Semi-Infinite Theory

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Abstract

In this seminar, we shall introduce semi-infinite manifolds and show how one may define cohomology theories dependent on this semi-infinite structure. In particular, we shall define a de Rham theory. With certain alterations, this is calculable for Wiener manifolds. The goal, however, is to calculate the theory for loop spaces and this is, as yet, an unsolved problem.

1 Semi-Infinite Theory

The term "semi-infinite theory" is a rather loose one. It describes a collection of work all of which have a common theme. This theme is the concept of being "half-way" between zero and infinity.

The principle behind "semi-infinity", or "infinity over two" as some prefer, is the following: we choose some well-defined point which represents half way between zero and infinity and consider finite differences to this point.

To illustrate this, consider $\mathbb{N}$ sitting inside $\mathbb{Z}$. This is a prime candidate for a semi-infinite subset of $\mathbb{Z}$ since it is infinite itself but also has an infinite complement. We consider subsets of $\mathbb{Z}$ which differ from $\mathbb{N}$ by a finite amount. Thus a semi-infinite subset of $\mathbb{Z}$ is a set which contains almost all the positive numbers and almost no negative ones.

En route to manifolds, the next step is to look at vector spaces. The analogue of subsets is closed subspaces and so the semi-infinite concept when translated into vector space theory tells us that we should be looking at objects of the form $X = X_- \oplus X_+$ where $X_\pm$ are closed, infinite dimensional subspaces. Choosing one particular such decomposition, we consider decompositions $X = X'_- \oplus X'_+$ which differ from the chosen one by a finite amount.

By finite amount, we mean that the projection onto X_+ restricted to X'_- is a finite rank operator, and similarly for X_- and X'_+. The class of such decompositions can be given various topologies but in the more natural ones, it is not complete. To complete it, we relax the finite rank operator condition to some ideal of compact operators. The most convenient is the space of Hilbert-Schmidt operators. Thus we define:

Definition 1.1 *A* polarisation *of a vector space X is a class of decompositions $X = X_- \oplus X_+$ such that any two decompositions differ by a Hilbert-Schmidt operator.*

There is a more usual definition in terms of polarising operators, namely that a polarisation is a class of endomorphisms $\mathcal{J}$ of X which are congruent modulo compact operators, satisfy the equation $J^2 = 1$ modulo compact operators and moreover $\pm 1 \notin \mathcal{J}$. This definition is often more useful to work with but for a beginner can hide the simplicity of the concept. The correspondence between the definitions comes from assigning to an operator J the decomposition into its positive and negative eigenspaces.

Corresponding to a polarisation is the group $\mathrm{Gl}_{\mathrm{res}}(X)$ of operators which preserve it. This acts transitively on the space of decompositions within the polarisation . There is also a Grassmannian manifold $\mathrm{Gr}_{\mathrm{res}}(X)$ consisting of all the positive parts of decompositions. This can also be characterised in the following way: choose one particular (or base) decomposition $X = X_- \oplus X_+$ and define $\mathrm{Gr}_{\mathrm{res}}(X)$ to be the space of all closed subspaces W of X with the properties:

1. $p_+ : W \to X_+$ is Fredholm,
2. $p_- : W \to X_-$ is Hilbert-Schmidt,

where $p_\pm$ are the projections onto $X_\pm$ defined by the decomposition. The group $\mathrm{Gl}_{\mathrm{res}}(X)$ acts transitively on this space.

This Grassmannian has connected components indexed by $\mathbb{Z}$. For a fixed element in the Grassmannian, say X_+, we can define the relative dimension of any other element:

Definition 1.2 *For $W \in \mathrm{Gr}_{res}(X)$, the* dimension of W relative to X_+ *is the index of the Fredholm operator $p_+|_W$ from W to X_+.*

The group $\mathrm{Gl}_{\mathrm{res}}(X)$ also splits into $\mathbb{Z}$ connected components and thus decomposes as the semi-direct product of $\mathbb{Z}$ and its identity component $\mathrm{Gl}_{\mathrm{res},0}(X)$. Under the relative dimension map into $\mathbb{Z}$, the action of $\mathrm{Gl}_{\mathrm{res}}(X)$ reduces to the standard additive action of $\mathbb{Z}$ on $\mathbb{Z}$.

If M is a manifold modelled on X we can ask whether the structure group of M reduces to $\mathrm{Gl}_{\mathrm{res}}(X)$. If is does, we say that the manifold is *polarised*[1]. There is thus a principle $\mathrm{Gl}_{\mathrm{res}}(X)$ bundle P over M. Quotienting out by the action of the identity component defines a $\mathbb{Z}$ bundle over M and thus an element b_1 of $H^1(M;\mathbb{Z})$. This defines the periodicity of the polarisation.

Another way to understand this is to consider the fibre bundle over M with fibre $\mathrm{Gr}_{\mathrm{res}}(T_pM)$ at $p \in M$. Picking a polarisation at p is essentially the same as choosing an element of this fibre bundle. When we transport this choice around a loop in M we may end up in a different component of $\mathrm{Gr}_{\mathrm{res}}(T_pM)$. If we wish to define the relative dimension of two local polarisations we run into the following problem: having picked the polarisations, we could leave one where it is and transport the other round a loop and end up with a different answer. The different answers will be congruent modulo the periodicity and thus the relative dimension can only be defined modulo periodicity.

The key examples of polarised manifolds are the loop spaces of almost complex closed manifolds. Other examples include the manifolds $\mathbb{P}H$ and $\mathrm{Gr}_k(H)$ where $H = L^2(S^1, \mathbb{C}^n)$ is a polarised Hilbert space. We shall examine the case of a loop space in more detail later.

2 Classifying Polarised Manifolds

Thus we have some nice structure on certain interesting infinite dimensional manifolds and we would like to use this to study those manifolds. In one sense, semi-infinite theory is just another tool to study loop spaces and as loop spaces crop up absolutely everywhere in Mathematics and Physics, any tools we can use will be put to good use. Semi-infinite theory does have a more direct link to other areas of mathematics. The earliest reference to anything remotely semi-infinite that I know of is Dirac's notion of the "electron sea". In that scenario we imagine that we already have an infinite number of electrons[2] and are looking at finite differences from this base point.

[1] The term "semi-infinite" is reserved for a technical refinement of this defintion.

[2] Or if you believe Feynman, the same electron an infinite number of times.

More recently, Floer theory uses the polarised structure of a loop space in a fundamental way to determine some structure of the original manifold. Floer theory is closely related to Quantum cohomology and to the theory of holomorphic spheres in symplectic manifolds. There are also links to string theory via BRST theory. The word "ghost" appears frequently in papers linked to that area.

There are two levels in which the extra structure of a polarisation can be used to give information about the underlying manifold. Firstly, there is ordinary information about extraordinary structure. Secondly, there is extraordinary information about ordinary structure. The first is reasonably simple as it uses techniques and definitions that have been in the literature for decades and applies them to polarisations. The second is more interesting as it involves discovering new techniques and new definitions and using them to look at the manifold - the "ordinary structure" - in a new way.

The main example of the first type is obstruction theory and characteristic classes. A polarised bundle can be represented by a homotopy class of maps $M \to B\mathrm{Gl}_{\mathrm{res}}(X)$. The rational cohomology of this classifying space is known for the usual choices of X and we can pull back the generators to get characteristic classes corresponding to the bundle. Certain of these can be identified and have interesting properties. One is the periodicity obstruction $b_1(M)$ mentioned above.

The main example of the second type is Floer theory. This is an extraordinary version of a well-known theory, namely Morse theory. However, instead of calculating the homology of the manifold, it calculates something else. What this something else is is something I'm interested in finding. The first step along the path to finding this is finding a version of de Rham theory which is to Floer theory as the ordinary de Rham theory is to Morse theory.

3 Semi-Infinite de Rham Cohomology

In order to define a de Rham cohomology theory we need two things. Firstly we need a suitable sequence of vector bundles over the manifold. Secondly we need a differential between appropriate sections of that bundle.

The exact definition of the vector bundles is reasonably technical. What we seek is a model for $\Lambda_{\mathrm{si}} T^* M$. A section of this bundle in local coordinates looks something like:

$$dx^S = dx^{s_1} \wedge dx^{s_2} \wedge dx^{s_3} \wedge \dots$$

where $S \subseteq \mathbb{Z}$ is a semi-infinite subset comparable to $\mathbb{N}$ and $\{x^k : k \in \mathbb{Z}\}$ are the local coordinates for M.

What we have there is a *semi-infinite form.* Clearly if S and T are comparable semi-infinite subset then dx^S and dx^T will differ by only a finite number of variables.

This bundle can be defined and sections taken. It turns out that the bundle can be graded according to the grading of $\mathrm{Gr}_{\mathrm{res}}(TM)$ and thus the spaces of sections can be similarly graded. Thus if the polarisation is periodic, the resulting de Rham theory will be also.

So far it seems as though we are mirroring the standard de Rham theory closely. However, when we take the next steps, we encounter two difficulties.

The first is how the semi-infinite forms transform. Suppose we have a form dx^S. This form is linked to a semi-infinite subspace X_S of X, namely the span of $\{x^s : s \in S\}$. There are elements of $\mathrm{Gl}_{\mathrm{res}}(X)$ which preserve this subspace so consider the action of one of them on dx^S. We should have a formula of the form:

$$A \circ dx^S = \det A dx^S$$

However, the determinant function is not defined on all infinite dimensional invertible operators. As an example, consider the invertible operator $2I$. This would have determinant 2^ω which is not allowed. There is, fortunately, a way round this which involves forcing operators to have determinants provided a certain characteristic class $b_3 \in H^3(M;\mathbb{Z})$ vanishes. This is actually an obstruction to the construction of the bundles $\Lambda_{\mathrm{si}}T^*M$ in the first place.

The second problem is more subtle. Although dx^S is a local section, it is not typical. The general form is:

$$\sum_S f_S dx^S$$

where the summation is over all subsets S of $\mathbb{Z}$ comparable to $\mathbb{N}$ (a countably infinite set). The de Rham derivative of this is:

$$\sum_S \frac{\partial f_S}{\partial x^i} dx^i \wedge dx^S$$

rearranging this gives:

$$\left(\sum_{t_i \in T} (-1)^{i-1} \frac{\partial f_{T \setminus t_i}}{\partial x^{t_i}} \right) dx^T$$

where $T = \{t_1, t_2, \dots\}$ is the enumeration of T in its natural order.

The problem here is the question of convergence of the sum. There is no reason why this sum should converge. What is happening here is the following: let U be an open subset of X. We have the bundle $U \times \Lambda_{\text{si}} X^*$ and so a section of this bundle is a smooth map $U \to \Lambda_{\text{si}} X^*$. The derivative is thus a map $U \to L(X, \Lambda_{\text{si}} X^*)$. The map $\wedge$ is defined with domain $X^* \otimes \Lambda_{\text{si}} X^*$. In finite dimensions, these two coincide but in infinite dimensions they are not even close. The tensor product sits inside the space of continuous linear maps as the space of finite rank operators.

Locally we can restrict to sections whose derivative happens to lie in the tensor product. These are the *tame* forms. This is not invariant under all diffeomorphisms, however. There is a class of diffeomorphisms which do preserve this concept, and all manifolds modelled on Hilbert spaces have smooth atlases with this property. But this does not apply to loop spaces, for example.

One thing we notice is that if a section s has this property and f is a smooth function then fs also has this property. Thus we can patch together tame forms locally to get a globally defined form which satisfies the tame property locally. Such a form is a *locally tame form* and is one which can be expressed as a sum of locally supported forms, each of which has the tame property in some chart. On such forms, we can define the de Rham differential.

This allows us to define a class of sections of the exterior powers. There is one further hurdle, namely that the de Rham differential of a locally tame form may not be locally tame. To surmount this problem we can do one of two things: either we can restrict further to those sections whose de Rham differential is also locally tame, or we can formally set $d^2 = 0$. In the first case, we need to check that $d^2 = 0$, whence if a section satisfies this property, so does its de Rham differential. In the second case we use the same calculation to check that, in the case that the de Rham differential of a section is actually locally tame, we also have $d^2 = 0$.

The homology of the complex is the same in both cases, but one gives a larger complex and thus more room to manœuvre when proving the cohomology axioms. We now have a de Rham complex of semi-infinite forms and we can show that these form a generalised cohomology theory.

4 How to Calculate de Rham Cohomology

The key tool in calculating de Rham cohomology is the Thom isomorphism. In finite dimensions, this allows us to extend a low dimension result to higher dimensions. An infinite dimensional version would allow us to extend a finite dimensional result to the semi-infinite case.

However, in order to define a semi-infinite Thom isomorphism, we need an integration theory and we need to ensure that our sections are integrable. This necessitates the use of Wiener manifolds and excludes loop spaces.

It does give us the nice result that $H_{\mathrm{si}}(\mathbb{P}H) = \mathbb{C}\left[u, u^{-1}\right]$, which agrees with Floer theory.

There is an alternative definition which relies on sequences of finite dimensional submanifolds which saturate the original manifold. Using this allows us to calculate:

$$H_{\mathrm{si}}(\mathrm{Gr}_k(H)) = \mathbb{C}\left[c_1, \ldots, c_k, {c_k}^{-1}\right]$$

which also agrees with Floer theory. However, this is only applicable to Banach manifolds modelled on a certain type of Banach space (a Hilbert space is enough) and so also excludes loop spaces.

5 Possible Lines of Attack for Loop Spaces

Loop spaces of almost complex manifolds are in many ways the natural objects in a semi-infinite theory. The loop groups $L\mathrm{Gl}_n(\mathbb{C})$ are prime examples of polarisation-preserving groups. If $H^{(n)} = L^2(S^1, \mathbb{C}^n)$ then there is an inclusion $L\mathrm{Gl}_n(\mathbb{C}) \to \mathrm{Gl}_{\mathrm{res}}(H^n)$ which, when restricted to $\Omega\mathrm{Gl}_n(\mathbb{C})$, is a homotopy isomorphism up to degree $2n-2$.

Let M be a simply connected, connected, closed, almost complex manifold of real dimension $2n$. A complex bundle over M of complex dimension r can be represented by an element ξ of the set $[M, B\mathrm{Gl}_r(\mathbb{C})]$. Applying the based loop function, Ω, gives an element $\Omega\xi$ of the set $[\Omega M, B\Omega\mathrm{Gl}_r(\mathbb{C})]$, in other words, a polarised vector bundle over ΩM. The construction of this bundle is given in the following way:

Let $E \to M$ be a complex bundle. Evaluation on the circle defines a map $e : S^1 \wedge \Omega M \to M$ which we use to pull back the bundle E. The bundle $\mathcal{E} \to \Omega M$ has fibre at γ $\mathcal{E}_\gamma = \Gamma_{S^1} e^* E$. We denote the combined functor by τ.

This method can be applied more generally than just to vector bundles. For example, in cohomology we have the map $e^* : H^*(M) \to H^*(S^1 \wedge \Omega M)$ and the slant product $\omega_{S^1} : H^*(S^1 \wedge \Omega M) \to H^{*-1}(\Omega M)$.

Given any generalised cohomology theory, h^*, we have $e^* : h^*(M) \to h^*(S^1 \wedge \Omega M)$ and an isomorphism $h^*(S^1 \wedge \Omega M) \to h^{*-1}(\Omega M)$ combining to give a map $\tau : h^*(M) \to h^{*-1}(\Omega M)$. The map on vector bundles was this map applied to K-theory.

The naturallity of these maps gives the identity $b_{2k-1}(\tau(E)) = \tau(c_k(E))$ connecting the characteristic classed of $\tau(E)$ with those from E. The map τ applied to the tangent space of M gives the tangent space of ΩM.

When dealing with loop spaces, it is convenient to pass to a certain cover which gets rid of the periodicity. Once this has been done, a recent preprint by Jack Morava gives the decomposition of the loop space as:

$$T\widetilde{LM} \cong TM \otimes L\mathbb{C}$$

as equivariant bundles, where TM is non-equivariantly isomorphic to the pull-back of the tangent bundle of M under the evaluation map at a fixed point in the circle. The vector space $L\mathbb{C}$ is canonically polarised and thus we have:

$$T\widetilde{LM} \cong TM \otimes L_-\mathbb{C} \oplus TM \otimes L_+\mathbb{C}$$

as a global decomposition. I believe that this decomposition is also integrable.

Another useful fact is the following about loop groups. The structure group of a loop space is $L\mathrm{Gl}_n(\mathbb{C})$ for some n. Although there is an inclusion from this group to $\mathrm{Gl}_{\mathrm{res}}(X)$ which is an isomorphism in homotopy up to a certain dimension, as groups of operators they have certain differences. The one I am interested in is the fact that if an operator in $L\mathrm{Gl}_n(\mathbb{C})$ has a determinant then it is the identity.

These two pieces of extra structure are important for the following reason: on a Hilbert manifold, the concept of a tame section makes sense globally because the structure group can be reduced to the space of finite rank operators. To extend this result globally, we introduced the idea of locally tame sections. However, if we throw out this concept and deal only with globally tame sections then on a loop space the only globally tame sections are the ones which pull back from the evaluation map. I can use this global decomposition to lever these up to semi-infinite forms and thus construct the semi-infinite de Rham cohomology of the loop space in such a way that it is naturally isomorphic to the ordinary cohomology of the original manifold.