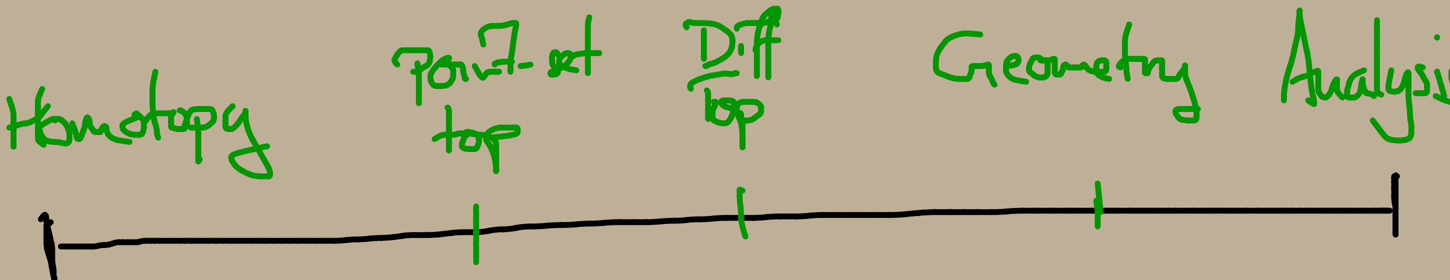
Manifolds of Piecewise Smooth Loops



Topology

$C^0(S^1, M) \simeq C^\infty(S^1, M)$ importance of models very different eg wrt $\int$

Differential topology - where we start to care

LM: tool to understand M

→ whatever works

object in its own right

→ more care needed

smooth

piecewise smooth

cts

	LM	$L_{ps}M$	$L^0 M$
transport	✓	✓	✗
gluing	✗	✓	✓
representation	✓	✓	✗

Suggests: worth a look

Q: <u>What is a manifold?</u>

A: Locally linear

Q: <u>When is $L^\times M$ a manifold?</u>

A: If $L^\times \mathbb{R}^n$ is "nice":

i/ LCTVS

ii/ $L\mathbb{R}^n \longrightarrow L^\times \mathbb{R}^n \longrightarrow L^\circ \mathbb{R}^n$

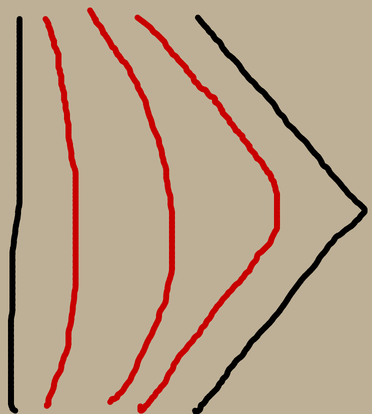
iii/ $L^\times$ — functorial on $\{\text{open} \subseteq \mathbb{R}^n, C^\infty\}$

iv/ $L^\times$ C^∞ — complete

Q: Does $L_{ps}\mathbb{R}^n$ satisfy these?

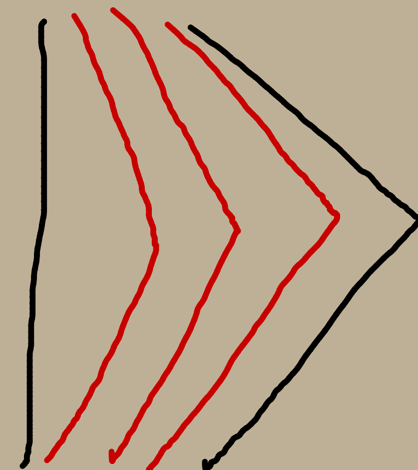
A: . . can you clarify the $q^{\underline{n}}$?

There are 2 (main) types of piecewise-smooth loop



breaks spontaneously appear

piecewise-smooth



breaks detectable

piecewise-smooth and bounded

Piecewise-Smooth

$$L_{ps}\mathbb{R} = \bigcup_F L_{ps,F}\mathbb{R}$$

$F \subseteq S^1,\ |F| < \infty$

breaks in F

In the category of LCTVS

$$L_{ps,F}\mathbb{R} \cong C^\infty([0,1],\mathbb{R})^{|F|} \quad \text{Fréchet}$$

Topology on $L_{ps}\mathbb{R}$ generated by

U : $U \cap L_{ps,F}\mathbb{R}$ open nbd of 0, all F

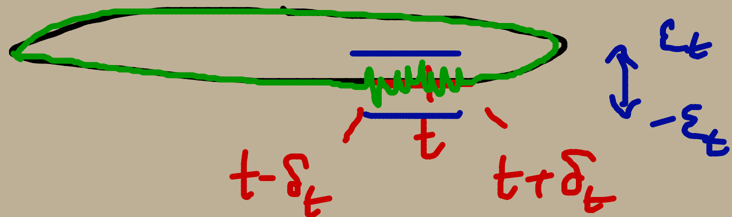
w/ U convex

Induced topology on $L\mathbb{R}$:

U as above.

$t \in S^1$, $U \cap L_{p;t}\mathbb{R}$ open nbd of 0

$\Rightarrow \exists\ \delta_t, \varepsilon_t$ st:



then $\gamma \in U$.

S^1 compact $\Rightarrow$ subordinate p of unity $\{e_1, \dots, e_n\}$

$\varepsilon_j = \varepsilon_{t_j}$ appropriate t_j

Put $\varepsilon = \frac{1}{n} \min\{\varepsilon_j\}$

If $|r| < \varepsilon$ then

1/ $ne_j r \in U$: $\operatorname{supp} ne_j r \subseteq (t_j - \delta_j, t_j + \delta_j)$

$|ne_j r| < \varepsilon_j$

2/ U convex $\Rightarrow$ $\sum \frac{1}{n}(ne_j r) \in U$

$\overset{\parallel}{r}$

So $r \in U$

General nonsense $\Rightarrow$

$\mathrm{Lps}\,\mathbb{R} \subseteq L^0\mathbb{R}$ topologically

Consequence. ∞-completion is $L^0\mathbb{R}$

Bad News

Integrals & derivatives need not exist when they should

What about piecewise-smooth & bounded?

Much happier start:

breaks detectable $\Rightarrow$ topology takes them into account

$$L\mathbb{R} \longrightarrow L_{psb}\mathbb{R} \longrightarrow \sum_{S^1} \mathbb{R}^{\mathbb{N}}$$

short exact sequence, <u>not</u> split

So complete, and the others as well.

<u>Theorem</u>. LpsbM <u>is</u> a manifold

<u>Bad news</u>: Not a very nice one.

<u>Conjecture</u>: Does not admit partitions of unity.

Even more badly behaved: $\mathrm{Diff}(S^1)$-action

$$\sigma \in \mathrm{Diff}(S^1) \rightsquigarrow \sigma^* : L_{psb}\mathbb{R} \to L_{psb}\mathbb{R}$$

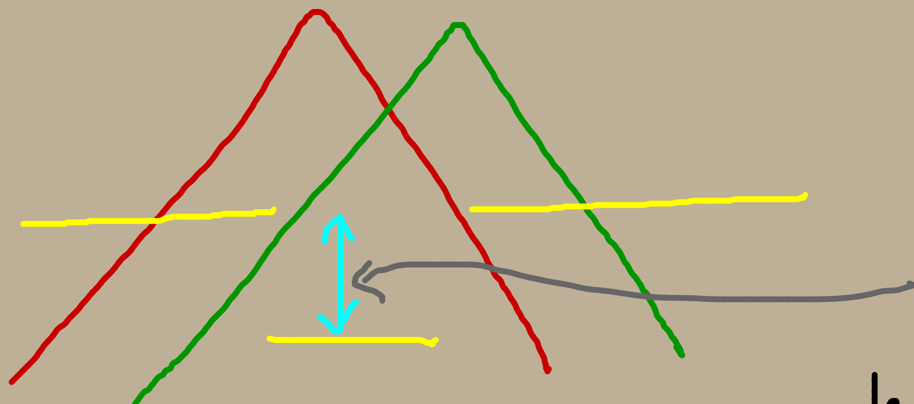
$$\sigma^*\gamma = \gamma \circ \sigma$$

Linear homeomorphism (diffeomorphism)

$$\mathrm{Diff}(S^1) \longrightarrow \mathcal{L}(L_{psb}\mathbb{R})$$

(Topology on $\mathcal{L}(\mathrm{Lpsb}\,\mathbb{R})$? Doesn't matter!)

<u>Theorem</u>: Image of $\mathrm{Diff}(S^1)$ in $\mathcal{L}(\mathrm{Lpsb}\,\mathbb{R})$ is <u>totally disconnected</u>, of S^1 is <u>discrete</u>.



significant, no matter how long

Practical Consequences

Reparametrisation homotopies

Do Not Work

	topology	S^1-action
LM	metrisable p of unity	smooth
L^0M	metrisable p of unity	continuous
$L_{psb}M$	not metrisable ?p of unity?	discrete

Conclusion:

"Better the manifold you know"

LpsM, LpsbM useful tools, nasty spaces