Research Proposal

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1 Introduction

My current and planned research divides into three areas: the relationship between stable and unstable cohomology operations, the differential topology of loop spaces, and the use of category theory in geometry. This research lies in the areas of algebraic topology, differential topology, and differential geometry and it has further application to mathematical physics. In addition to the techniques of the areas already mentioned, my research also uses theory from functional analysis.

2 Stable and Unstable Cohomology Operations

Overview

My research in this area started as part of an EPSRC–funded project, grant number GR/S76823/01. The principal investigator on this grant was Sarah Whitehouse of Sheffield University and this work continues in collaboration with her.

Algebraic topology splits into two pieces: stable and unstable. The break is not clean, however, and many interesting problems require techniques from both pieces in their solution. Therefore it is of considerable interest to study the relationship between the two parts.

The original project involved certain cohomology theories called the Morava K–theories. Algebraic topologists use cohomology theories to convert topological problems into algebraic ones. Different cohomology theories "see" different parts of algebraic topology and the Morava K–theories are particularly interesting due to what they "see" and to their good properties.

The project has now expanded to a detailed study of the relationship between stable and unstable phenomena in a wider class of cohomology theories.

Background

Part of the structure that comes with a generalised cohomology theory are the operations and co-operations. These contain considerable information about the cohomology theory. These operations divide into three types: stable, additive, and unstable. The original project was to study the relationship between these for the Morava K–theories.

In the 1970s, Morava introduced a family of cohomology theories now known as the Morava K–theories. For each prime p there is a sequence of theories $K(0), K(1), K(2), \ldots$. Each is periodic and all have nice properties which aid calculation. Their importance in stable homotopy theory has been well-established, [24, 34, 49, 50, 51].

One can associate to a cohomology theory E a *localisation* functor, L_E, on the stable homotopy category. This is essentially a projection onto the part of the stable homotopy category that this cohomology theory "sees". In general localisation is quite complicated but for the Morava K–theories (at least for $n \geq 1$), work of Bousfield [10] (for $n = 1$) and Kuhn [38] (for all $n \geq 1$) has given a surprising description of $K(n)$–localisation, namely that it factors through the zeroth space functor, Ω^∞. In other words, they have constructed a functor Φ_n from the homotopy category of p–local based spaces to the homotopy category of $K(n)$–local spectra such that $L_{K(n)} = \Phi_n \Omega^\infty$. This has many interesting consequences which have been often exploited, for example in [11, 12, 39, 52]. One consequence is that the natural map from stable operations to unstable operations is split for the Morava K–theories.

In [66] we studied the sets of stable and unstable operations for the Morava K–theories using the Bousfield–Kuhn functor and the algebraic descriptions of the co-operations in [7, 8, 73]. Our main result was a topological description for this splitting map.

Our current focus builds on this work. In order to translate our description to the algebraic side, we wanted descriptions of the structure on the sets of operations of the "generator–relation" type. Such a description exists, and was given in [7], for the stable operations. For unstable operations the best descriptions were in [8] but these were not of a form suited to our needs.

We therefore sought a better context to describe this structure. This involved using techniques from universal algebra and led us to devise the notion of a graded, completed *Tall–Wraith monoid*.

Briefly, for a variety of algebras, $\mathcal{V}$, a Tall–Wraith monoid is a set with precisely the algebraic structure required for it to act on $\mathcal{V}$–algebras. One example is very familiar: a ring is a Tall–Wraith monoid for the category of abelian groups. This notion was originally introduced by Tall and Wraith in [70] for the specific example of commutative unital rings; they used the term *biring triple*. Recently, Borger and Weiland [9] rediscovered this and extended it to the case of commutative unital k–algebras under the name *plethory*. Thus a plethory is that-which-acts-on-algebras. This is clearly relevant to our purposes as unstable cohomology operations of multiplicative cohomology theories act on the cohomology algebras of spaces.

However, cohomology algebras are both graded and topologised and this requires some care to handle. The full description of graded, complete Tall–Wraith monoids is in [65].

The benefit of this description is that it will allow us to describe the structure of the set of unstable operations in very simple terms, analogous to the description of a ring in terms of generators and relations. Our purpose is to use this description to study unstable operations and the stable–unstable interface.

As well as continuing with the broad aims of the original research project, I intend to look at the more topological aspects of this area. Much of the current study of Morava K–theories uses their algebraic properties and descriptions but their original definition was firmly based on the geometric ideas of Baas and Sullivan [4, 5, 69]. The result in [66] strongly suggests that this will prove a fruitful area of research.

Planned Research

This work is joint with Sarah Whitehouse of Sheffield University.

1. We shall study the algebraic properties of Tall–Wraith monoids. Our primary focus will be on describing the sets of unstable operations for particular cohomology theories. Our goal will be to give descriptions of the form of "generators and relations", however as Tall–Wraith monoids do not form a variety of algebras this type of description is more subtle than it first appears. We shall therefore look for particularly simple examples of Tall–Wraith monoids to compare against the ones that come from cohomology theories.

2. Our first application of this theory will be a description of the unstable operations of a Morava K–theory in terms of its stable operations. This will enable us to set the work of [66] in its proper context.

Our aim is a description of the unstable operations of Morava K–theories analogous to that of Bousfield wherein a method is given for p–adic K–theory to generate unstable operations from their stable counterparts.

We shall then extend the above to the Morava K–theory of a spectrum in terms of the Morava K–theory of its zeroth space. This is also done by Bousfield for p–adic K–theory and is related to work of Rezk [52].

3. The Morava K–theory $K(1)$ is closely related to ordinary K–theory mod p. The algebras of stable and unstable operations for ordinary K–theory have concrete algebraic descriptions which have also been studied in the p–local case, [16, 17, 18, 19]. We shall also relate this work to these descriptions.

4. The prime $p = 2$ has some subtleties not present at other primes. It is often convenient work at odd primes to avoid difficulties. We shall examine the case of $p = 2$ to see which of our results remains valid.

5. Our second aim is to develop a theory for Tall–Wraith monoids similar to that available for rings and their modules. This will be invaluable in studying how unstable operations act on the cohomology theories.

Another aspect of this is to describe algebraically the relationships between additive and stable operations with unstable operations.

6. I also intend to look for links between this aspect of my research and the others. I shall so this by examining the interaction between geometry and algebra in the Morava K–theories. The initial aim of this is to derive topological explanations for known algebraic phenomena.

3 The Differential Topology of Loop Spaces

Overview

The primary aims of my research in this area are twofold: to develop an index theory for loop spaces and to define a calculable semi-infinite de Rham cohomology. This research will help advance the study of loop spaces as smooth manifolds.

Often the most powerful theories in mathematics are those which bridge the gaps between different areas. The original index theory was one of these in that it provided a passage between algebraic topology on the one hand and geometry and mathematical physics on the other. This has produced many important theorems in all the areas that it links. Index theory for loop spaces has already proved itself extremely useful even though it is not yet a rigorous theory. Just by knowing the form it should take, mathematicians and physicists have been able to transfer ideas from one area to another. A full theory would enable us to transfer results as well. I have already made significant progress on this project: in [60] I showed how to construct the operators that are the fundamental objects of study in index theory.

Semi-infinite de Rham cohomology has the potential to be of similar importance. Ordinary de Rham theory provides a bridge between algebraic topology and geometry and this should also hold for the semi-infinite variety. With its links to Floer homology and loop groups, semi-infinite de Rham theory will also bring in such diverse topics as symplectic geometry, mathematical physics, representation theory, string topology and others. My thesis, [57], contained a basic definition of semi-infinite de Rham cohomology but there are many ways to vary this construction. Recent work in the field of Floer theory, my work on loop spaces, and the emergence of parametrised homotopy theory and generalised complex structures means that this is an apposite time to revisit this topic.

Background

Loop spaces occur throughout mathematics. They have always been fundamental objects in algebraic topology and functional analysis, though in markedly different ways. Recently, they have also moved to the centre-stage in differential topology, geometry, and mathematical physics. The intersection is the study of loop spaces as smooth manifolds and therefore this has the potential to be a rich theory with many applications to other areas of mathematics.

From the early days of geometry it has been convenient to view a space of loops in a manifold as a manifold in its own right, though this is not always a rigorous concept. The primary motivation for doing so was usually to gain some insight into the original manifold through studying particular paths. The proof of Bott periodicity using Morse theory is a classical example of this, see [45] for an exposition.

This changed in the 1980s with the rise of string theory. A central idea of string theory is that space is a loop space. Loop spaces thus moved from being auxiliary to primary objects. Again motivated by considerations from string theory, one purpose of the study of loop spaces is to repeat the successes of finite dimensional geometry in the infinite realm. Often in so doing new aspects come to light. For example, the theory of loop groups has proved a fertile ground for research with its links to twisted K–theory, [25, 26, 28, 27, 48].

There are usually difficulties in extending some technique or construction from finite dimensions to infinite. This is even the case with the definition of an infinite dimensional manifold, [36, 37, 40, 46, 47]. The difficulties often lie in differential topology or geometry and require techniques from functional analysis in their solution.

A prime example of this is the problem of doing index theory on loop spaces. As already mentioned, index theory is a highly successful theory in finite dimensions, providing links between algebraic topology, geometry, and mathematical physics; see [41, 55]. Thus information and insights can freely move between the subject areas. The key steps in index theory are the following.

1. The input is a differential operator $D : \Gamma(E) \to \Gamma(F)$ where E, F are vector bundles over a manifold. There is a technical condition that this operator needs to satisfy called *ellipticity*. Many problems in geometry and physics give rise to such an operator.

2. To this operator we assign a *topological index*. The method of doing this is to start with a class in some K–theory, called the *symbol* of the operator, which we push-forward to the coefficient ring. This results in an integer.

3. We also consider the *analytical index*: $\operatorname{Index} D := \dim \ker D - \dim \operatorname{coker} D$. The aforementioned ellipticity condition is to ensure that this is defined.

4. The index theorem states that these two indices are the same.

Two footnotes to this are that there is often a simple cohomological formula for the topological index, and that in the presence of a group action there is an equivariant version of the above wherein the indices become characters of the group. Moreover, if there is a group action then the cohomological formula is in terms of the fixed point set.

In [74], Witten formally applied this to the Dirac operator on a loop space with the natural circle action. As the fixed point set of this action is the constant loops, which is just the original manifold, the fixed point form of the topological index is in terms of finite dimensional things. The resulting expression, now known as the Witten genus, can therefore be rigorously defined and has since been extensively studied; for example [2, 23, 33, 35, 42, 64, 67, 71]. It has proved to be an important object in many areas of mathematics. Many of its properties were conjectured from its original construction by analogy with the problem in finite dimensions but as that construction was purely formal the analogies did not lead to proofs. Other methods had to be developed to fill in the gaps. This, of course, led to new and interesting mathematics but a genuine index theory for loop spaces would provide a deeper understanding of the mathematics involved.

The problems in doing index theory on loop spaces are many. Firstly, there is the basic problem of defining the operator. In the case of the Dirac operator one can set up the ingredients necessary for its definition exactly as in finite dimensions but two steps in its definition in finite dimensions fail when extended to infinite dimensions. There have been many partial solutions proposed to this problem which work by exploiting a particular feature of some special space, for example [13, 56]. In [60], I found a solution which worked for all loop spaces. One of the strengths of this method is that none of the initial data is changed, it is just used in a slightly different way.

Secondly, there is the question as to which K–theory should hold the symbol of the operator. This question is further obscured by the plethora of properties that this symbol possesses. Thus one can consider "high-level" K–theories which preserve as much of this structure as possible, and "low-level" ones which preserve just as much as is needed to define a reasonable theory. There are several current proposals for the high-level K–theory: [68] and [3] being the two front-runners. At the low-level, I have been looking for a suitable K–theory of equivariant, positive-energy vector bundles. The challenge is to build a theory which is non-trivial and has a fixed point theory. Using such a theory one would be able to give a rigorous construction of the Witten genus as the topological index of the Dirac operator.

Thirdly, there is the analytical index. Although one could attempt a basic definition, many of the techniques used to study the index in finite dimensions rely on the fact that there are enough Hilbert spaces around to impose additional regularity when it is needed. These in turn depend for their definition on the existence of a measure on the manifold. This introduces considerable difficulties when working with loop spaces.

Finally, once one has defined all of the structure there is the proof of the index theorem itself to consider.

Another example of extending finite dimensional structure to infinite dimensions is Floer theory, [44, 54]. For loop spaces of certain manifolds, including symplectic, it is possible to identify a "middle dimension". When adapting a construction from finite to infinite dimensions one can often choose to use this "middle dimension" in a non-trivial way. The term "semi-infinite" is used to distinguish these from standard theories. One such theory is Floer homology, originally introduced to solve the Arnold conjecture in finite dimensional symplectic geometry. It can be viewed as a semi-infinite version of Morse theory. There are many different ways of constructing the cohomology of a manifold, each with its strengths and areas of application. The fact that they all compute the same thing means that they can be used to pass information from one area to another. Each method should have a semi-infinite version for loop spaces.

I am particularly interested in semi-infinite de Rham cohomology. In defining this one encounters similar problems to those encountered in defining the Dirac operator. Due to technical considerations one cannot use the same techniques for the solution. In my thesis, [57], I proposed a solution to this problem. When applying this theory to certain simple test cases I found that to get a calculable theory I had to use some additional structure particular to those cases which is not available for loop spaces. Specifically, I used either a measure or a filtration by finite dimensional submanifolds. Therefore there are still many questions as to the best construction of semi-infinite de Rham cohomology for loop spaces.

These two areas, index theory and semi-infinite theory, form the focus of my current research on loop spaces. In working in these areas I have had to use results and techniques from a broad range of mathematics; particularly functional analysis, differential topology, and geometry. Moreover, I have found that most occurrences of loop spaces can be linked to these areas and thus come within this remit. Thus while providing me with a definite focus to my work I also have the freedom to investigate and use techniques from many areas of mathematics. The papers [58, 59, 63] are examples of how this has happened.

As remarked earlier, even the question of defining the structure of a smooth manifold on a loop space is problematic. In [59] I considered the general problem of doing so on an arbitrary space of loops in a smooth manifold. There are many types of loops that can be defined, ranging from smooth to continuous, and often one wants to use a specific type of loop for a specific problem. It is known that certain spaces of loop form a smooth manifold, such as smooth and continuous loops. The method of proving this result is essentially the same in each case and in [59] I considered this construction to determine the conditions under which it worked in general. In [63] I applied that to the specific case of piecewise–smooth loops. These loops are often used as a compromise between smooth and continuous loops, having some of the advantages of each. However the work of [63] showed that there are also significant disadvantages of using these loops. The techniques used in [63] were a mixture of topology and functional analysis. There are several open questions: having a smooth manifold structure on a space of loops is certainly desirable but more is often needed to use standard techniques of differential topology. One of the most important pieces of extra structure is the existence of smooth partitions of unity. It would be useful to determine when a loop space admits such.

Planned Research

1. Having defined the Dirac operator, the next stage is to define the topological index. To define this, I intend looking for a suitable class of infinite dimensional vector bundles. This class must have certain good properties to make it into a cohomology theory and so that it has a fixed point theory. The obvious candidate of Hilbert spaces has certain problems when taking the circle action into account so I am considering various other types of model space.

2. Having defined the Dirac operator and the topological index, the next stage is to consider the analytical index. I shall start this investigation by looking at the space of smooth sections of a vector bundle over a loop space. This investigation will require a thorough understanding of what it means for a section to be "smooth". I intend to use the calculus of [37] in this. In order to find a reasonable topology on this space it may be necessary to limit the sections in some fashion.

3. Once the topological and analytical indices have been defined, the next step is to formulate an index theorem and develop a programme for its proof. The guide for this programme will be the proof of the index theorem in finite dimensions but it is anticipated that there will be many twists in fitting this guide to infinite dimensions. It is likely that these twists will themselves be of independent interest.

4. On another tack, I intend to examine the properties of the Dirac operator itself. One key property is its behaviour with regard to the action of the diffeomorphism group of the circle. In [61] I showed that the natural action of this diffeomorphism group does not act on the operator but I also showed that this action could be modified so that it did. The natural action has been much studied and it will be interesting to see what carries over to the modified action.

5. I also intend studying the properties of the Dirac operator when there is a group acting on the original manifold in a suitable way. Under my supervision, a student from Lausanne University, Emanuele Dotto, has recently completed a masters thesis on this topic. Many interesting ideas arose in our discussions that I would like to see developed further. One goal of this is to link the Dirac operator on loop spaces to the work of [26] on twisted K–theory of loop groups.

6. The construction of the Dirac operator revealed considerable geometrical structure on loop spaces that had not hitherto been seen. The key ingredient in its construction is what I have called a *co-Riemannian structure*, see [61]. This is similar to the ordinary notion of a Riemannian structure except that certain maps go in different directions. One of the common problems in extending constructions from finite dimensional geometry to infinite dimensions is that the easy transition between tangent and cotangent vectors breaks down in infinite dimensions. The notion of a co-Riemannian structure goes partway to fixing this and I shall examine what from finite dimension geometry extends to the infinite. In particular, it should be possible to define Morse theory for general infinite dimensional manifolds rather than just for Hilbert manifolds as is the current case.

7. With regard to semi-infinite de Rham cohomology, the recent work of [1, 53, 72] has suggested a test case that I did not consider in my thesis: the loop space of a cotangent bundle. As it is a loop space, this case is more applicable to the general case than other test cases; as it is a vector bundle it may be more tractable to analysis than a generic loop space. An essential part of any successful semi-infinite de Rham theory is an infinite dimensional Thom isomorphism. This will involve an integration theory on the fibres of the cotangent bundle. I shall investigate how this may be done, starting by considering the work of Gel'fand and Vilenkin [29].

8. One unfortunate aspect of the semi-infinite structure of the loop space of a symplectic manifold is that it is based on a choice of compatible complex structure for the original symplectic manifold. Unless the manifold is Kähler this will not be integrable and thus the semi-infinite structure is not integrable. The solution to this will probably lie with the theory of *generalised complex structures*, introduced by Hitchin in [31] and continued by Gualtieri and others. See [6, 22, 30, 32, 43, 75, 77, 76] for a flavour of the research in this area. Specifically, there is an interesting relationship between the spinor space of the generalised complex bundle and the exterior power of the cotangent bundle.

9. One aspect of the theory of loop spaces that occurs throughout is the notion of *locality*. It is often desirable that some construction on a loop space depend locally on the original manifold. This can easily be achieved by ensuring that it depend *pointwise*, but this is a stronger notion than is often required. For example, the tangent bundle has pointwise dependence but the cotangent bundle only has local dependence. Although much can be done purely with objects with pointwise dependence, this approach does not reveal the full range of what is achievable. Whilst working on loop spaces I shall examine this issue to see where the boundary between pointwise and local objects lies.

10. The differential topology of loop spaces has recently developed very rapidly in many different directions. The basics of this theory can be unfamiliar to a researcher in topology as they involve quite intricate concepts from functional analysis. I have written seminar notes on the fundamentals of this subject,

[62]. These are intended as an introduction to the differential topology of loop spaces for those familiar with ordinary differential topology but less familiar with infinite dimensional calculus. I intend to continue to write these notes with a view to publishing them as a book.

11. Whilst occupied in the above research I shall look for ways to forge links with other topics in this area. In particular, the recent theory of string topology [14, 15, 20, 21] promises rich structure which should interact well with the objects that I study.

4 Category Theory in Geometry

Overview

In both of the main areas of my research I have found it necessary to use the language and techniques of category theory.

That categorical terminology is needed to describe *Tall–Wraith monoids* is evident. Indeed, part of our goal in this area is to find a balance between the formal categorical language of Lawvere theories and monads, and the more concrete description in terms of generators and relations.

In studying infinite dimensional manifolds such as the space of smooth loops one naturally considers the question of extending techniques of differential topology to spaces for which they were not originally designed. This leads one to consider enlarging the category of smooth manifolds to something larger where many of the techniques of ordinary manifolds still apply.

In addition, it can be useful to view certain infinite dimensional manifolds, such as loop and path spaces, as categories themselves. The categorical structure emphasises the extend to which these depend on the original manifold. It seems to be the case that the more interesting aspects of the theory of loop and path spaces is that which respects this categorical structure.

As part of the online collaborative community, the *nLab*, I have been examining both of these aspects of the interaction of category theory with geometry.

Background

Uses of category theory in other areas of mathematics can be loosely divided into two types: *big* and *small*. The former tends to deal with finding suitable categories in which to place objects of interest, and in studying the structure of said categories, and related functors, natural transformations, and so forth. The latter deals more with things as categories and considers how the structure of being a category can be used to study the object in question. A simple example to show the difference is that of studying a group. In the "big" context, one could study the category of groups, or a category of objects with an action of a particular group. In the "small" context, one could study a particular group as an example of a category.

When studying infinite dimensional manifolds, one naturally encounters an obvious question of the "big" type: what is the correct category for differential topology? Smooth manifolds are very structured spaces and as such are extremely useful objects to study. The category of smooth manifolds, on the other hand, is an extremely badly behaved category and it is very easy to do things to manifolds that land outside this category. Thus there has been a search for a suitable category containing that of smooth manifolds which is nicer in the categorical sense but close enough to that of smooth manifolds that much of differential topology still works there. There have been various attempts to do this and, prompted by discussions on the weblog, the *n*–Category Café, I have begun a study of these attempts. The current state of this is in [58] where I have given a description of the relationships between the various extensions of the category of manifolds. This has provoked much debate on the *n*–Category Café regarding the various properties of these extensions and regarding how to extend techniques of differential topology from manifolds to the extensions.

On the other end of the scale, loop and path spaces can themselves be viewed as categories. The description is not clean, however, in that the composition rule, which is composition of paths, is not quite as well-behaved as it ought to be. There are various solutions to this problem, some of which involve invoking higher categorical structures. Whilst these solutions are well-known and have been extensively studied for topological spaces, translating them to smooth spaces involves some subtleties.

The advantage in considering a path space as a category is that the relationship to the original manifold is explicit. The idea is that functorial structure on the path space is more closely related to structure on the original manifold than arbitrary structure is. An important example of this is the notion of an *elliptic object*. All of the proposed interpretations of an elliptic object have used categorical language to describe some part of the structure. These are clearly related to my own work on index theory on loop spaces. Up to now, I have not included this structure in my work.

Much of this work will be done as part of and in collaboration with the online *nLab* community.

Planned Research

1. I am particularly interested in examining which of the basic definitions of differential topology can be extended to the various versions of "smooth space". My main focus will be on Frölicher spaces, but building on [58] I shall also look at how the different theories interact.

2. In seeking for a way to compare the different notions of a "category of smooth spaces", I was led to the notion of a category being determined by a subcategory of "test objects". The roots of this idea can be found in the work of Lawvere and Isbell and extends the idea of a sheaf. I shall look for other places to apply these ideas in the hope of being able to appropriate techniques from other areas of mathematics to apply to smooth spaces.

3. In studying path and loop spaces as categories, I shall begin by looking at how to use that structure in my work on index theory. In particular, concatenation of paths forces more homogeneity around the spaces involved and it may be possible to exploit this in order to reduce the complexity of the structures.

4. One important feature of viewing path and loop spaces as categories is the passage from the original manifold to the path or loop space. This has particular relevance for the theories of elliptic objects, and especially for two-vector bundles and gerbes, as to how those objects give rise to new objects on the loop space, and exactly what those new objects are. In studying the interaction between the categorical and differential notions of path spaces, I hope to gain new insight as to how to lift structures from the original manifold to its path space.

5 Funding and Projects

I am keen to seek funding for several aspects of my research. In particular I would look for funding for a project entitled "Index Theory on Loop Spaces" based on the second part of my research plan. One of the aims of this project would be to establish and develop collaborations with researchers in my own and other institutions. There are many mathematicians interested in this subject due to the broad range of mathematics that it involves and so I anticipate great interest in such a project. Also due to the range of techniques needed in the study, there are many mathematicians who could usefully contribute to this study whether or not they are currently engaged with loop spaces.

There are also several aspects of the study of loop spaces that would make good doctoral research projects. From my planned research, the study of the properties of the Dirac operator would be suitable, as would the semi-infinite structure of the loop space of cotangent bundles. Both of these are important problems which are finite in scope but which could lead to other more open questions.

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