How to Construct a Dirac Operator in Infinite Dimensions
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Abstract
I shall describe how to construct the Dirac operator on suitable loop spaces (i.e. loop spaces of string manifolds). There are two important aspects to the construction: firstly, that we use the space of smooth loops; secondly, that we first construct a cometric on the loop space. I shall also indicate how this method might or might not be adapted to define other operators on loop spaces, with particular interest on a semi-infinite de Rham operator.

Input A vector bundle $E \rightarrow M$ with a spin structure. (Riemannian) metric.
Output Two Hilbert bundles $S^+, S^-$; a bilinear map $c: E \times S^+ \rightarrow S^-$, a covariant differential operator $\nabla$ on $S^+$.

$\partial / : \Gamma(S^+) \rightarrow \Gamma(S^-)$. 

$\Gamma(S^+) \xrightarrow{\nabla} \Gamma(L(TM, S^+_T))$

$\Gamma(T^*M \otimes S^+_T) \xrightarrow{c} \Gamma(S^-_T)$

$\partial / : \Gamma(S^+_T) \rightarrow \Gamma(S^-_T)$. 

Spinor

Input

Output

$\Gamma(S^+_T) \xrightarrow{\nabla} \Gamma(L(TM, S^+_T))$

$\Gamma(T^*M \otimes S^+_T) \xrightarrow{c} \Gamma(S^-_T)$

$\partial / : \Gamma(S^+_T) \rightarrow \Gamma(S^-_T)$. 

Spinor

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\[ \Gamma(S^+_T) \xrightarrow{\nabla} \Gamma(L(TX, S^+_T)) \]
\[ \xrightarrow{V \otimes W \rightarrow L(V, W)} \]
\[ \Gamma(T^*X \otimes S^+_T) \xrightarrow{V \rightarrow V^*} \Gamma(TX \otimes S^+_T) \]
\[ c \xrightarrow{\sim} \Gamma(S^-_T) \]

\[ V \rightarrow V^* \, \, \, v \mapsto \langle - , v \rangle. \]
\[ V^* \otimes W \rightarrow L(V, W), \, \, f \otimes w \mapsto (v \mapsto f(v)w) \]
The image of the tensor product is all finite rank operators.

\[ c: TX \otimes S^+_T \rightarrow S^-_T \]
The image of the completed tensor product is trace class operators.

\[ V \] such that all \( T: V \rightarrow H \) are trace-class
Nuclear spaces.

\[ LR^n \text{ is nuclear.} \]

\[ T^\ast X \rightarrow \text{Spinor} \rightarrow S^+, S^- \]

\[ \Gamma(S^+_{T^\ast}) \xrightarrow{\nabla} \Gamma(L(TX, S^+_{T})) \]

\[ \Gamma(T^\ast X \otimes S^+_{T^\ast}) \xrightarrow{\gamma} \Gamma(S^-_{T^\ast}) \]

Remark

1. Tangent method uses inner products on \( TX \) and \( T^\ast X \).
2. Cotangent method uses inner products on \( T^\ast X \) only.

Theorem

Let \( X \) be an infinite dimensional manifold. Suppose that \( X \) admits a cospin structure and is modelled on a complete, reflexive, nuclear space. Then the Dirac operator \( \partial : \Gamma(S^+) \rightarrow \Gamma(S^-) \) can be defined.
- Signature Operator: $E \mapsto \Lambda^{\text{even}} E, \Lambda^{\text{odd}} E$
- de Rham Operator: $E \mapsto \Lambda^* E$
- co–de Rham Operator: $E \mapsto \Lambda^{\infty} E = \Lambda^* E^*$
- semi-infinite de Rham Operator: $E \mapsto \Lambda^{\infty/2} E$.

**What is a cospin structure?**

$E \to X$ with structure group $G$

Replace $G$ by Spin$_n$ or Spin$_J (= \text{Spin}_{\text{res}})$.  

1. Fix an equivalence class of inner products on $V$ such that $G$ fixes this class.
2. Fix a polarising operator $J$ such that $G$ fixes the polarisation.
3. Fix an actual inner product such that $G$ acts by isometries.
4. Reduce to oriented isometries.
5. Lift to the spin group.
Spin$_J$($L^2\mathbb{R}^n$) → Spin$_J$($L\mathbb{R}^n$) → SO$_J$($L^2\mathbb{R}^n$) → O$_J$($L^2\mathbb{R}^n$) → Gl$_J$($L^2\mathbb{R}^n$) →

L$_0$SO$_n$ → SO$_J$($L\mathbb{R}^n$) → O$_J$($L\mathbb{R}^n$) → Gl$_J$($L\mathbb{R}^n$) →

L$_0$O$_n$ → LO$_n$ → L Gl$_n$

$T_y LM \cong \Gamma_{S^1}(\gamma^*TM)$

$\langle \alpha, \beta \rangle_{\gamma} = \int_{S^1} \langle \alpha(t), \beta(t) \rangle_{\gamma(t)} dt$

**bad news** No “natural” Hilbert completion of ($L\mathbb{R}^n$)$^*$.  
**good news** There are at least some completions!  
**bad news** None on which $L$ Gl$_n$ act.  
**good news** But some on which $L$ pol O$_n$ act.  
**bad news** None on which $L$ pol O$_n$ act isometrically.  
**good news** But can alter the action by homotopies so that it does.  
**good news** The rest follows easily.

$D_y : T_y LM \cong \Gamma_{S^1}(\gamma^*TM) \rightarrow \Gamma_{S^1}(\gamma^*TM) \cong T_y LM$

$\cos D_y : T_y LM \cong \Gamma_{S^1}(\gamma^*TM) \rightarrow \Gamma_{S^1}(\gamma^*TM) \cong T_y LM$

$\cos D_y : T_y^*LM \cong \Gamma_{S^1}(\gamma^*TM) \rightarrow \Gamma_{S^1}(\gamma^*TM) \cong T_y LM$

$\langle \zeta, \xi \rangle_{\gamma} = \langle \cos D_y \zeta, \cos D_y \xi \rangle_{\gamma}$