

# Mechanics to Manifolds

## An Introduction to Quantum Mechanics

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### §1 Introduction

This seminar was originally given as the first of a series of three which aimed to explain some of the connections between Mechanics and Geometry. They were given last term by myself and Paul Cooper. The aim of this talk was to be an introduction to Mechanics.

The original text of this seminar is still available on the web via the graduate geometry home page. Also available is the text of the third seminar and the way we organised the three seminars means that the first and third stand alone without reference to the second.

In that talk I gave an introduction to various of the theories of Mechanics, concentrating on Newtonian and Quantum Mechanics. Today I want to concentrate on Quantum Mechanics alone.

There are a few theories of Mechanics, and it can sometimes be hard to separate them in the literature and applications. There are three main principles behind them. These are the Newtonian<sup>1</sup>, Relativistic and Quantum Theories. I have thus divided Mechanics into six areas:

1. Newtonian Mechanics.

Applicable to systems on a large scale which are moving slowly<sup>2</sup>.  
Newton published *Principia* in 1687.

2. Special Relativity.

Not so much a theory of mechanics, but it has implications for mechanical theories. Applicable to systems on a large scale which are moving fast in the absence of a gravitational field.  
Einstein published his first papers on Special Relativity in 1905.  
Minkowski published *Space and Time* in 1908.

3. General Relativity.

This includes gravitation into relativity.  
Einstein published his first papers on General Relativity in 1915.

4. Quantum Mechanics.

Applicable to systems on a small scale which are moving slowly.

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<sup>1</sup>Often called Classical.

<sup>2</sup>“Large” and “small” are in comparison to the nanometre ( $1 \times 10^{-9} m$ ). “Slow” and “fast” are in comparison to the megametre per second ( $1 \times 10^6 ms^{-1}$ ).

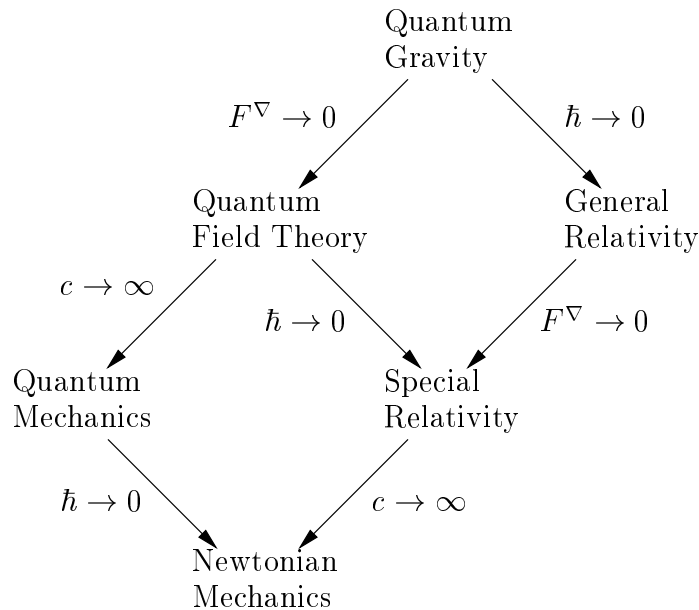


Figure 1: Theories of Mechanics

Planck proposed that energy came in quanta in 1900.  
 Einstein proposed his Quantum Theory of light in 1905.  
 de Broglie proposed the principle of wave-particle duality in 1923.  
 Schrödinger published his theory of wave mechanics in 1926.

5. Quantum Field Theory.  
 Represents a unification of Special Relativity with Quantum Mechanics.  
 Dirac published his equation for a relativistic charged particle in 1926.  
 Pauli and Heisenberg formed a QFT based on Lagrangians in 1929.
6. Quantum Gravity.  
 Represents a unification of General Relativity with Quantum Field Theory.  
 Various theories have been proposed (twistors, superstrings, etc.) but none  
 have yet been widely accepted.

Diagrammatically, one could represent these six as in Figure 1. The theories are always formulated with the fundamental constants included by symbol. This is mainly due to the fact that writing out the exact speed of light becomes an annoyance after about the first time. One effect of this is that we can regard these constants as variables in the theory and so consider the limiting theories. What this would mean in application is that the scale of the problem is such that the substitution of the limit induces no significant inaccuracy in the solution.

I want to provide you with an introduction to Quantum Theory. Although I shall be concentrating on Quantum Mechanics, most of what I say holds true for Quantum Field Theory as well. Although I shan't try to justify Quantum Theory in any way, I shall describe some experiments that have been done which show Quantum phenomena. Although Quantum Theory can seem very strange

to the newcomer, it does provide incredibly accurate descriptions of the world as we see it.

## §2 Why bother with Physics?

The short answer to this question is that a lot of new concepts and ways of thinking originate in Physics. Physics is a major part of the interface between Mathematics and the Real World. Thus one way to motivate Real Mathematics - i.e. beyond that needed to fill in a tax invoice - to the non-mathematician is to show how useful it is in science, then we leave it up to the scientists to explain why science is worth anything.

The boundary between Maths and Physics allows information to cross both ways. Since so much crosses from us to them it is natural to ask whether a fair amount comes back again, and this is indeed the truth. In disciplines where there is a clear application to the Real World, such as Mechanics, then this is more true than anywhere else.

Nearly all of the major intuition jumps in Mechanics have come from Physics. The way that this usually works is something like this:

First someone notices something odd about the universe. Maybe something doesn't move quite the way expected, or a careful observation of a planet reveals that it's not anywhere near predicted<sup>3</sup>. Then someone tries to develop a model for this new behaviour. Using this model, new experiments are devised and done until it is fairly well tested and accepted by the establishment. At some point, while the Physicists are arguing about whether some constant should have a 4 or a 5 in its thirtieth decimal place, someone else comes along and abstracts a general principle. This is then in the realms of Mathematics. The principle is used and tested, and can lead to a new branch of Mathematics. Finally a Psychologist gets hold of the new lingo and tries to use it to explain why Julius Caesar could never conquer Britain.

This is the general process, some times it goes faster than other times. Occasionally then the same person is involved at every step. More often, not. It's also hard to separate where the Physics ends and the Mathematics begins, but once someone tries extending the principle to infinite dimensions, it's a safe bet that that person has never seen a laser outside a disco.

## §3 Mechanics

Mechanics is that branch of Mathematics and Physics which deals with motion. The idea is that given the state of the universe now, what will it be like in 30 seconds', 10 minutes', 5 million years' time? There are two parts to this problem. The first is the amount of information required to specify the problem - so-called Cauchy data - and the second is how that information evolves in time - so-called dynamics.

Usually, knowing the dynamics tells you what data you need to know, but it is often through considering the data that the laws of dynamics are initially derived.

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<sup>3</sup>If you want to be at the forefront of the next big idea then probability says you should be an Astronomer.

A general principle tells one how to set up the problem, what data to look for. Then using this principle, one sets up the model. Often this involves simplifications which one hopes are negligible. For example, in planetary motions then the physical size of the planets is neglected because it is small compared to the radii of the orbits. One model is better than another if it predicts the future of the universe more accurately than the second. One general principle is better than another if it allows one to construct better models or to construct models for a wider variety of problems.

## §4 The Language of Mechanics

There are several words which get flung around in mechanical theories, a few of which I want to describe here. These are *states*, *observables*, *particles* and *fields*.

### §4.1 States and Observables

The concepts of *states* and *observables* are very important in considering the theories of mechanics.

A pure state is a *possible* configuration of the universe. The key idea is that this state, however it is Mathematically formulated, contains all the information needed to describe the universe, as exactly as theoretically possible.

An observable is a function on states which gives the result of an experiment done on the universe. It can be a measurement of displacement or momentum in a particular direction, it may be the total energy or the charge. Mathematically, the result must always be a real number. Of course, one can always interpret that real number in any way one likes and so this isn't restrictive at all. It should be emphasised that these are *real* experiments. Physically, a function qualifies as an observable if you can conceive of an experiment which would allow you to measure something which gives the value of this function on all states. Mathematically, that's not a workable definition and so we work with a set which contains all observables whether anyone has conceived of experiments or not.

This leads to a difference in definitions for quantities which are observable. A Mathematician would say that a quantity is defined by the function which corresponds to it, a Physicist would say that a quantity is defined by the experiment which observes it.

### §4.2 Particles and Fields

In Classical Mechanics, there are two very different objects called *particles* and *fields*. They are very easy to distinguish between because a particle has a very definite position whereas a field is spread out over the entire universe.

A particle is basically a billiard ball. It can have different properties, ranging from zero volume to infinite elasticity, but the basic picture you should have is of a billiard ball.

A field is more complicated to visualise, but easier to describe mathematically. It is a function on the universe which describes some global phenomenon.

Examples are the force felt due to a gravitational attraction or a magnet, or the density of matter or the temperature.

Classical Mechanics deals with these as separate objects and a lot of the complicated stuff comes when you try to deal with the interaction of particles and fields.

What Quantum Mechanics says is that all particles are fields. What Quantum Field Theory says is that all fields are particles. Thus we can use whichever picture seems most appropriate at the time, knowing that both are allowed<sup>4</sup>.

## §5 Quantum Mechanics

### §5.1 The Formulation of Quantum Theory

Newtonian Mechanics is all very well when the objects are large and slow, but when things get very small or very fast then the predictions become unacceptably inaccurate.

For example, the classical model of the Hydrogen atom is of an electron orbiting a proton much as a planet orbits the sun. However, the electron is an accelerating charged particle and classical theory says that such an object should radiate energy. Thus energy is constantly being emitted from the system and it must eventually collapse. Atoms do not collapse beyond a certain size, and they do not continuously radiate energy.

Quantum Mechanics allows one to consider situations in which the objects are very small. They still have to be moving slowly with respect to light, one needs Quantum Field Theory to deal with that problem. However, the *basic* principles involved in Quantum Field Theory are the same as those for Quantum Mechanics.

From the Physics end, the principles of Quantum Mechanics are as follows: every particle exhibits wave-like properties and thus can be assigned a wave function. A wave function is just a normalised field, which is a map from configuration space to the complex numbers. These wave functions are then the states of the universe. The time evolution of a state is controlled by the equation:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}$$

where  $H$  is a Hermitian operator obtained from considering the classical problem. This is the mystical process of Dirac quantisation. It is not well defined; there are choices to be made. In fact, this method of quantisation is the reverse of a well defined procedure - that of letting  $\hbar$  tend to zero.

For a free wave we have the celebrated Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi.$$

The definition of Quantum Mechanics from the point of view of Mathematics is as follows:

1. The pure states of the universe are rays in a Hilbert space (that is, unit vectors of arbitrary phase).

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<sup>4</sup>However, you should also be aware that neither is actually *correct*. These are just helpful pictures which must be discarded the moment they cease to be helpful.

2. The observables in a Quantum system are Hermitian operators on this space. The expectation value of an observable  $B$  of a state  $\psi$  is  $\langle\psi|B\psi\rangle = \langle\psi|B|\psi\rangle$ . This is the mean of many observations of  $B$  all carried out on the same state  $\psi$ .
3. The Hamiltonian  $H$  is the infinitesimal generator of the time evolution group. The momentum  $P$  is the infinitesimal generator of the space translation group. The angular momentum  $J$  is the infinitesimal generator of the spatial rotation group.

That is, the wave function propagates according to the rules:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \quad i\hbar\frac{\partial\psi}{\partial x} = P\psi, \quad i\hbar\frac{\partial\psi}{\partial\theta} = J\psi.$$

Thus the problem becomes finding the Hilbert space and the operators  $H, P, J$  and this is where the method of quantisation comes in. Usually the Hilbert space is  $L^2(M, \mathbb{C})$ , where  $M$  is a manifold corresponding to the universe under consideration - i.e. to the configuration space. The inner product is:

$$\langle\psi|\phi\rangle = \int_M \psi^* \phi \, d\tau$$

The operators  $H, P, J$  are determined from the classical observables by the process of quantisation.

## §5.2 States and Observables

The Mathematical formulation of Quantum Mechanics defines the states and observables. However, it is worth looking at what information a state can carry and how this is observed. There are two points that I wish to discuss: quantisation and indeterminacy.

Quantisation here refers to the original discovery which lead to Quantum Theory, namely that it is found, and experimentally verified, that the observed value of an observable is an eigenvalue of the operator. One tends not to have too many conceptual problems with this when considering observables like position with spectrum the whole real line, but it can cause interesting results in other cases. For example, the angular momentum of an electron orbiting a nucleus can only take discrete values.

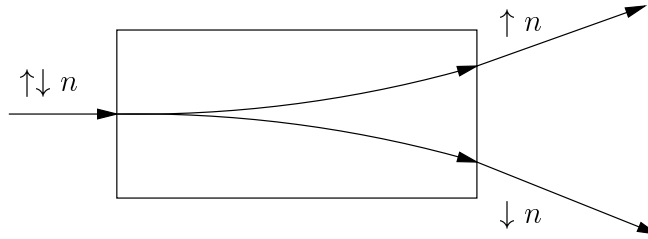
Closely linked to this is the fact that the action of observation interferes with the universe. That this is so qualitatively can be seen by considering the act of observing an electron. To observe it, we need to bounce some light off it. However, bouncing the light off it gives it a push and so it is no longer moving along in the original direction.

Quantum Mechanics says that this can be made quantitative as well. The act of observation forces the universe into a state which is an eigenvector of that observable - hence the fact that the result of an observation is always an eigenvalue.

A considerable conceptual problem now arises because eigenvectors of one operator are not necessarily eigenvectors of another. The condition for there to be simultaneous eigenvectors is that the operators commute.

If this does not happen, then when we make one observation, followed by another, followed by the original one again, then there is no reason whatsoever for the first and last observations to match.

The Basic Experiment:



An Experiment to Investigate Spin:

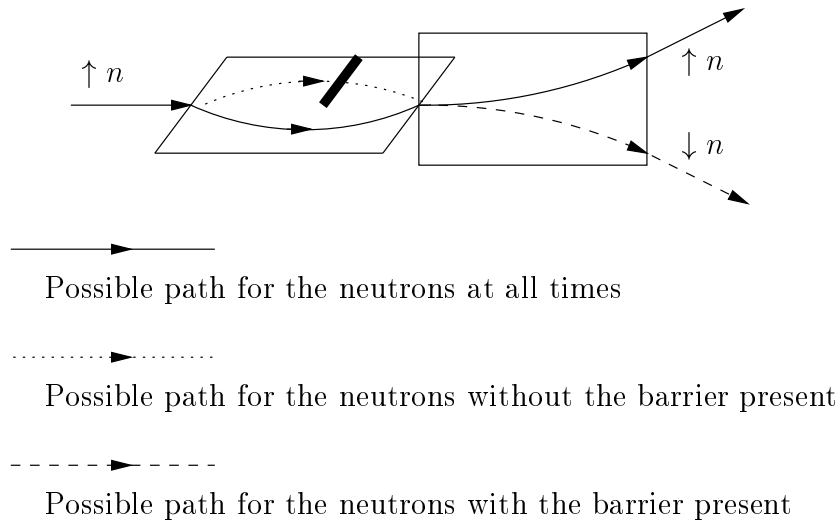


Figure 2: The Stern-Gerlach Experiment

### §5.3 The Stern-Gerlach Experiment and Spin

An example bringing out the significance of quantisation and indeterminacy is the Stern-Gerlach apparatus for measuring the spin of a particle (1924), Figure 2. The apparatus separates a beam of particles according to their spin about a particular axis. It is observed that the beam of particles splits into a finite number of beams, rather than a spread, indicating that the operator corresponding to spin has a finite discrete spectrum.

In the case of the neutron, the beam would split in two and we call neutrons in the upper beam spin-up, and those in the lower, spin-down. Since the apparatus has a particular orientation, we additionally specify this axis.

We suppose that we have a similar apparatus that separates the beam and then recombines it again, allowing us to block off one beam in the middle if we so desire. It should be noted that this is a hypothetical apparatus which, although feasible, has a few difficulties in practice. However, the principle is

sound.

We consider an experiment where we have a source of neutrons moving from the left to the right which we know are spin-up with respect to the z-axis. We feed this beam into the apparatus above which is oriented with respect to the x-axis. We feed the output from this apparatus into the original Stern-Gerlach apparatus oriented with respect to the z-axis. Thus we can measure the spin of emerging neutrons with respect to the z-axis by looking at which of the two possible beams are present.

We will do this with and without an obstruction in the first apparatus. Note that there are two possible places a neutron could end up and two possible paths for it to have travelled in order to arrive there.

With no obstruction, what we measure is that there are no spin-down neutrons emerging from the other end. All the neutrons emerge in the upper beam.

If we now block off one of the paths in the first apparatus, thus forcing the neutrons to be, say, spin-up with respect to the x-axis, then we get a different result. Suddenly, we get both spin-up and spin-down particles out of the end, and in equal proportions.

Classically, one thinks of the neutrons as particles separated into two beams. Thus a neutron arriving at the left hand side must choose<sup>5</sup> which path it travels along in the first apparatus and then choose again which it travels along in the second. Now we can develop a theory of spin which accounts for either of these results, but not for both. Either we can say that spin in perpendicular axes is independent. Thus although our neutron was forced to choose whether it was spin-up or spin-down with respect to the x-axis, it was still spin-up with respect to the z-axis. This neatly explains the first result. However, this theory says that blocking off one path in the first apparatus makes no difference to the z-axis spin so we should get the same result the second time. Or we could say that there is a unique axis of spin and when we force the neutron to pass through the second apparatus then we rotate this axis to align with the x-axis, which it does so with equal probability as up or down. But then when we force the neutron through the second apparatus, this should happen again and it should be spin-up or spin-down with equal probability. This explains the second result, but not the first.

However, Quantum Theory predicts when a particle has several possible paths between two points then it behaves as if it travelled along all of them. Provided we do not disturb the system in between, say by making a measurement, then the initial state determines the final state. In this case, although we *know* that an individual neutron “had” to travel along one path or the other, *so long as we don't know which* then the neutron emerges in exactly the same state as it went in, i.e. as spin-up with respect to the z-axis.

Thus we have an experiment which cannot be explained classically and which shows firstly the principle of quantization and secondly that of indeterminacy.

## §6 Conclusion

The purpose of this talk was to introduce you to some of the basics of Quantum Mechanics in order that you have a basis of knowledge for the talks this term which will be aimed at understanding some of the links between Geometry and

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<sup>5</sup>Please forgive my anthropomorphic tendencies here.

Physics - in particular Quantum Theory. I have not tried to justify the theories of Mechanics, if you want to know how they came about and why they are now considered as the best theories going then I suggest you read an elementary Physics book or a History of Science book.

## References

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