Describing Unstable Operations

Joint Algebra and Topology Seminar Sheffield

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19th October 2006

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Outline

Preliminaries: Operations

- Graded multiplicative cohomology theory

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- Graded multiplicative cohomology theory contravariant functors

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- Unstable Operations: $E^k_U(-) \to E^l_U(-)$

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- Operations are natural transformations
- Forget structure: $E^k_U(-) : \mathrm{hTop} \to \mathrm{Set}$
- Unstable Operations: $E^k_U(-) \to E^l_U(-)$
- Appears to disregard the structure of $E^*(X)$

Preliminaries: Representation

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- $E^*(\underline{E}_k)$ certainly has some structure

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- Unstable Co-operations: $E_*(\underline{E}_k)$
- Considerable structure, but does not act on $E_*(X)$.

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Also want to factor $E_*(\underline{E}_k)$ and $E_*(X)$ in to the story.

Algebras and Modules

$$A \times M \to M$$

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Intuitive

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Monads and Co-monads

Definition

A co-monad on a category C consists of a functor

$$T : C \to C$$

and natural transformations

$$\mu : T \to TT \qquad \epsilon : T \to I$$

satisfying the obvious co-associativity and co-unit diagrams.

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A co-module for a co-monad T is an object X of C with a morphism

$$\rho : X \to T(X)$$

satisfying the obvious co-module diagrams.

Examples

- A an algebra. $A_+ : \mathrm{Mod} \to \mathrm{Mod}$ by

$$A_+(M) := \mathrm{Mod}(A, M)$$

Natural transformations:

$$\mathrm{Mod}(A, M) \to \mathrm{Mod}(A, \mathrm{Mod}(A, M))$$

$$\Big(f : A \to M\Big) \mapsto \Big(a_1 \mapsto \Big(a_2 \mapsto f(a_1 a_2)\Big)\Big)$$

$$\Big(f : A \to M\Big) \mapsto f(1) \in M$$

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- M an A-module. $\hat{\rho} : M \to A_+(M)$ by

$$m \mapsto \Big(a \mapsto \rho(a \otimes m)\Big).$$

Examples (contd.)

- C a co-algebra. $C_! : \mathrm{Mod} \to \mathrm{Mod}$ by

$$C_!(M) := C \otimes M$$

Natural tranformations:

$$C \otimes M \xrightarrow{\Delta \otimes 1} C \otimes C \otimes M$$

$$C \otimes M \xrightarrow{\epsilon \otimes 1} k \otimes M \cong M$$

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- M a C-co-module.

$$M \to C \otimes M$$

Operations as Co-monads

Theorem (Boardman, Johnson, Wilson)

$E^*(\underline{E}_*)$ *represents a co-monad in* GAlg.

$E^*(X)$ *is a co-module for this co-monad.*

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$$\left(\alpha \in E^k(X) = \mathrm{hTop}(X, \underline{E}_k)\right) \to \left(\alpha^* : E^*(\underline{E}_k) \to E^*(X)\right)$$

Co-operations

$$E^k(X) = \mathrm{hTop}(X, \underline{E}_k) \to \mathrm{GAlg}(E^*(\underline{E}_k), E^*(X))$$
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Theorem (Ravenel, Wilson)

$E_*(\underline{E}_*)$ *is a Hopf ring*

Question: What is a Hopf ring?
Answer: A co-algebra H such that the contravariant functor

$$H^+ : C \to \mathrm{Coalg}(C, H)$$

actually ends up in Alg.

Module Functors

Definition

Let T be a co-monad on a category $\mathcal{C}$. A (left) co-module functor of T is a functor

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with a natural transformation

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Silly Example

$$A_+(-) = \mathrm{Mod}(A, -), M_+(-) = \mathrm{Mod}(M, -)$$

$$\Big(f : M \to N\Big) \mapsto \Big(a \mapsto \Big(m \mapsto f(am)\Big)\Big)$$

Module Functors (contd.)

Proposition

A co-module functor factors through the subcategory of co-modules.

Enriched Hopf Rings

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The Monoidal Story

The problem . . . is . . . the tensor product . . . that is simply unavailable for operations that are not additive (not that this has stopped us from trying).

Boardman, Johnson, Wilson

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Let $\mathcal{C}$ be a category with small colimits; $\mathcal{V}$ a variety of algebras; $F : \mathcal{C} \to \mathcal{V}$ a covariant functor. The following are equivalent.

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3. *$F_U : \mathcal{C} \to \mathrm{Set}$ is representable.*

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Corollary: Compositions of covariant representable functors are representable

▸ Skip example ▸ Skip all examples

Examples

Variety: groups
Source: hTop' (*based*)
The circle is a co-group object in hTop' with maps

$$S^1 \xrightarrow{\mu} S^1 \vee S^1, \qquad \text{pinch}$$
$$S^1 \xrightarrow{\nu} S^1, \qquad \text{reverse}$$
$$S^1 \xrightarrow{\varepsilon} \mathrm{pt}.$$

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These make $\pi_1(X) := \text{hTop}'(S^1, X)$ into a group:

$$f + g \text{ is } S^1 \xrightarrow{\mu} S^1 \vee S^1 \xrightarrow{f \vee g} X \vee X \to X$$
$$-f \text{ is } S^1 \xrightarrow{\nu} S^1 \xrightarrow{f} X$$
$$1 \text{ is } S^1 \xrightarrow{\varepsilon} \text{pt} \to X$$

Examples (contd.)

Variety: k-algebras
Source: arbitrary, C; initial object I.
Operations: $\lambda \in k$

$$X \xrightarrow{\alpha} X \amalg X \qquad \text{co-addition}$$

$$X \xrightarrow{\mu} X \amalg X \qquad \text{co-multiplication}$$

$$X \xrightarrow{\lambda} I \qquad \text{co-}\lambda\text{-action.}$$

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Structure: $f, g \in C(X, Y)$

$$f + g\colon\ X \xrightarrow{\alpha} X \amalg X \xrightarrow{f \amalg g} Y \amalg Y \to Y$$
$$fg\colon\ X \xrightarrow{\mu} X \amalg X \xrightarrow{f \amalg g} Y \amalg Y \to Y$$
$$\lambda\colon\ X \xrightarrow{\lambda} I \xrightarrow{\iota_Y} Y$$

Examples: Birings

Variety: k-algebras
Source: k-algebras
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$$X \xrightarrow{\alpha} X \otimes X \qquad \text{co-addition}$$
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Structure: $f, g \in \mathrm{Alg}(X, Y)$

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Structure: $f, g \in \mathrm{Alg}(X, Y)$

$$f + g\colon\ x \xrightarrow{\alpha} \sum_i x_i^{[1]} \otimes x_i^{[2]} \xrightarrow{f \otimes g} \sum_i f(x_i^{[1]}) \otimes g(x_i^{[2]})$$
$$\xrightarrow{m} \sum_i f(x_i^{[1]}) g(x_i^{[2]})$$

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Operations: The biring structure of $E^*(\underline{E}_*)$ is:

- Co-addition: H-map $\underline{E}_k \times \underline{E}_k \to \underline{E}_k$
- Co-multiplication: ring maps $\underline{E}_k \times \underline{E}_l \to \underline{E}_{k+l}$
- Co-E^*-action: $v \in E^k = E^k(\mathrm{pt}) = \mathrm{hTop}(\mathrm{pt}, \underline{E}_k)$ induces $v^* : E^*(\underline{E}_k) \to E^*(\mathrm{pt}) = E^*$.

Applications pt I: Products

Compositions of representable functors are representable.

$$A_+B_+ : \mathrm{Alg} \to \mathrm{Set} \qquad B_{1+}B_{2+} : \mathrm{Alg} \to \mathrm{Alg}$$

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Proposition (Tall, Wraith (1970))

There is a product

$$\mathrm{Biring} \times \mathrm{Alg} \to \mathrm{Alg}, \qquad (B, A) \to B \odot A$$

and a natural isomorphism

$$\mathrm{Alg}(B \odot A, A') \cong \mathrm{Alg}(A, \mathrm{Alg}(B, A'))$$

and if B_1, B_2 are birings then $B_1 \odot B_2$ is a biring.

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- Linear in B but not in A
- Not symmetric

Applications pt II: Plethories

Representable co-monad: $P_+ \to P_+P_+$, $P_+ \to I$.

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A plethory consists of a biring P and maps of birings

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satisfying the obvious diagrams.

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- I is the initial biring, $k[e]$

Applications pt II: Plethories

Representable co-monad: $P_+ \to P_+P_+$, $P_+ \to I$.

Definition (Tall and Wraith, Borger and Wieland)

A plethory consists of a biring P and maps of birings

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Applications pt III: Modules

Co-module over a representable co-monad:

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Example: If G acts on an algebra A then A is a $P(G)$-module.

Unstable Operations

Theorem

For "good" cohomology theories, the set of unstable operations of a cohomology theory is a graded, completed plethory.

The cohomology of a space is a module for this plethory.

The Problem and Answers pt I

To give a straightforward description of the algebraic structure of

Unstable operations

$E^*(\underline{E}_*)$ represents a co-monad on GAlg
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Freyd's Theorem pt II

Theorem (Freyd)

Let $\mathcal{C}$ be a category with small colimits; $\mathcal{V}$ a variety of algebras; $F : \mathcal{C} \to \mathcal{V}$ a covariant functor. The following are equivalent.

1. *F has a left adjoint*
2. *F is representable by a co-$\mathcal{V}$-object in $\mathcal{C}$*
3. *$F_U : \mathcal{C} \to \mathrm{Set}$ is representable.*

Corollary: Compositions of covariant representable functors are representable

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Corollary: Composition of a contravariant representable functor followed by a covariant one is representable

Pairings

$C \mapsto \mathrm{Alg}(A, \mathrm{Coalg}(C, H))$ is representable

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There is a pairing

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Caveat:
The pairing is contravariant in A and covariant in H.

Pairing Properties

From:

$$\mathrm{Alg}(B, \mathrm{Coalg}(C, H)) = \mathrm{Coalg}(C, B \boxtimes H)$$

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▸ Skip appalling pun

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Question: When does $H_* : C \mapsto \mathrm{Coalg}(C, H)$ land in the subcategory of P-algebras?

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Lemma

An enriched Hopf ring is a Hopf ring with an action of a plethory.

Problem: $P \boxtimes H$ is contravariant in P so tricky to work with.

Enriched Hopf Rings

Theorem

There is a pairing $\mathrm{Biring} \times \mathrm{Hopf} \to \mathrm{Hopf}$, $(B, H) \mapsto B \circledcirc H$, *covariant in both, and a natural isomorphism*

$$\mathrm{Hopf}(H_1, B \boxtimes H_2) \cong \mathrm{Hopf}(B \circledcirc H_1, H_2).$$

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which does come from Freyd's theorem.

- A plethoric action on a Hopf ring is now in the more usual form:

$$P \circledcirc H \to H$$

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