How to Construct a Dirac Operator in Infinite Dimensions

Talk given to the Topology Group at Oslo

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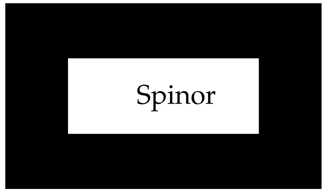


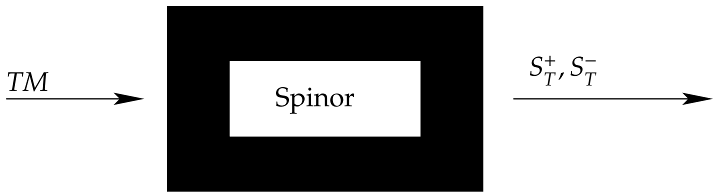
Input A vector bundle $E \to M$ with a *spin structure*. (Riemannian) metric.

Output Two Hilbert bundles S^+, S^-; a bilinear map $c\colon E \times S^+ \to S^-$, a covariant differential operator ∇ on S^+.

$$\partial\!\!\!/\colon \Gamma(S^+) \to \Gamma(S^-).$$

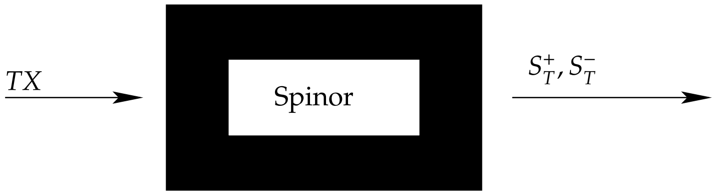
Abstract

I shall describe how to construct the Dirac operator on suitable loop spaces (i.e. loop spaces of string manifolds). There are two important aspects to the construction: firstly, that we use the space of *smooth* loops; secondly, that we first construct a *cometric* on the loop space. I shall also indicate how this method might or might not be adapted to define other operators on loop spaces, with particular interest on a *semi-infinite* de Rham operator.



$$\begin{array}{ccc}
\Gamma(S_T^+) & \xrightarrow{\nabla} & \Gamma(\mathcal{L}(TM, S_T^+)) \\
 & & \uparrow \cong \; V^* \otimes W \to \mathcal{L}(V,W) \\
 & & \Gamma(T^*M \otimes S_T^+) \\
 & & \uparrow \cong \; V \to V^* \\
 & & \Gamma(TM \otimes S_T^+) \xrightarrow{c} \Gamma(S_T^-)
\end{array}$$

$$\partial\!\!\!/\colon \Gamma(S_T^+) \to \Gamma(S_T^-).$$



$$
\begin{array}{ccccc}
\Gamma(S^+_T) & \xrightarrow{\nabla} & \Gamma(\mathcal{L}(TX, S^+_T)) & & \\
 & & \uparrow{\scriptstyle V^*\otimes W \to \mathcal{L}(V,W)} & & \\
 & & \Gamma(T^*X \otimes S^+_T) & & \\
 & & \uparrow{\scriptstyle V \to V^*} & & \\
 & & \Gamma(TX \otimes S^+_T) & \xrightarrow{c} & \Gamma(S^-_T)
\end{array}
$$

$s \in \Gamma(S^+_T)$ for which $\nabla s \in \Gamma(T^*X \otimes S^+_T)$

$$V \to V^*, \quad v \mapsto \langle -, v \rangle.$$

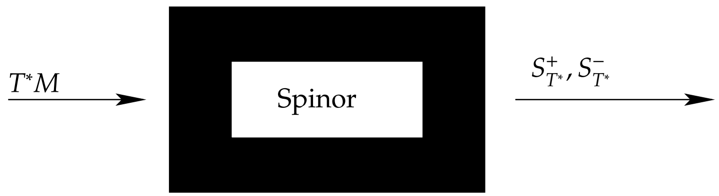
$$V^* \otimes W \to \mathcal{L}(V, W), \quad f \otimes w \mapsto (v \mapsto f(v)w)$$

The image of the tensor product is all *finite rank* operators.

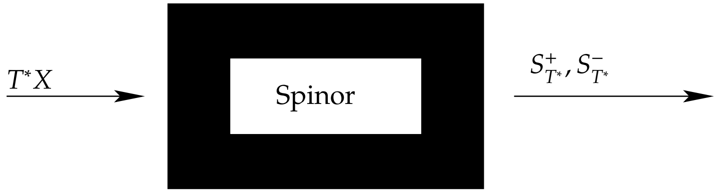
$$c\colon TX \widetilde{\otimes} S^+_T \to S^-_T$$

The image of the **completed** tensor product is *trace class* operators.

V such that all $T\colon V \to H$ are trace-class



$$
\begin{array}{ccccc}
\Gamma(S^+_{T^*}) & \xrightarrow{\nabla} & \Gamma(\mathcal{L}(TM, S^+_{T^*})) & & \\
 & & \cong\uparrow{\scriptstyle V^*\otimes W \to \mathcal{L}(V,W)} & & \\
 & & \Gamma(T^*M \otimes S^+_{T^*}) & \xrightarrow{c} & \Gamma(S^-_{T^*})
\end{array}
$$



$$\begin{array}{ccc} \Gamma(S^+_{T^*}) & \xrightarrow{\nabla} & \Gamma(\mathcal{L}(TX, S^+_{T^*})) \\ & & \uparrow {\scriptstyle V^*\otimes W \to \mathcal{L}(V,W)} \\ & & \Gamma(T^*X \widetilde{\otimes} S^+_{T^*}) \xrightarrow{c} \Gamma(S^-_{T^*}) \end{array}$$

V reflexive

$$V^* \to V^{**} \cong V$$

$$\begin{array}{ccc} \Gamma(S^+_T) & \xrightarrow{\nabla} & \Gamma(\mathcal{L}(TX, S^+_T)) \\ & & {\scriptstyle \cong} \uparrow {\scriptstyle V^*\otimes W \to \mathcal{L}(V,W)} \\ & & \Gamma(T^*X \widetilde{\otimes} S^+_T) \\ & & {\scriptstyle V \to V^*} \downarrow \\ & & \Gamma(TX \widetilde{\otimes} S^+_T) \xrightarrow{c} \Gamma(S^-_T) \end{array}$$

Remark

1. *Tangent method uses inner products on TX* ***and*** *T^*X.*
2. *Cotangent method uses inner products on T^*X* ***only****.*

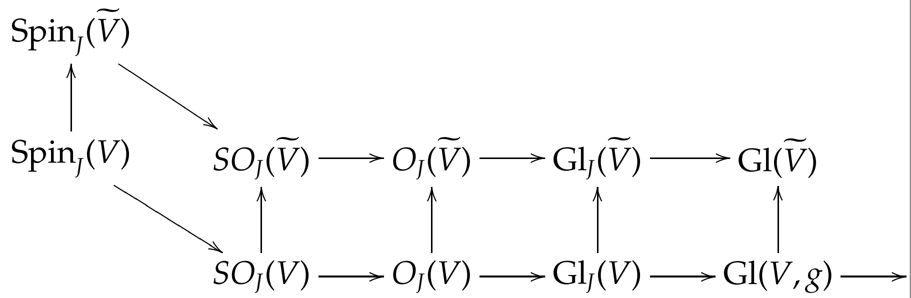
Nuclear spaces.

$L\mathbb{R}^n$ is nuclear.

Theorem

Let X be an infinite dimensional manifold. Suppose that X admits a cospin structure and is modelled on a complete, reflexive, nuclear space. Then the Dirac operator $\partial\!\!\!/ \colon \Gamma(S^+) \to \Gamma(S^-)$ can be defined.

- Signature Operator: $E \mapsto \Lambda^{\text{even}} E, \Lambda^{\text{odd}} E$
- de Rham Operator: $E \mapsto \Lambda^{\bullet} E$
- co–de Rham Operator: $E \mapsto \Lambda^{\infty - \bullet} E = \Lambda^{\bullet} E^{*}$
- semi-infinite de Rham Operator: $E \mapsto \Lambda^{\infty/2 + \bullet}$.

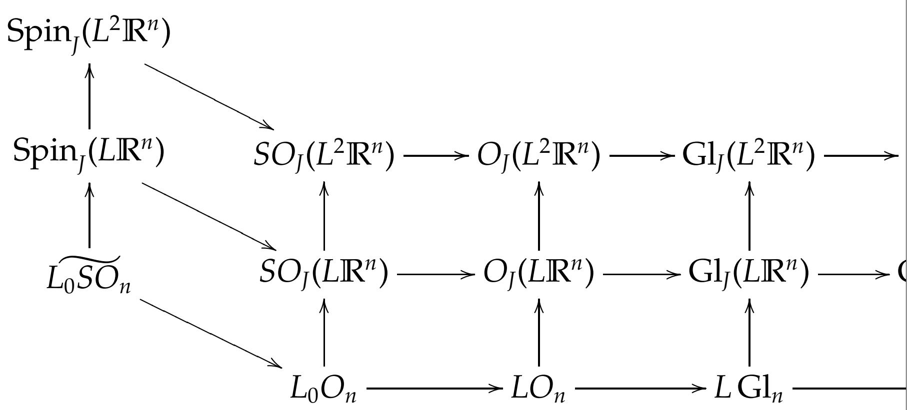


What is a **cospin** structure?

$E \to X$ with structure group G

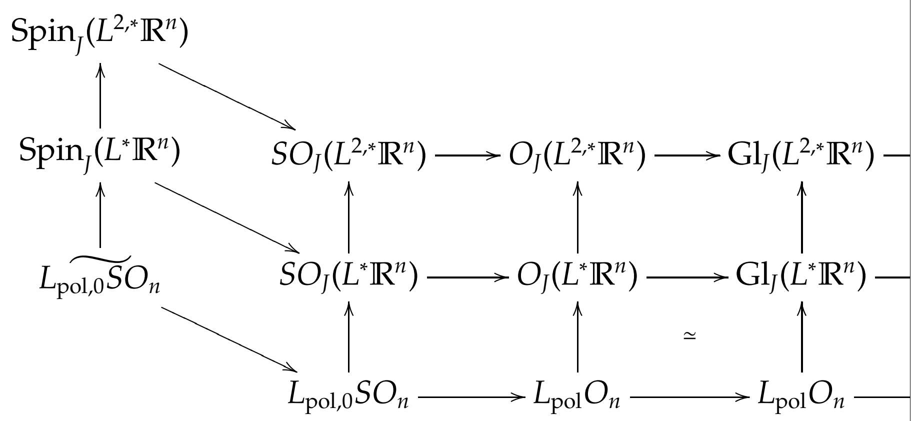
Replace G by Spin_n or $\mathrm{Spin}_J (= \mathrm{Spin}_{\text{res}})$.

1. Fix an equivalence class of inner products on V such that G fixes this class.
2. Fix a polarising operator J such that G fixes the polarisation.
3. Fix an actual inner product such that G acts by isometries.
4. Reduce to oriented isometries.
5. Lift to the spin group.



$$T_\gamma LM \cong \Gamma_{S^1}(\gamma^* TM)$$

$$\langle \alpha, \beta \rangle_\gamma = \int_{S^1} \langle \alpha(t), \beta(t) \rangle_{\gamma(t)} dt$$



bad news No “natural” Hilbert completion of $(L\mathbb{R}^n)^*$.
good news There are at least some completions!
bad news None on which $L\,\mathrm{Gl}_n$ act.
good news But some on which $L_{\mathrm{pol}} O_n$ act.
bad news None on which $L_{\mathrm{pol}} O_n$ act isometrically.
good news But can alter the action by homotopies so that it does.
good news The rest follows easily.

$$D_\gamma \colon T_\gamma LM \cong \Gamma_{S^1}(\gamma^* TM) \to \Gamma_{S^1}(\gamma^* TM) \cong T_\gamma LM$$

$$\cos D_\gamma \colon T_\gamma LM \cong \Gamma_{S^1}(\gamma^* TM) \to \Gamma_{S^1}(\gamma^* TM) \cong T_\gamma LM$$

$$\cos D_\gamma \colon T^*_\gamma LM \cong \Gamma_{S^1}(\gamma^* TM) \to \Gamma_{S^1}(\gamma^* TM) \cong T_\gamma LM$$

$$\langle \zeta, \xi \rangle_\gamma = \langle \cos D_\gamma \zeta, \cos D_\gamma \xi \rangle_\gamma$$