Describing Unstable Operations

Joint Algebra and Topology Seminar
Sheffield

Andrew Stacey[1] Sarah Whitehouse[2]

University of Sheffield
[2]Partially and [1]fully supported by the EPSRC, grant no.: GR/S76823/01

19th October 2006

The Problem

To give a straightforward description of the algebraic structure of

Unstable operations

and

Unstable co-operations

Outline

Preliminaries: Operations

- Graded multiplicative cohomology theory contravariant functors

$$E^*(-) : \mathrm{hTop} \to \mathrm{GAlg}$$

- Operations are natural transformations
- Forget structure: $E_U^k(-) : \mathrm{hTop} \to \mathrm{Set}$
- Unstable Operations: $E_U^k(-) \to E_U^l(-)$
- Appears to disregard the structure of $E^*(X)$

Preliminaries: Representation

- $E^*(-)$ is representable
  Spaces $\underline{E}_k$, $k \in \mathbb{Z}$, classes $\iota_k \in E^k(\underline{E}_k)$.

$$\begin{aligned} \mathrm{hTop}(X, \underline{E}_k) &\xrightarrow{\cong} E^k(X) \\ \alpha &\mapsto \alpha^* \iota_k \end{aligned}$$

- Structure of $E^*(-)$ ↔ structure of $(\underline{E}_k)_{k \in \mathbb{Z}}$
- Yoneda's Lemma:
  Unstable operations $E^k(-) \to E^l(-)$ are

$$\mathrm{hTop}(\underline{E}_k, \underline{E}_l) \cong E^l(\underline{E}_k)$$

- $E^*(\underline{E}_k)$ certainly has some structure

Preliminaries: Homology

- Associated homology:
  covariant functor

$$E_*(-) : \mathrm{hTop} \to \mathrm{GMod} \quad (\mathrm{GCoalg})$$

- In “good” cases $E^*(X)$ is E^*-dual to $E_*(X)$.

- Unstable Co-operations: $E_*(\underline{E}_k)$

- Considerable structure, but does not act on $E_*(X)$.

The Problem

To give a straightforward description of the algebraic structure of

Unstable operations

$E^*(\underline{E}_k)$ acting, somehow, on the algebra $E^*(X)$ but not by morphisms of algebras.

Unstable co-operations

Also want to factor $E_*(\underline{E}_k)$ and $E_*(X)$ in to the story.

Algebras and Modules

Monoid		Co-monad
$A \otimes M \to M$	$A \to \mathrm{Mod}(M, M)$	$M \to \mathrm{Mod}(A, M)$
Pros: Intuitive	Pros: Simple	Pros: Least “specials”
Cons: Monoidal	Cons: Algebras	Cons: Least intuitive

Monads and Co-monads

Definition

A co-monad on a category $\mathcal{C}$ consists of a functor

$$T : \mathcal{C} \to \mathcal{C}$$

and natural transformations

$$\mu : T \to TT \qquad \epsilon : T \to I$$

satisfying the obvious co-associativity and co-unit diagrams.
A co-module for a co-monad T is an object X of $\mathcal{C}$ with a morphism

$$\rho : X \to T(X)$$

satisfying the obvious co-module diagrams.

Examples

- A an algebra. $A_+ : \text{Mod} \to \text{Mod}$ by
$$A_+(M) := \text{Mod}(A, M)$$
Natural transformations:
$$\text{Mod}(A, M) \to \text{Mod}(A, \text{Mod}(A, M))$$
$$\big(f : A \to M\big) \mapsto \big(a_1 \mapsto \big(a_2 \mapsto f(a_1 a_2)\big)\big)$$
$$\big(f : A \to M\big) \mapsto f(1) \in M$$
- M an A-module. $\hat{\rho} : M \to A_+(M)$ by
$$m \mapsto \big(a \mapsto \rho(a \otimes m)\big).$$

Examples (contd.)

- C a co-algebra. $C_!$: Mod $\to$ Mod by

$$C_!(M) := C \otimes M$$

  Natural tranformations:

$$C \otimes M \xrightarrow{\Delta\otimes 1} C \otimes C \otimes M$$

$$C \otimes M \xrightarrow{\epsilon\otimes 1} k \otimes M \cong M$$

- M a C-co-module.

$$M \to C \otimes M$$

Operations as Co-monads

Theorem (Boardman, Johnson, Wilson)

$E^*(\underline{E}_*)$ *represents a co-monad in* GAlg.

$E^*(X)$ *is a co-module for this co-monad.*

Co-module structure: need a map

$$E^{\bigstar}(X) \to \mathrm{GAlg}(E^{\bigstar}(\underline{E}_{\bigstar}), E^{\bigstar}(X))$$

$$\left(\alpha \in E^k(X) = \mathrm{hTop}(X, \underline{E}_k)\right) \to \left(\alpha^* : E^*(\underline{E}_k) \to E^*(X)\right)$$

Co-operations

$$E^k(X) = \mathrm{hTop}(X, \underline{E}_k) \to \mathrm{GAlg}(E^*(\underline{E}_k), E^*(X))$$
$$\alpha \mapsto \alpha^*$$
$$E^k(X) = \mathrm{hTop}(X, \underline{E}_k) \to \mathrm{GCoalg}(E_*(X), E_*(\underline{E}_k))$$
$$\alpha \mapsto \alpha_*$$

Theorem (Ravenel, Wilson)

$E_*(\underline{E}_*)$ is a Hopf ring

Question: What is a Hopf ring?
Answer: A co-algebra H such that the contravariant functor

$$H^+ : C \to \mathrm{Coalg}(C, H)$$

actually ends up in Alg.

Module Functors

Definition

Let T be a co-monad on a category $\mathcal{C}$. A (left) co-module functor of T is a functor

$$F : \mathcal{D} \to \mathcal{C}$$

with a natural transformation

$$\rho : F \to TF$$

satisfying the obvious diagrams.

Silly Example

$A_+(-) = \mathrm{Mod}(A, -)$, $M_+(-) = \mathrm{Mod}(M, -)$

$$\Big(f : M \to N\Big) \mapsto \Big(a \mapsto \big(m \mapsto f(am)\big)\Big)$$

Module Functors (contd.)

Proposition

A co-module functor factors through the subcategory of co-modules.

Enriched Hopf Rings

Theorem (Boardman, Johnson, Wilson)

$E_*(\underline{E}_*)$ *is an enriched Hopf ring*

Proposition

The enriched bit means that the functor GCoalg $\to$ GAlg

$$C_* \mapsto \mathrm{GCoalg}(C_*, E_*(\underline{E}_*))$$

is a co-module functor for the co-monad represented by $E^*(\underline{E}_*)$. *The map*

$$E^*(X) \to \mathrm{GCoalg}(E_*(X), E_*(\underline{E}_*))$$

is a morphism of co-modules.

The Problem and Answers pt I

To give a straightforward description of the algebraic structure of

Unstable operations

$E^*(\underline{E}_*)$ represents a co-monad on GAlg
$E^*(X)$ is a co-module.

Unstable co-operations

$E_*(\underline{E}_*)$ represents a co-module functor.
$E^*(X) \to \mathrm{GCoalg}(E_*(X), E_*(\underline{E}_*))$
is a morphism of co-modules.

The Monoidal Story

The problem ... is ... the tensor product ... that is simply unavailable for operations that are not additive (not that this has stopped us from trying).

Boardman, Johnson, Wilson

Algebras and Modules

Monoid

$A \otimes M \to M$

Pros:
Intuitive

Cons:
Monoidal

Co-monad

$M \to \mathrm{Mod}(A, M)$

Pros:
Least "specials"

Cons:
Least intuitive

Freyd's Theorem

Theorem (Freyd)

Let $\mathcal{C}$ be a category with small colimits; $\mathcal{V}$ a variety of algebras; $F : \mathcal{C} \to \mathcal{V}$ a covariant functor. The following are equivalent.

1. *F has a left adjoint*
2. *F is representable by a co-$\mathcal{V}$-object in $\mathcal{C}$*
3. *$F_U : \mathcal{C} \to \mathrm{Set}$ is representable.*

Corollary: Compositions of covariant representable functors are representable

Examples

Variety: groups
Source: hTop′ (*based*)
The circle is a co-group object in hTop′ with maps

$$S^1 \xrightarrow{\mu} S^1 \vee S^1, \qquad \text{pinch}$$
$$S^1 \xrightarrow{\nu} S^1, \qquad \text{reverse}$$
$$S^1 \xrightarrow{\varepsilon} \mathrm{pt}.$$

These make $\pi_1(X) := \mathrm{hTop}'(S^1, X)$ into a group:

$$f + g \text{ is } S^1 \xrightarrow{\mu} S^1 \vee S^1 \xrightarrow{f \vee g} X \vee X \to X$$
$$-f \text{ is } S^1 \xrightarrow{\nu} S^1 \xrightarrow{f} X$$
$$1 \text{ is } S^1 \xrightarrow{\varepsilon} \mathrm{pt} \to X$$

Examples (contd.)

Variety: k-algebras
Source: arbitrary, $\mathcal{C}$; initial object I.
Operations: $\lambda \in k$

$$X \xrightarrow{\alpha} X \amalg X \qquad \text{co-addition}$$

$$X \xrightarrow{\mu} X \amalg X \qquad \text{co-multiplication}$$

$$X \xrightarrow{\lambda} I \qquad \text{co-}\lambda\text{-action.}$$

Operations: The biring structure of $E^*(\underline{E}_*)$ is:

- Co-addition: H-map $\underline{E}_k \times \underline{E}_k \to \underline{E}_k$
- Co-multiplication: ring maps $\underline{E}_k \times \underline{E}_l \to \underline{E}_{k+l}$
- Co-E^*-action: $v \in E^k = E^k(\text{pt}) = \text{hTop}(\text{pt}, \underline{E}_k)$ induces $v^* : E^*(\underline{E}_k) \to E^*(\text{pt}) = E^*$.

Applications pt I: Products

Compositions of representable functors are representable.

$A_+ B_+ : \mathrm{Alg} \to \mathrm{Set} \qquad B_{1+} B_{2+} : \mathrm{Alg} \to \mathrm{Alg}$

Proposition (Tall, Wraith (1970))

There is a product

$$\mathrm{Biring} \times \mathrm{Alg} \to \mathrm{Alg}, \qquad (B, A) \to B \odot A$$

and a natural isomorphism

$$\mathrm{Alg}(B \odot A, A') \cong \mathrm{Alg}(A, \mathrm{Alg}(B, A'))$$

and if B_1, B_2 are birings then $B_1 \odot B_2$ is a biring.

- Linear in B but not in A
- Not symmetric

Applications pt II: Plethories

Representable co-monad: $P_+ \to P_+ P_+$, $P_+ \to I$.

Definition (Tall and Wraith, Borger and Wieland)

A plethory consists of a biring P and maps of birings

$$P \odot P \to P, \qquad I \to P$$

satisfying the obvious diagrams.

- I is the initial biring, $k[e]$
- For a ring R, Set(R, R) is a plethory
- For a group G there is a notion of a free plethory $P(G)$

Applications pt III: Modules

Co-module over a representable co-monad:

Definition

Let P be a plethory. A P-module is an algebra A with a map

$$P \odot A \to A$$

satisfying the obvious diagrams.

Example: If G acts on an algebra A then A is a $P(G)$-module.

Unstable Operations

Theorem

For "good" cohomology theories, the set of unstable operations of a cohomology theory is a

graded, completed plethory.

The cohomology of a space is a module for this plethory.

The Problem and Answers pt III

To give a straightforward description of the algebraic structure of

Unstable operations

$E^*(\underline{E}_*)$ is a graded, complete plethory
$E^*(X)$ is a plethoric module.

Unstable co-operations

$E_*(\underline{E}_*)$ represents a co-module functor.
$E^*(X) \to \mathrm{GCoalg}(E_*(X), E_*(\underline{E}_*))$
is a morphism of co-modules.

Freyd's Theorem pt II

Theorem (Freyd)

Let $\mathcal{C}$ be a category with small limits; $\mathcal{V}$ a variety of algebras; $F : \mathcal{C} \to \mathcal{V}$ a contravariant functor. The following are equivalent.

1. *F is one of a mutually right adjoint pair*
2. *F is representable by a $\mathcal{V}$-object in $\mathcal{C}$*
3. *$F_U : \mathcal{C} \to \mathrm{Set}$ is representable.*

Corollary: Composition of a contravariant representable functor followed by a covariant one is representable

Pairings

$C \mapsto \mathrm{Alg}(A, \mathrm{Coalg}(C, H))$ is representable

Lemma

There is a pairing

$$\mathrm{Alg} \times \mathrm{Hopf} \to \mathrm{Coalg}, \qquad (A, H) \to A \boxtimes H$$

and a natural bijection

$$\mathrm{Alg}(A, \mathrm{Coalg}(C, H)) \cong \mathrm{Coalg}(C, A \boxtimes H)$$

Caveat:
The pairing is contravariant in A and covariant in H.

Pairing Properties

From:

$$\mathrm{Alg}(B, \mathrm{Coalg}(C, H)) = \mathrm{Coalg}(C, B \boxtimes H)$$

$$\mathrm{Alg}(A, \mathrm{Alg}(B, \mathrm{Coalg}(C, H))) = \mathrm{Coalg}(C, A \boxtimes (B \boxtimes H))$$
$$\mathrm{Alg}(B \odot A, \mathrm{Coalg}(C, H)) = \mathrm{Coalg}(C, (B \odot A) \boxtimes H)$$

We deduce:

Lemma

1. *If B is a biring, $B \boxtimes H$ is a Hopf ring*
2. *There is a natural isomorphism* $(B \odot A) \boxtimes H \cong A \boxtimes (B \boxtimes H)$.

Plethories and Hopf Rings

Question: When does $H_* : C \mapsto \mathrm{Coalg}(C, H)$ land in the subcategory of P-algebras?

Answer: When H is a P-co-module.

$$H \to P \boxtimes H$$

Lemma

An enriched Hopf ring is a Hopf ring with an action of a plethory.

Problem: $P \boxtimes H$ is contravariant in P so tricky to work with.

Enriched Hopf Rings

Theorem

There is a pairing $\mathrm{Biring} \times \mathrm{Hopf} \to \mathrm{Hopf}$, $(B, H) \mapsto B \circledcirc H$, *covariant in both, and a natural isomorphism*

$$\mathrm{Hopf}(H_1, B \boxtimes H_2) \cong \mathrm{Hopf}(B \circledcirc H_1, H_2).$$

Remarks

- Existence does not come from Freyd's theorem.
- But related to pairing
$$\mathrm{Alg} \times \mathrm{Coalg} \to \mathrm{Hopf}$$
from the representable functor
$$H \mapsto \mathrm{Alg}(A, \mathrm{Coalg}(C, H))$$
which does come from Freyd's theorem.
- A plethoric action on a Hopf ring is now in the more usual form:
$$P \circledcirc H \to H$$

The Problem and Answers pt III

To give a straightforward description of the algebraic structure of

Unstable operations

$E^*(\underline{E}_*)$ is a graded, complete plethory
$E^*(X)$ is a plethoric module.

Unstable co-operations

$E_*(\underline{E}_*)$ is a plethoric module.
$E^*(X) \to \mathrm{GCoalg}(E_*(X), E_*(\underline{E}_*))$
is a morphism of plethoric modules.