Describing Unstable Operations

Geometry and Topology Seminar, Glasgow

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30th October 2006

1 The Problem

To give a good description of the structure on the set of unstable operations of a nice cohomology theory.

"Good" description:

- Simple
- Practical
- Intuitive
- Complete
- Elegant

2 Preliminaries

Our fundamental object of study is $E^*(-)$; a graded, multiplicative, commutative, generalised cohomology theory. It is a *contravariant* functor

$$E^*(-)\colon \text{hTop} \to \text{GAlg}$$

The coefficient ring is $E^* := E^*(\text{pt})$.

Unstable operations on $E^*(-)$ are natural transformations $E^k_U(-) \to E^l_U(-)$ where $E^k_U(-)\colon \text{hTop} \to \text{Set}$ is the underlying set of kth component.

Algebraic topologists use cohomology theories to convert topological problems into algebraic ones. One common problem is to distinguish between two spaces; which turns into detecting a difference between $E^*(X)$ and $E^*(Y)$. Even if they are isomorphic as algebras, they may have different actions of operations. Unstable operations are the largest set of operations that can act, and so are the most powerful tools. Thus studying unstable operations tells us about the power of the theory $E^*(-)$.

Examples 2.1. Some examples of unstable operations:

- $v \in E^*$, $x \mapsto vx$;
- $x \mapsto 1_X$;
- $x \mapsto x^2$;
- More generally, for $p(t) \in E^*[t]$, get $x \mapsto p(x)$;
- On $K^0(X) = K$(iso classes of vector bundles over X)

$$(E \to X) \mapsto \begin{cases} (E^* & \to X) \\ (E \otimes E & \to X) \\ (\hom(E, E) & \to X) \\ (\Lambda^k E & \to X) \\ (S^k E & \to X) \end{cases}$$

- On $K^0(X)$, define the kth Adams operation Ψ^k by:

$$\Psi^k(L) = L^{\otimes k}, \qquad \Psi^k(E \oplus F) = \Psi^k(E) \oplus \Psi^k(F)$$

What makes this problem tractable is the fact that $E^*(-)$ is *representable*. There are spaces $\underline{E}_k$ and classes $\iota_k \in E^k(\underline{E}_k)$ such that the map $\alpha \mapsto \alpha^*\iota_k$ is a natural bijection

$$\mathrm{hTop}(X, \underline{E}_k) \to E^k(X)$$

Yoneda's Lemma tells us that:

$$\{\text{Unstable operations: } E^k_U(-) \to E^l_U(-)\} \cong E^l(\underline{E}_k)$$

and that the structure on $E^*(-)$ is determined by maps on the $(\underline{E}_k)$.

This tells us

1. The underlying sets
2. How to determine the structure

3 Answer I: The Structure

The first answer to our problem is simply to list all the available structure. This answer is certainly complete; one could even make a case for it being practical. However it is certainly not simple, elegant, or intuitive.

1. $E^*(\underline{E}_k)$ is a graded algebra over E^*
2. $E^*(X)$ is an abelian group, so $\underline{E}_k$ is an H-space:

$$\underline{E}_k \times \underline{E}_k \to \underline{E}_k$$

3. $E^*(X)$ is a graded algebra, so get maps

$$\underline{E}_k \times \underline{E}_l \to \underline{E}_{k+l}$$

4. $E^*(X)$ is an E^*-module, so for each $v \in E^k$ have a map

$$\xi v\colon \mathrm{pt} \to \underline{E}_k$$

5. $E^l(\underline{E}_k)$ acts on $E^k(X)$, so we have composition

$$\mathrm{hTop}(\underline{E}_l, \underline{E}_k) \times \mathrm{hTop}(\underline{E}_m, \underline{E}_l) \to \mathrm{hTop}(\underline{E}_m, \underline{E}_k)$$

In summary we get the following maps, which have to satisfy certain compatibility relationships that we won't specify:

The Structure

1. $E^*(\underline{E}_k)$ is a graded algebra
2. Co-addition: $\Delta^+\colon E^*(\underline{E}_k) \to E^*(\underline{E}_k)\widetilde{\otimes}E^*(\underline{E}_k)$
3. Co-multiplication: $\Delta^\times\colon E^*(\underline{E}_{k+l}) \to E^*(\underline{E}_k)\widetilde{\otimes}E^*(\underline{E}_l)$
4. Co-linear: $\epsilon^v\colon E^*(\underline{E}_k) \to E^*(\mathrm{pt}) = E^*$
5. Composition: $E^k(\underline{E}_l) \times E^l(\underline{E}_m) \to E^k(\underline{E}_m)$

4 Answer II: Co-monads

This structure is certainly rich, but simply listing it does not offer any great insight as to how it fits together or what it is for. Thus we search for some simple statement that encapsulates all of this structure. The first such answer is due to Boardman, Johnson, and Wilson in their paper in the Handbook of Algebraic Topology in 1995.

Theorem 4.2 (Boardman, Johnson, Wilson)**.** *$E^*(\underline{E}_*)$ represents a* co-monad *in* GAlg*. $E^*(X)$ is a* co-module *for this co-monad.*

What this means is

1. The hom-functor GAlg $\to$ Set

   $$A^* \mapsto \mathrm{GAlg}(E^*(\underline{E}_*), A^*)$$

   has a natural lift to a functor $U\colon \mathrm{GAlg} \to \mathrm{GAlg}$
2. There are natural transformations $\mu\colon U \to U^2$, $\epsilon\colon U \to I$ satisfying obvious co-associativity and co-unit diagrams.
3. There is a co-action map $\rho\colon E^*(X) \to U(E^*(X))$ satisfying the obvious diagrams.

We'll look at the how the lift works a bit later. The action map is straightfoward; it is

$$E^k(X) \cong \mathrm{hTop}(X, \underline{E}_k) \ni \alpha \mapsto \alpha^* \colon E^*(\underline{E}_k) \to E^*(X)$$

This description is . . . complete, simple, elegant, . . . impractical.

This is authors' opinion of this answer:

> This is our elegant but extremely terse answer, . . . [later] we translate this answer into practical language, in the context of Hopf rings, that we can use for computation.
>
> *Boardman, Johnson, Wilson*

5 Answer III: Hopf Rings

The "practical language" that Boardman, Johnson, and Wilson refer to is one that uses the associated homology theory, $E_*(-)$. This is a *covariant* functor

$$E_*(-) \colon \mathrm{hTop} \to \mathrm{GMod}$$

For "good" spaces, $E^*(X)$ is the E^*-linear dual of $E_*(X)$ (and $E_*(X)$ is a graded coalgebra). We shall call a cohomology theory "good" if each $\underline{E}_k$ is a "good" space in this sense.

So $E_*(\underline{E}_k)$ is "pre-dual" to $E^*(\underline{E}_k)$ and is simpler in some respects. Analogous to the co-action map

$$E^k(X) \cong \mathrm{hTop}(X, \underline{E}_k) \ni \alpha \mapsto \alpha^* \colon E^*(\underline{E}_k) \to E^*(X)$$

we have

$$E^k(X) \cong \mathrm{hTop}(X, \underline{E}_k) \ni \alpha \mapsto \alpha_* \colon E_*(X) \to E_*(\underline{E}_k)$$

and this shows how $E_*(\underline{E}_*)$ might fit into the picture.

Just as $E^*(\underline{E}_*)$ has considerable structure, so also does $E_*(\underline{E}_*)$. Boardman, Johnson, and Wilson call it an *enriched Hopf ring*. Its structure is as follows.

Enriched Hopf Ring Structure

1. $E_*(\underline{E}_k)$ is a graded E^*-coalgebra
2. $*$-multiplication $E_*(\underline{E}_k) \otimes E_*(\underline{E}_k) \to E_*(\underline{E}_k)$
3. $\circ$-multiplication $E_*(\underline{E}_k) \otimes E_*(\underline{E}_l) \to E_*(\underline{E}_{k+l})$
4. extra linear structure, $v \in E^k$ $(\xi v)_* \colon E^* = E_*(\mathrm{pt}) \to E_*(\underline{E}_k)$
5. co-composition ("mposition"), $r \in E^l(\underline{E}_k)$ $r_* \colon E_*(\underline{E}_k) \to E_*(\underline{E}_l)$

(1 - 4): Hopf ring *aka* algebra object in coalgebras

(5): enriched structure

The "enriched" part is slightly messy and complicated to work with – but it is necessary if one is interested in operations.

A Hopf ring is usually specified by giving a list of generators and relations. Elements of the Hopf ring are sums of $*$-products of $\circ$-products of these generators subject to these relations. A typical element thus has the form

$$\sum_i \bigstar_j \bigcirc_k g_{ijk}$$

The structure of an enriched Hopf ring is dual to that on the set of unstable operations. For most of structure there is not overmuch to choose between the two pictures, though one might prefer the Hopf ring as it has one comultiplication and two multiplications against the operations having two comultiplications and one multiplication. However, what really distinguishes them is composition.

Given two operations, r, s, expressed as linear functionals on the Hopf ring, the composition sr is determined by the formula

$$\langle sr, c\rangle = \langle s, r_* c\rangle$$

so we need to know $r_* c$ for all c.

Generally there is some explicit – but complicated – formula for $r_* c$ for each of the generators and then one uses certain formulae for $\circ$-products and $*$-products to get $r_* c$ for all c.

Push Forward Formulae

Example: $K(1)^0(\underline{K(1)}_0)$

$*$-generators: b^J, $J = (j_0, j_1, \dots)$ with $0 \le j_i \le p-1$, almost all zero, and $\sum j_i = 0$ mod $p-1$

relations: $(b^J)^{*p} = 0$

$r_* b^J$: b^J built by $\circ$-multiplication of elements b_k; $r_* b_k$ is the coefficient of x^k in the formal identity:

$$r_* b(x) = [\langle r, 1_2\rangle] * \bigstar_{j=1}^{\infty} b(x)^{\circ j} \circ [\langle r, b_j\rangle]$$

After which, we use the formulae:

$$r_*(a \circ c) = \sum_i \sum_j \pm \bigstar_{\alpha} r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} c_{j,\alpha}$$

$$r_*(a * c) = \sum_i \sum_j \pm \bigstar_{\alpha} r'_{\alpha *} a_{i,\alpha} \circ r'''_{\alpha *} c_{j,\alpha}$$

6 Interlude: Algebras and Modules

Boardman, Johnson, and Wilson take some care to justify their choice of co-monads to describe the structure of unstable operations. At one point, they make the following remark

> The problem . . . is . . . the tensor product . . . that is simply unavailable for operations that are not additive **(not that this has stopped us from trying)**.

Boardman, Johnson, Wilson (emphasis added)

This, together with the earlier quote, suggests that they are aware of the disadvantages of the co-monad description and looked for one more akin to the usual action of an algebra on a module but without success.

Co-monads are certainly less familiar than algebras so let us look at a simpler example to see what is going on.

Let Ab be the category of abelian groups. Let $T\colon \text{Ab} \to \text{Ab}$ be a representable co-monad on Ab with representing object R.

The functor T^2 is also representable:

$$T^2(M) = \text{Ab}(R, \text{Ab}(R, M)) = \text{Ab}(R \otimes R, M)$$

so the co-monad structure of T translates into morphisms:

$$\begin{aligned} T &\to T^2 & \qquad R \otimes R &\to R \\ T &\to I & \mathbb{Z} &\to R \end{aligned}$$

The diagrams translate into the axioms for R to be a ring.

If M is a T-co-module, we have a morphism of abelian groups

$$\begin{aligned} \rho \in{} & \text{Ab}(M, T(M)) \\ &= \text{Ab}(M, \text{Ab}(R, M)) \\ &= \text{Ab}(R \otimes M, M) \end{aligned}$$

so M is an R-module in the usual sense.

Thus we can either talk of representable co-monads and their co-modules or of rings and their modules. The latter is more intuitive.

The key feature of the correspondence was the fact that T^2 was representable, equivalently the existence of a left adjoint to the hom-functor $M \to \text{Ab}(R, M)$.

7 Birings and Plethories

Now let us return to algebras over a coefficient ring R. We want to consider a representable functor $T\colon \text{Alg} \to \text{Alg}$, with representing object B.

The first thing to observe is that with abelian groups, the hom-sets were again naturally abelian groups so we could consider things like

$$\mathrm{Ab}(M_1, \mathrm{Ab}(M_2, M_3))$$

for arbitrary abelian groups M_1, M_2, M_3. This *doesn't* hold for algebras. So if $T\colon \mathrm{Alg} \to \mathrm{Alg}$ is representable, the representing object must be somewhat special for the hom-functor to lift to algebras.

Given $f, g\colon B \to A$ and $\lambda \in R$ we need to define

$$\begin{aligned} f + g\colon B &\to A \\ fg\colon B &\to A \\ \lambda\colon B &\to A \end{aligned}$$

The techniques of universal algebra tell us that we need the following maps to exist on B

$$\begin{aligned} \text{co-addition: } \Delta^+\colon B &\to B \otimes B \\ \text{co-multiplication: } \Delta^\times\colon B &\to B \otimes B \\ \text{co-scalar: } \epsilon^\lambda\colon B &\to R \end{aligned}$$

writing $\Delta^+ b = \sum_i b_i^{(1)} \otimes b_i^{(2)}$ and $\Delta^\times b = \sum_i b_i^{[1]} \otimes b_i^{[2]}$, these give us the required operations as follows

$$\begin{aligned} f + g\colon B &\xrightarrow{\Delta^+} B \otimes B \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A \\ f + g(b) &= \sum_i f(b_i^{(1)}) g(b_i^{(2)}) \\ fg\colon B &\xrightarrow{\Delta^\times} B \otimes B \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A \\ fg(b) &= \sum_i f(b_i^{[1]}) g(b_i^{[2]}) \\ \lambda\colon B &\xrightarrow{\epsilon^\lambda} R \xrightarrow{\iota_A} A \\ \lambda(b) &= \epsilon^\lambda(b) 1_A \end{aligned}$$

Note: the tensor product appears because it is the categorical coproduct of algebras.

The structure maps have to satisfy some obvious compatibility relations which we won't state here. The object B is officially known as a *coalgebra object in the category of algebras* and more commonly as a *biring*. It was first identified by Tall and Wraith in 1970.

Examples 7.1.

1. Let R be a finite ring; $\mathrm{Set}(R, R)$ is a biring with co-structure coming from the first factor of R.

2. For a set X, the free algebra on X is a biring with co-structure defined by:

$$\begin{aligned} \Delta^+ x &= 1 \otimes x + x \otimes 1 \\ \Delta^\times x &= x \otimes x \\ \epsilon^\lambda(x) &= \lambda \end{aligned}$$

(x is called *ring-like*)

3. in particular the polynomial algebra, $R[x]$, is a biring.

The next question is: can we represent T^2? The techniques of universal algebra say that certain combinations of representable functors are representable and T^2 is one of these so, yes, T^2 is representable. Let us write $B \odot B$ for the representing object, it must again be a biring.

The structure of a co-monad on T induces biring maps

$$B \odot B \to B, \qquad I \to B$$

where $I = R[x]$ (which represents the identity functor); these satisfy the obvious diagrams.

A biring with this structure is called, variously, a *biring triple*, a *Tall-Wraith biring triple*, or a *plethory*.

Similarly, there is a product $B \odot A$ for an algebra A and the structure of a T-co-module on A induces an action map

$$B \odot A \to A$$

Thus we have a picture much like the intuitive picture of an algebra acting on a module, only it is a plethory acting on an algebra.

What does $B \odot A$ look like? Tall and Wraith gave the first construction as the free algebra on symbols (b, a) subject to the relations:

$$\begin{aligned}
(b_1 + \lambda b_2, a) &= (b_1, a) + \lambda(b_2, a) \\
(b, a_1 + a_2) &= \sum_i (b_i^{(1)}, a_1)(b_2^{(2)}, a_2) \\
(b, a_1 a_2) &= \sum_i (b_i^{[1]}, a_1)(b_2^{[2]}, a_2) \\
(b, \lambda 1_A) &= \epsilon^\lambda(b)1
\end{aligned}$$

These should be compared with the formula for $(f + g)(b)$ and $(fg)(b)$ which make $\mathrm{Alg}(B, A)$ into an algebra.

Unlike the tensor product, this is neither symmetric nor bilinear (though it is linear in B).

Examples 7.3.

1. Let R be a finite ring, $\mathrm{Set}(R, R)$ is a plethory with composition as the action map. For any set X, $\mathrm{Set}(X, R)$ is a module over this plethory with action map composition.

2. $R[x]$ is a plethory by composition. It is, in fact, the initial plethory and there is a canonical isomorphism $R[x] \odot A \cong A$ for an R-algebra A. Thus all R-algebras are $R[x]$-modules.

3. Let M be a monoid; the free algebra on the underlying set of M is a plethory with $m \in M$ ring-like and composition from the monoid product: $m_1 \odot m_2 \to m_1 m_2$. This is the "free plethory on M". Let us write this as $P(M)$.

   Any algebra with an action of M by algebra morphisms is a module for this plethory.

   Examples 7.4.

   (a) $M = \mathbb{N}_0$; $P(M) = R[x_0, x_1, x_2, \ldots]$. A P-module is an algebra with a specified algebra endomorphism, (A, ψ). x_j acts as ψ^j thus, say,

   $$x_1^2 x_2(a) = \psi(a)^2 \psi^2(a) = \psi(a)\psi(a)\psi(\psi(a))$$

   (b) $M = \mathbb{Z}$; $P(M) = R[\ldots, x_{-1}, x_0, x_1, \ldots]$. A P-module is an algebra with a specified algebra automorphism.

   (c) $M = \mathbb{Z}/2$; $P(M) = R[x_{-1}, x_1]$. A P-module is an algebra with a specified involution.

8 Answer IV: Plethories and Modules

Theorem 8.1 (S-Whitehouse)**.** *$E^*(\underline{E}_*)$ is a* graded, completed, plethory*; $E^*(X)$ is an $E^*(\underline{E}_*)$-module.*

Graded Plethories

1. P_k^* is a graded algebra
2. Co-addition: $\Delta^+ \colon P_k^* \to P_k^* \widetilde{\otimes} P_k^*$
3. Co-multiplication: $\Delta^\times \colon P_{k+l}^* \to P_k^* \widetilde{\otimes} P_l^*$
4. Co-linear: $\epsilon^\lambda \colon P_k^* \to k^*$
5. Composition: $P_l^k \odot P_m^l \to P_m^k$

A graded plethory is actually bigraded, P_*^*. Following the example of cohomology, we call the upper degree the *homological* degree and the lower one the *spacial* degree. The identity functor is represented by

$$I_*^* \coloneqq R[\iota_k \colon k \in \mathbb{Z}]$$

with $\iota_k \in I_k^k$; thus it is polynomial in even spacial degree and exterior in odd.

The "completed" refers to the fact that the algebras are completed with respect to the filtration topology, and in particular tensor products must be completed appropriately.

Examples 8.3.

1. There is a natural extension from ungraded plethories to positively graded plethories. Assume that the coefficient ring is concentrated in degree 0. Let P be an ungraded plethory. Define P^*_* as the plethory

$$P^k_l := \begin{cases} P & \text{if } (k,l) = (0,0) \\ I^k_l & \text{otherwise} \end{cases}$$

(note that $I^k_0 = \{0\}$ unless $k = 0$). Examining the structure of a graded plethory, we can see that the only interaction between the two parts of the structures is as follows

$$\Delta^\times : P^l_k \to P^l_k \widetilde{\otimes} P^0_0$$
$$P^0_0 \odot P^0_m \to P^0_m$$

In the former, if $k \neq 0$ we use the canonical map $I^0_0 = R[\iota_0] \to P$

$$\Delta^\times : I^*_k \to I^*_k \widetilde{\otimes} I^0_0 \to I^*_k \widetilde{\otimes} P$$

In the latter, P^0_m is canonically isomorphic to R and there is a canonical isomorphism $P \odot R \to R$ which we use.

Now let A^* be a positively graded algebra such that A^0 is a P-module. Then A^* is a P^*_* module with P acting on A^0 and I^*_* acting via the identity on the rest.

2. $H = H\mathbb{Q}$. This is a specific case of the previous extension with $P = \text{Set}(\mathbb{Q}, \mathbb{Q})$. Since $H^0(X) = \text{Set}(\pi_0(X), \mathbb{Q})$, $H^*(X)$ satisfies the required conditions.

3. $K(1)^0(\underline{K(1)}_0)$. This is the free completed plethory on the submonoid of $\{\mathbb{N}_0, \times\}$ generated by $\{0, \tilde{q}\}$; equivalently,

$$\langle x, y | xy = yx = y^2 = y \rangle$$

Then x acts as $\Psi^{\tilde{q}}$, y as Ψ^0. The completion is slightly non-standard as we take formal sums in $x - 1$ rather than x.

9 Bonus: Enriched Hopf Rings

Finally, as an added bonus we get a simpler description of an enriched Hopf ring.

Question: What is a Hopf ring?
Answer: An algebra object in coalgebras.

That is, to say that a coalgebra H is a Hopf ring is the same as saying that the contravariant hom-functor $H^* : C \to \text{Coalg}(C, H)$ lifts to a functor into algebras.

Suppose we have a plethory P representing a functor $P_* :$ Alg $\to$ Alg; then we have a category $P -$ Alg of P-modules, which is a subcategory of Alg.

Question: When does $H^* :$ Coalg $\to$ Alg actually land in $P -$ Alg?
Answer: When we have a natural transformation $H^* \to P_* H^*$.

This answer is co-monadic in form so we would like to convert it to a module-like setting. We need to represent the functor $P_*H^*\colon \mathrm{Coalg} \to \mathrm{Alg}$. Again, universal algebra tells us that we can. Write the representing object as $P \boxtimes H$.

This is covariant in H and contravariant in P; these variances mean that the action map is:

$$H \to P \boxtimes H.$$

This is like saying that H is a P-module, except that P and H now lie in different categories. There is no problem with this, however.

We can reformulate our answer:

Question: When does $H^*\colon$ Coalg $\to$ Alg actually land in $P-$Alg?
Answer: When H is a P-module.

There are two issues with this answer:

1. The assignment $(P, H) \to P \boxtimes H$ is covariant in H but *contravariant* in P. This makes it somewhat tricky to work with.
2. It is constructed using limits in the category Coalg, but arbitrary small limits in Coalg are complicated.

   As an example, let $C_r := \mathbb{Z}\langle \beta_0, \ldots, \beta_{2^r}\rangle$ with co-multiplication

   $$\Delta\beta_k = \sum_{i+j=k} \beta_i \otimes \beta_j$$

   Define maps $f_r\colon C_r \to C_{r-1}$ by $f_r(\beta_{2k}) = \beta_k$ and $f_r(\beta_{2k+1}) = 0$.

   The limit of this system as abelian groups is

   $$\prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}$$

   but the limit as coalgebras is just $\mathbb{Z}$.

Theorem 9.1 (S-Whitehouse)**.** *The functor $H \to P \boxtimes H$ has a left adjoint, $H \to P \circledast H$.*

This converts the action map $H \to P \boxtimes H$ into the more familiar form $P \circledast H \to H$. It is covariant in both arguments and uses colimits in Coalg which are much nicer.

Theorem 9.2 (S-Whitehouse)**.** *The* enriched *part of $E_*(\underline{E}_*)$ says that it is an $E^*(\underline{E}_*)$-module. The natural map*

$$E^*(X) \to \mathrm{GCoalg}(E_*(\underline{E}_*), E_*(X))$$

is a morphism of $E^*(\underline{E}_*)$-modules.

In fact, the most general statement is that the following is a map of $E^*(\underline{E}_*)$-modules:

$$E^*(X) \to \mathrm{GCoalg}(F_*(\underline{E}_*), F_*(X))$$
$$E^*(X) \ni \alpha \mapsto \alpha_*\colon F_*(\underline{E}_*) \to F_*(X).$$