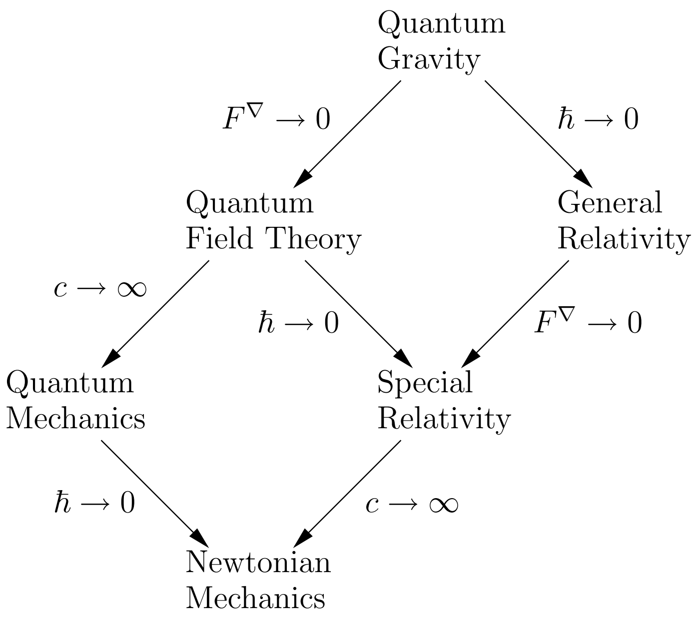

Figure 1: Theories of Mechanics

5. Quantum Field Theory.
   Represents a unification of Special Relativity with Quantum Mechanics. Dirac published his equation for a relativistic charged particle in 1926. Pauli and Heisenberg formed a QFT based on Lagrangians in 1929.

6. Quantum Gravity.
   Represents a unification of General Relativity with Quantum Field Theory. Various theories have been proposed (twistors, superstrings, etc.) but none have yet been widely accepted.

Diagrammatically, one could represent these six as in Figure 1. The theories are always formulated with the fundamental constants included by symbol. This is mainly due to the fact that writing out the exact speed of light becomes an annoyance after about the first time. One effect of this is that we can regard these constants as variables in the theory and so consider the limiting theories. What this would mean in application is that the scale of the problem is such that the substitution of the limit induces no significant inaccuracy in the solution.

In this short series of seminars, we shall be concentrating on Newtonian and Quantum Mechanics since these are the main theories of Mechanics on which everything else rests. I shall mention Relativity briefly in connection with manifolds but shall leave Quantum Field Theory to another talk - if not another seminar series - and Quantum Gravity to another century.

I shall end by describing one of the more surprising results that has been observed by experiment and which can be accounted for by Quantum Theory.

2 Why bother with Physics?

The short answer to this question is that a lot of new concepts and ways of thinking originate in Physics. Physics is a major part of the interface between Mathematics and the Real World. Thus one way to motivate Real Mathematics - i.e. beyond that needed to fill in a tax invoice - to the non-mathematician is to show how useful it is in science, then we leave it up to the scientists to explain why science is worth anything.

The boundary between Maths and Physics allows information to cross both ways. Since so much crosses from us to them it is natural to ask whether a fair amount comes back again, and this is indeed the truth. In disciplines where there is a clear application to the Real World, such as Mechanics, then this is more true than anywhere else.

Nearly all of the major intuition jumps in Mechanics have come from Physics. The way that this usually works is something like this:

First someone notices something odd about the universe. Maybe something doesn't move quite the way expected, or a careful observation of a planet reveals that it's not anywhere near predicted[3]. Then someone tries to develop a model for this new behaviour. Using this model, new experiments are devised and done until it is fairly well tested and accepted by the establishment. At some point, while the Physicists are arguing about whether some constant should have a 4 or a 5 in its thirtieth decimal place, someone else comes along and abstracts a general principle. This is then in the realms of Mathematics. The principle is used and tested, and can lead to a new branch of Mathematics. Finally a Psychologist gets hold of the new lingo and tries to use it to explain why Julius Caesar could never conquer Britain.

This is the general process, some times it goes faster than other times. Occasionally then the same person is involved at every step. More often, not. It's also hard to separate where the Physics ends and the Mathematics begins, but once someone tries extending the principle to infinite dimensions, it's a safe bet that that person has never seen a laser outside a disco.

3 Mechanics

Mechanics is that branch of Mathematics and Physics which deals with motion. The idea is that given the state of the universe now, what will it be like in 30 seconds', 10 minutes', 5 million years' time? There are two parts to this problem. The first is the amount of information required to specify the problem

---

[3] If you want to be at the forefront of the next big idea then probability says you should be an Astronomer.

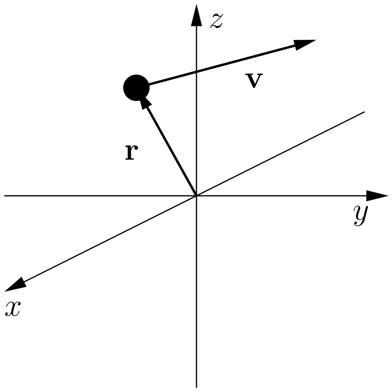

Figure 2: A Free Particle

- so-called Cauchy data - and the second is how that information evolves in time - so-called dynamics.

Usually, knowing the dynamics tells you what data you need to know, but it is often through considering the data that the laws of dynamics are initially derived.

A general principle tells one how to set up the problem, what data to look for. Then using this principle, one sets up the model. Often this involves simplifications which one hopes are negligible. For example, in planetary motions then the physical size of the planets is neglected because it is small compared to the radii of the orbits. One model is better than another if it predicts the future of the universe more accurately than the second. One general principle is better than another if it allows one to construct better models or to construct models for a wider variety of problems.

4 Manifolds

4.1 Coordinates

Before looking at the principles involved in Mechanics, we can already find a link to the study of manifolds. Consider a free particle[4] moving in $\mathbb{R}^3$ as in Figure 2. Just from intuition, we would expect a complete characterization of the states of this universe to at least include the initial position and velocity of the particle. This space is just $\mathbb{R}^3 \times \mathbb{R}^3$. Thus it is a reasonable assumption to make that the set of all possible sets of Cauchy data of the universe is a manifold. However, there is no reason at this stage to assume that this manifold is other than some Euclidean space and so there is no reason to bring in the whole of Geometry to solve a Euclidean problem.

---

[4] That is a point mass which is not acted on by anything.

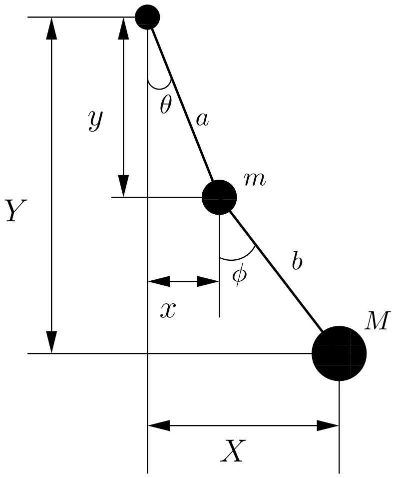

Figure 3: A Double Pendulum

So consider the problem of a double pendulum moving in two dimensions, as in Figure 3. What data might comprise Cauchy data for this problem? The answer to that of course depends on the dynamical laws, but let us see what makes sense.

Consider the action of taking a photograph of the pendulum. Whatever my Cauchy data is, it gives as much information as is possible about the configuration of the pendulum for all time. Thus it must contain enough information to specify that photograph. Let us consider, then, the simpler problem of what data is necessary to exactly specify the photograph.

In order to describe this photograph to you I would have to tell you the positions of the bobs relative to some origin. Taking the fixed point as the origin, one way to do this would be to tell you the displacements x, y and X, Y of the inner and outer bobs, respectively. Thus a first guess at the set of all possible data for the photograph is $\mathbb{R}^4$.

This is clearly far too big. Given a particular x value then there can only be two y values since x and y have to satisfy $x^2+y^2 = a^2$. Also, x cannot lie outside the interval $[-a, a]$. We could try lots of restrictions on x, y, X, Y and find some subset of $\mathbb{R}^4$ which did give us all possible configurations of our photograph and no more than those, but the simplest specification of the photograph is the angles θ and ϕ. Thus a far more reasonable guess at the set of all possible data is $S^1 \times S^1$, which is more commonly known as a torus.

Thus the operation of taking a photograph of the universe provides a map from our set of all Cauchy data to the torus. So although the Cauchy data may well be Euclidean, we need to consider maps from it to the torus and thus we force our way into the subject of Geometry in its generality.

4.2 Relativity

I want to deal quickly with Relativity here as we only have time to go into detail about Newtonian and Quantum theories. Relativity starts as a theory about how you relate what I see to what you see. One of its implications for theories of Mechanics is the idea that time is no longer special. According to relativity then time is another coordinate and space-time looks decidedly strange.

Until Einstein's theory of relativity was formulated, people had been working over a manifold chosen reasonably arbitrarily as a subspace of Euclidean space with a metric induced by that on Euclidean space as a measure of distance. Time was involved as a parameterization of paths on this manifold. However, the theories of Special and General Relativity say that time must be considered as another coordinate, equal in status with those of space. The Euclidean metric is then also wrong, the correct one being $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.[5] General Relativity also says that the universe is not a Euclidean space but space-time itself is a manifold with curvature defined to take account of the effects of gravity.

Thus if we were to consider a planet orbiting a star in 2 dimensions as in Figure 4, using Newtonian Mechanics we find that it moves in an ellipse with the star at one focus. Using General Relativity, we find that the presence of the star curves space into a cone and the planet follows a geodesic path in this cone which is almost an ellipse.

5 States and Observables

The concepts of *states* and *observables* are very important in considering the theories of mechanics.

A pure state is a *possible* configuration of the universe. The key idea is that this state, however it is Mathematically formulated, contains all the information needed to describe the universe, as exactly as theoretically possible.

An observable is a function on states which gives the result of an experiment done on the universe. It can be a measurement of displacement or momentum in a particular direction, it may be the total energy or the charge. Mathematically, the result must always be a real number. Of course, one can always interpret that real number in any way one likes and so this isn't restrictive at all. It should be emphasized that these are *real* experiments. Physically, a function qualifies as an observable if you can conceive of an experiment which would allow you to measure something which gives the value of this function on all states. Mathematically, that's not a workable definition and so we work with a set which contains all observables whether anyone has conceived of experiments or not.

This leads to a difference in definitions for quantities which are observable. A Mathematician would say that a quantity is defined by the function which corresponds to it, a Physicist would say that a quantity is defined by the experiment which observes it.

---

[5] There are ways and means to get around the fact that this is no longer positive definite.

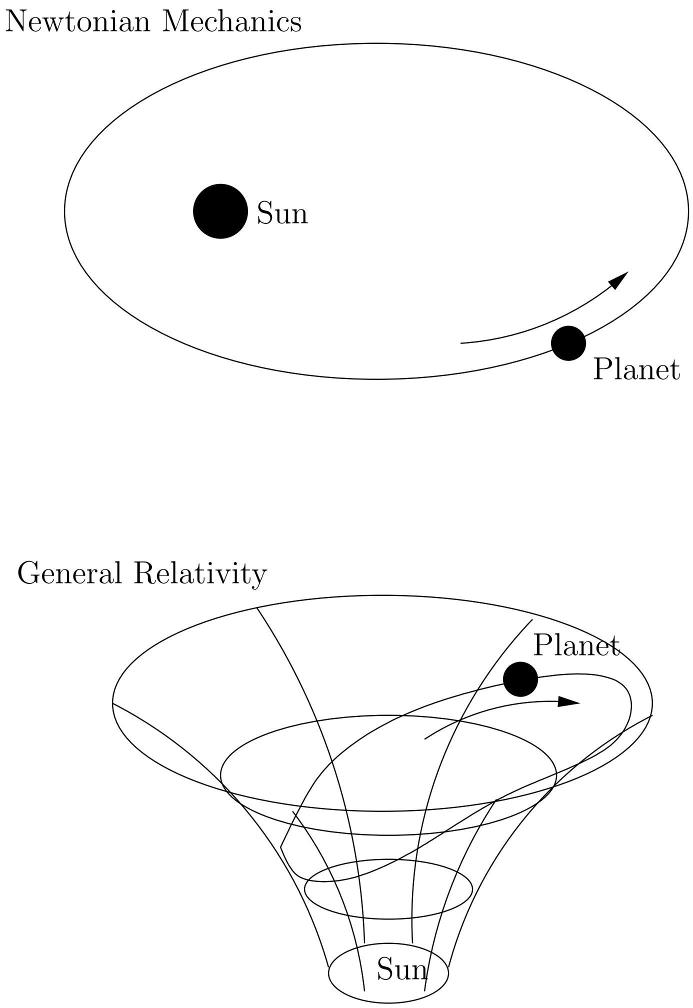

Figure 4: The Path of a Planet

6 Newtonian Mechanics

6.1 The General Principle

Newtonian mechanics deals with situations involving a number of bodies which are large compared to the atom and traveling slowly compared to light[6]. It introduces the concept of a force to explain the dynamics. Usually the required information is the initial position and velocity of each body together with information about the forces involved. With this information, Newton's laws of Motion can be applied. These are:

1. A free body moves with constant velocity.
2. If acted on by a net force $\mathbf{F}$ then $\dot{\mathbf{p}} = \mathbf{F}$, where $\mathbf{p}$ is the momentum of the body.
3. Every action has an equal an opposite reaction.

This last really means that if one particle exerts a force by some means on another particle then there is a force exerted on the first by the second of equal magnitude and opposite direction. It should also be noted that the word action has many different meanings in mechanics.

Physically, momentum is just the product of mass and velocity. If the mass is held constant then Law 2 reduces to the more commonly known $F = ma$. I shall come to the Mathematical definition shortly.

So to develop a model for a situation, one need only completely specify the forces that are involved and then the motion of the bodies involved is determined as a differential equation.

It should be noted that Law 2 *defines* the concept of a force. A force is that which causes a change in momentum of some body. One needs other laws to determine what these forces are. Some of them are:
Gravitation (mk I): For two bodies of masses M and m of distance d apart, each experiences a force of $\frac{GMm}{d^2}$ towards the other.
Gravitation (mk II): For a body of mass m near the surface of the earth, there is a force mg downwards. This is an approximation to the above.
Electric charge: For two charged bodies of charges Q and q, of distance d apart, each experiences a force of $-\frac{1}{4\pi\epsilon_0}\frac{Qq}{d^2}$ towards the other. This is similar to gravity, but note that if Q and q are of the same sign, the force is a repulsion.

The concept of a field[7] is also necessary in Newtonian Mechanics. It is often used to define these forces. But in the Newtonian formulation, fields are a means to an end rather than objects worth study in their own right.

---

[6] Strictly speaking, whenever one can neglect the non-zero size of Planck's constant h and the reciprocal of the speed of light c^{-1}.

[7] A field is a section of a vector bundle over the configuration space.

6.2 Phase Space

According to Newtonian Mechanics, the Cauchy data for the universe is the positions and momenta or velocities of the bodies and knowledge of the forces and how they evolve. Often the forces are either fixed or a function of the positions and velocities and so the Cauchy data becomes just the positions and velocities.

In the case of our double pendulum, the positions are the angles, so the initial velocities are the angular velocities. Thus the set of all Cauchy data is the total space of the tangent bundle to the torus.

In general this is the case. The positions take values on some manifold and the set of all Cauchy data is the total space of the tangent bundle. This is called the Phase Space.

This isn't the only possible phase space. For example, we could identify the tangent and cotangent bundles somehow and use the total space of the cotangent bundle. Why we would want to do this will hopefully become clear.

The observables, Mathematically, are then just smooth functions from the phase space to the real numbers.

6.3 Lagrangians and Hamiltonians

There are many equivalent ways of expressing the principle behind Newtonian Mechanics. The most important two are the Lagrangian and Hamiltonian formulations. These allow one to set up a system for which the evolution of the universe minimizes a function. This is useful as it allows one to use variational principles to study and solve a vast array of problems. The key idea in variational methods is to start with a basic problem for which we know the solution exactly and then to look at problems which are in some sense close to this one in the hope that the solutions are similarly close.

There are also Quantum Mechanical reasons why the Lagrangian and Hamiltonian formulations are so important. I will mention these later.

For these formulations, one has functions L and H from particular phase spaces to the real numbers. The dynamics are then controlled by equations involving the differentials of these functions.

For the Lagrangian formulation, the Lagrangian function L is a map with domain the tangent space of the configuration space. Thus it is a map involving positions[8] and velocities. It may also explicitly involve time. The path taken by the state of the universe through phase space must then satisfy the relations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Here q runs through a set of coordinates and $\dot{q}$ are the velocities relating to those coordinates. A solution is a path through the configuration space with a particular parameterization, which gives rise to the velocities upon differentiation.

---

[8] Here position is used to denote any coordinate on the configuration space.

For the Hamiltonian formulation, the Hamiltonian function H is a map with domain the cotangent space of the configuration space. Again, it may explicitly involve time. The path taken by the state of the universe through phase space must then satisfy the relations:

$$\frac{\partial H}{\partial p} = \frac{dq}{dt} \qquad \frac{\partial H}{\partial q} = -\frac{dp}{dt}$$

A solution to this is directly a path through phase space.

The problem then is to determine the functions L or H for the problem involved. In many cases, they can be written down directly from considering the energy. If the forces are of a certain type, then a quantity called the potential energy can be defined as a function of position only. This is often written as V and is usually defined to be the energy needed to move a particle from infinity to that point under the given forces. It is related to the forces via the equations $\mathbf{F} = -\nabla V$. Then the kinetic energy, T, can also be defined, either as a function of tangent or cotangent vectors. In this case we have $L = T - V$ and $H = T + V$.

In these cases, in cartesian coordinates we have:

$$L = \frac{1}{2} m v^2 - V$$

and so Lagrange's equations give:

$$\frac{d}{dt} m v = -\frac{dV}{dx}$$

The Hamiltonian is:

$$H = \frac{1}{2m} p^2 + V$$

and so Hamilton's equations give:

$$\frac{dx}{dt} = \frac{1}{m} p \quad \frac{dp}{dt} = -\frac{dV}{dt}$$

Now we need to compare these two systems. The Hamiltonian is a function on the cotangent bundle and so involves this identification of the tangent and cotangent bundles which is not precisely defined as yet. However, the Hamiltonian formulation involves only first-order differentials and is defined on the cotangent space which is a symplectic manifold. Then the methods of symplectic geometry can be employed which greatly aid matters.

So the Lagrangian is easier to write down but the Hamiltonian is easier to solve. Fortunately, one can get from one to the other.

For each point x in the configuration space, we have a map $L_x : T_x M \to \mathbb{R}$. This can be differentiated to produce a map $DL_x : T_x M \to T_x^* M$, which then gives rise to a bundle map $DL : TM \to T^* M$. The image of the velocity vector under this map is the momentum vector corresponding to that coordinate. There is then a process whereby the Hamiltonian can be calculated from the Lagrangian[9], and thus the problem can be more easily solved than directly from the Lagrangian.

---

[9] This isn't always possible, but in most cases where Newtonian Mechanics is applied, it is.

6.4 States and Observables

Now in Newtonian Mechanics, a pure state consists of the position and momentum of every object under consideration. It is theoretically possible to know this information, and given this information it is theoretically possible to know the exact state of the universe at all times in the future.

The only problem with the above is the accuracy of the information. In practice, it is impossible to measure anything exactly. We currently know Newton's gravitational constant, G, to 128 parts per million but we do not know G precisely. Even given that we are happy with the accuracy of our measurement, it may be that there are so many things to measure that no computer has the storage space to keep all the information and no computer has the computational power to calculate the time evolution of all this information.

Alone, this problem may seem to have little to do with the theory; as time goes by then we develop more and more accurate ways of measurement and hope that as our measurements get more accurate, so will our predictions[10]. Or we develop ways of applying the principles in such a way that we can get practical information out of a smaller set of data.

Thus the theory of Statistical Mechanics uses the same principles as Newtonian Mechanics, but because of the vast numbers of objects involved, it deals with observables which depend on properties of the whole system rather than individual objects.

For example, it is not inconceivable that someone would wish to consider the dynamics of a gas in a container. At room temperature and pressure, one cubic foot of oxygen contains approximately 6×10^{23} molecules. Writing down every single position and momentum would take some time. Thus for most purposes we don't bother, since we only deal with observables which are averages, such as the pressure, volume and temperature of the gas.

7 Quantum Mechanics

7.1 The Formulation of Quantum Theory

Newtonian Mechanics is all very well when the objects are large and slow, but when things get very small or very fast then the predictions become unacceptably inaccurate.

For example, the classical model of the Hydrogen atom is of an electron orbiting a proton much as a planet orbits the sun. However, the electron is an accelerating charged particle and classical theory says that such an object should radiate energy. Thus energy is constantly being emitted from the system and it must eventually collapse. Atoms do not collapse beyond a certain size, and they do not continuously radiate energy.

Quantum Mechanics allows one to consider situations in which the objects are very small. They still have to be moving slowly with respect to light, one

---

[10] Provided someone shoots that butterfly.

needs Quantum Field Theory to deal with that problem. However, the *basic* principles involved in Quantum Field Theory are the same as those for Quantum Mechanics.

From the Physics end, the principles of Quantum Mechanics are as follows: every particle exhibits wave-like properties and thus can be assigned a wave function. A wave function is just a normalized field, which is a map from configuration space to the complex numbers. These wave functions are then the states of the universe. The time evolution of a state is controlled by the equation:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}$$

where H is a Hermitian operator obtained from the classical Hamiltonian by substituting operators for the variables q, p. Up to multiplicative constants, q is replaced by the operator Q which multiplies the wave function by q and p is replaced by the covariant differential operator ∇ in the direction conjugate to p. This is the mystical process of Dirac quantization. It is not well defined; there are choices for the covariant differential operator, though not a free choice. In fact, this method of quantization is the reverse of a well defined procedure - that of letting $\hbar$ tend to zero.

One can understand these substitutions by considering a function proportional to that representing a beam of free particles[11], $\psi(\mathbf{x}, t) = \exp i(\mathbf{k}.\mathbf{x} - \omega t)$. Thus $\triangle\psi = -k^2\psi$ and $\frac{\partial\psi}{\partial t} = -i\omega\psi$. Now for a free particle $H = \frac{1}{2m}p^2 = E$. For a free wave $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$. Thus:

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi = H\psi = \frac{1}{2m}p^2\psi = -\frac{\hbar^2}{2m}\triangle\psi$$

This is the celebrated Schrödinger equation.

The definition of Quantum Mechanics from the point of view of Mathematics is as follows:

1. The pure states of the universe are rays in a Hilbert space (that is, unit vectors of arbitrary phase).
2. The observables in a Quantum system are Hermitian operators on this space. The expectation value of an observable B of a state ψ is $\langle\psi|B\psi\rangle = \langle\psi|B|\psi\rangle$. This is the mean of many observations of B all carried out on the same state ψ.
3. The Hamiltonian H is the infinitesimal generator of the time evolution group. The momentum P is the infinitesimal generator of the space translation group. The angular momentum J is the infinitesimal generator of the spatial rotation group.

---

[11] We need to consider an infinite number of non-interacting particles since otherwise when we normalize this function we get zero.

Thus the problem becomes finding the Hilbert space and the operators H, P, J and this is where the method of quantization comes in. Usually the Hilbert space is $L^2(M, \mathbb{C})$ with inner product:

$$\langle\psi|\phi\rangle = \int_M \psi^* \phi \; d\tau$$

The operators H, P, J are determined from the classical observables by the process of quantization.

Suppose our universe consists of a single particle moving in Euclidean 3-space. Thus our state is a function of three position and one time variable. Suppose it moves according to the dynamical rule:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle\psi + V\psi$$

where V is a real valued function of space only and $V\psi$ stands for pointwise multiplication of V and ψ, so $H = \frac{1}{2m}P^2 + V$ where $P = i\hbar\nabla$.

The expectation value of the x position of the particle is given by the expectation value of the operator X which multiplies by x. This is:

$$\langle x\rangle = \langle\psi|X|\psi\rangle = \int_{\mathbb{R}^3} \psi^* x\psi \; d\tau$$

It is a fairly easy calculation to show that the dynamics of this expectation value are given by:

$$\frac{d\,\langle x\rangle}{dt} = \frac{1}{m}\,\langle p_x\rangle$$

and the dynamics of the expectation value of p_x are given by:

$$\frac{d\,\langle p_x\rangle}{dt} = -\left\langle\frac{\partial V}{\partial x}\right\rangle$$

which is precisely what we would expect from Newtonian Mechanics.

As we let $\hbar$ tend to zero then the wave function tends to a delta function about $\langle x\rangle$ and $\langle p_x\rangle$ and so Newtonian Mechanics is the limit as $\hbar$ tends to zero of Quantum Mechanics.

7.2 States and Observables

The Mathematical formulation of Quantum Mechanics defines the states and observables. However, it is worth looking at what information a state can carry and how this is observed. There are two points that I wish to discuss: quantization and indeterminacy.

Quantization here refers to the original discovery which lead to Quantum Theory, namely that it is found, and experimentally verified, that the observed value of an observable is an eigenvalue of the operator. One tends not to have too many conceptual problems with this when considering observables like position with spectrum the whole real line, but it can cause interesting results in other cases. For example, the angular momentum of an electron orbiting a nucleus can only take discrete values.

Closely linked to this is the fact that the action of observation interferes with the universe. That this is so qualitatively can be seen by considering the act of observing an electron. To observe it, we need to bounce some light off it. However, bouncing the light off it gives it a push and so it is no longer moving along in the original direction.

Quantum Mechanics says that this can be made quantitative as well. The act of observation forces the universe into a state which is an eigenvector of that observable - hence the fact that the result of an observation is always an eigenvalue.

A considerable conceptual problem now arises because eigenvectors of one operator are not necessarily eigenvectors of another. The condition for there to be simultaneous eigenvectors is that the operators commute.

If this does not happen, then when we make one observation, followed by another, followed by the original one again, then there is no reason whatsoever for the first and last observations to match.

7.3 The Stern-Gerlach Experiment and Spin

An example bringing out the significance of quantization and indeterminacy is the Stern-Gerlach apparatus for measuring the spin of a particle (1924), Figure 5. The apparatus separates a beam of particles according to their spin about a particular axis. It is observed that the beam of particles splits into a finite number of beams, rather than a spread, indicating that the operator corresponding to spin has a finite discrete spectrum.

In the case of the neutron, the beam would split in two and we call neutrons in the upper beam spin-up, and those in the lower, spin-down. Since the apparatus has a particular orientation, we additionally specify this axis.

We suppose that we have a similar apparatus that separates the beam and then recombines it again, allowing us to block off one beam in the middle if we so desire. It should be noted that this is a hypothetical apparatus which, although feasible, has a few difficulties in practice. However, the principle is sound.

We consider an experiment where we have a source of neutrons moving from the left to the right which we know are spin-up with respect to the z-axis. We feed this beam into the apparatus above which is oriented with respect to the x-axis. We feed the output from this apparatus into the original Stern-Gerlach apparatus oriented with respect to the z-axis. Thus we can measure the spin of emerging neutrons with respect to the z-axis by looking at which of the two possible beams are present.

We will do this with and without an obstruction in the first apparatus. Note that there are two possible places a neutron could end up and two possible paths for it to have traveled in order to arrive there.

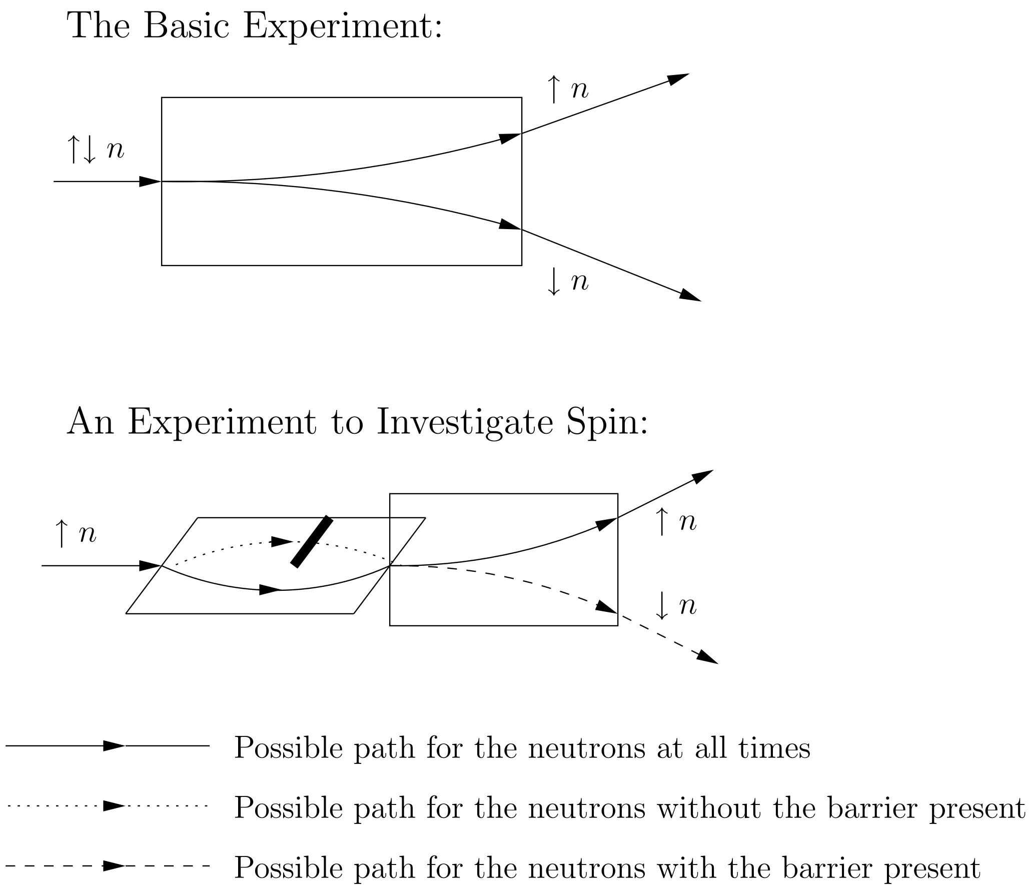

Figure 5: The Stern-Gerlach Experiment

With no obstruction, what we measure is that there are no spin-down neutrons emerging from the other end. All the neutrons emerge in the upper beam.

If we now block off one of the paths in the first apparatus, thus forcing the neutrons to be, say, spin-up with respect to the x-axis, then we get a different result. Suddenly, we get both spin-up and spin-down particles out of the end, and in equal proportions.

Classically, one thinks of the neutrons as particles separated into two beams. Thus a neutron arriving at the left hand side must choose[12] which path it travels along in the first apparatus and then choose again which it travels along in the second. Now we can develop a theory of spin which accounts for either of these results, but not for both. Either we can say that spin in perpendicular axes is independent. Thus although our neutron was forced to choose whether it was spin-up or spin-down with respect to the x-axis, it was still spin-up with respect to the z-axis. This neatly explains the first result. However, this theory says that blocking off one path in the first apparatus makes no difference to the z-axis spin so we should get the same result the second time. Or we could say that there is a unique axis of spin and when we force the neutron to pass through the second apparatus then we rotate this axis to align with the x-axis, which it does so with equal probability as up or down. But then when we force the neutron through the second apparatus, this should happen again and it should be spin-up or spin-down with equal probability. This explains the second result, but not the first.

However, Quantum Theory predicts when a particle has several possible paths between two points then it behaves as if it traveled along all of them. Provided we do not disturb the system in between, say by making a measurement, then the initial state determines the final state. In this case, although we *know* that an individual neutron “had” to travel along one path or the other, *so long as we don’t know which* then the neutron emerges in exactly the same state as it went in, i.e. as spin-up with respect to the z-axis.

Thus we have an experiment which cannot be explained classically and which shows firstly the principle of quantization and secondly that of indeterminacy.

8 Conclusion

The purpose of this talk was to introduce you to some of the basics of Newtonian and Quantum Mechanics in order that you have a basis of knowledge for the subsequent talks which will show how some of these things extend to problems in Geometry. I have not tried to justify the theories of Mechanics, if you want to know how they came about and why they are now considered as the best theories going then I suggest you read an elementary Physics book or a History of Science book.

We have seen how consideration of a problem in Mechanics can directly lead to a Geometrical problem by looking at Cauchy data and how Relativity gives a more direct link. In the later talks we will see how this correspondence leads

---

[12] Please forgive my anthropomorphic tendencies here.

from specific Mechanical to specific Geometrical problems and thence to more general Geometrical situations.

Appendices

A Fundamental Constants

Some of the relevant fundamental constants of the universe are:

Constant	Value	Precision
Speed of light in a vacuum	$c = 299792458 ms^{-1}$	Exact
Planck's constants	$h = 6.6260755(40) \times 10^{-34} Js$	0.60ppm
	$\hbar = 1.05457266(63) \times 10^{-34} Js$	0.60ppm
Gravitational constant	$G = 6.67259(85) \times 10^{-11} m^3 kg^{-1} s^{-2}$	128ppm
Permeativity of free space	$\epsilon_0 = 8.854187817\ldots \times 10^{-12} Fm^{-1}$	Exact

B Accuracy of Newtonian Mechanics

It is possible to determine the accuracy of Newtonian Mechanics in terms of the other Mechanical theories and corrections. Thus if we agree that Relativity and Quantum Theory are sufficiently accurate for calculation, we can calculate the errors would be due to Newtonian Mechanics if we were to use that for our predictions. We can then determine conditions under which Newtonian Mechanics is also sufficiently accurate for calculation. It should be noted that the Newtonian formulation is far easier to use that either the Relativistic or Quantum counterparts.

In relativity, the quantity γ often occurs as a multiplicative correction factor. That is, a quantity which is predicted classically requires multiplication by γ before it is correct relativistically. The formula for this is:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The linear speed of the earth around the sun is approximately $3 \times 10^4 ms^{-1}$. At this speed, the correction factor is $\gamma = 1 + 5 \times 10^{-9}$.

The escape speed from the earth's surface is approximately $11000 ms^{-1}$. At this speed, the correction factor is $\gamma = 1 + 7 \times 10^{-10}$.

There is no direct analogue in Quantum Theory, so to calculate the inaccuracy of Newtonian Mechanics involves a little more work. The Heisenberg Uncertainty Principle states that it is *theoretically* impossible to measure the position and momentum of a body at the same time. The relevant formula is:

$$\Delta x \Delta p \geq \frac{1}{2}\hbar$$

There is often a quantity which one would hope the error is small in comparison with so to get a quantity for which we can compare different situations, we divide the error by this quantity.

If we have a spherical body of diameter d moving with approximate momentum p then we would hope that the error in position was small in comparison to d and the error in momentum small in comparison to p. Thus we calculate the quantity Δ given by:

$$\Delta = \left(\frac{\Delta x}{d} \frac{\Delta p}{p} \right)^{\frac{1}{2}}$$

As we are concerned with theoretical accuracy, we can substitute $\frac{1}{2}\hbar$ for the numerator in this.

We get the following values:

Situation	Δ
Helium atom at 300K, 1atm	0.29
Radon atom at 300K, 1atm	0.12
A tennis ball at 100mph	4×10^{-32}
The Earth (relative to the sun)	5×10^{-71}

Thus it can be seen that for “everyday” use, Newtonian Mechanics has a high degree of accuracy.

References

[1] Leighton: Principles of Modern Physics.
McGraw-Hill (1959).

[2] Abrahams and Marsden: Fountations of Mechanics.
Addison-Wesley, (1978).

[3] Glimm and Jaffe: Quantum Physics. A Functional Integral Point of View.
Spinger-Verlag (1981).