

Comparative Smootheology

Workshop on Smooth Structures in Logic, Category
Theory, and Physics, Ottawa

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3rd May 2009

Contents

1. Smootheology

Aim: Frölicher spaces are “obvious”
generalisation of manifolds.

2. Comparative

Aim: how the various categories of “smooth
objects” fit together.

Smootheology

There now follows a party political broadcast
for the
Frölicher Party

Goal

Build a **category** extending **smooth manifolds**.

To contain:

- ▶ Orbifolds: **quotients**
- ▶ Stratifolds: **push-outs**
- ▶ Loop spaces: **mapping objects**
- ▶ $P_1(M)$: **mapping objects AND quotients**
- ▶ Embedding spaces: **mapping objects AND subobjects**
- ▶ And more . . .

C^5 : complete, co-complete, cartesian closed category.

Guiding principle:

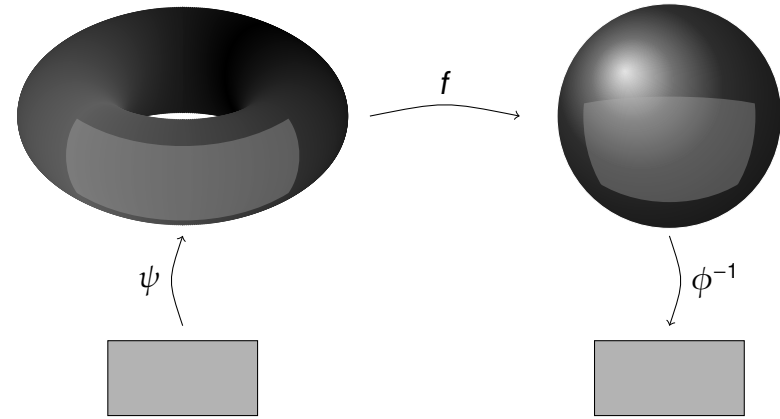
The Smooth,
the Whole Smooth,
and Nothing But the Smooth.

The Smooth: Building a Category

A category has

Objects
and
Morphisms

Morphism of Manifolds



$f: M \rightarrow N$ smooth if $\phi^{-1}f\psi$ is C^∞

Charts

- ▶ Charts **control** smooth structure
- ▶ **Domains** are “standard” spaces
- ▶ Use **both** charts and inverses
- ▶ Charts are **homeomorphisms**

Lose homeomorphism: **too fragile.**

Retain:

$$I(U) = \{\psi: \mathbb{R}^m \supseteq U \rightarrow M\} \quad \text{input test functions}$$

$$O(V; \mathbb{R}^m) = \{\phi: M \supseteq V \rightarrow \mathbb{R}^m\} \quad \text{output test functions}$$

First Candidate

Definition (First Attempt)

A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- ▶ X is a **topological space**
- ▶ $\mathcal{I}(U) \subseteq \text{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- ▶ $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

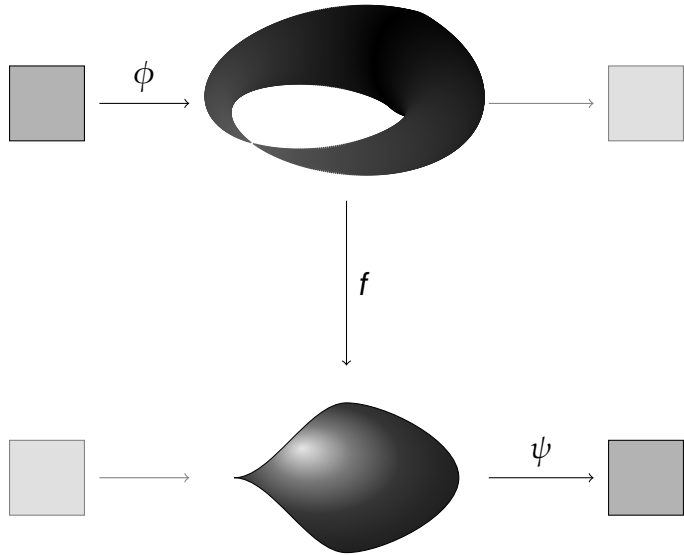
A **morphism** is a continuous map $f: X \rightarrow Y$ such that

$$\phi f \psi \text{ is } C^\infty \text{ for } \psi \in \mathcal{I}(U), \phi \in \mathcal{O}(V; \mathbb{R}^m)$$

Notation:

- ▶ smooth map = morphism
- ▶ $\psi \in \mathcal{I}$, $\phi \in \mathcal{O}$, $\theta \in C^\infty$

Morphisms



Categorical Construct?

- Identities:
 $1_X: X \rightarrow X$ must be smooth.

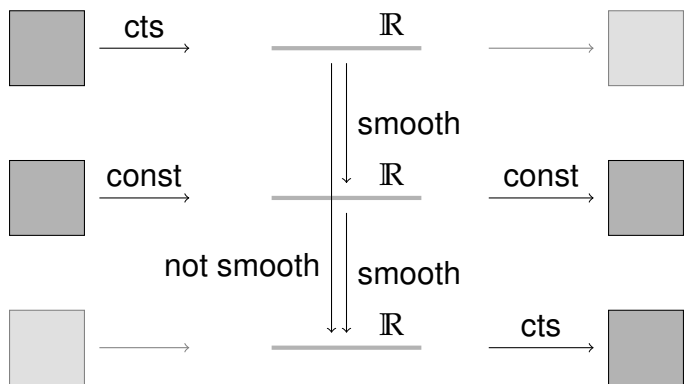
$$\psi \in \mathcal{I}, \phi \in \mathcal{O} \text{ then } \phi\psi = \phi 1_X \psi \in \mathcal{C}^\infty$$



Compatibility Condition

Categorical Construct?

- Composition:

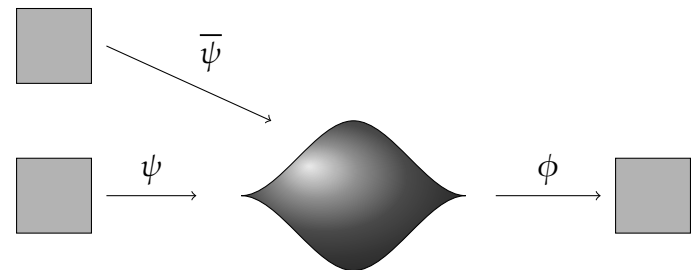


Composition

Definition

The **completed** inputs and outputs are

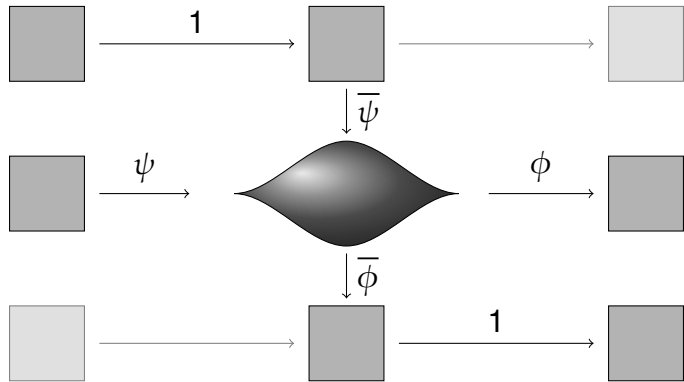
- $\bar{\mathcal{I}}(U) = \{\bar{\psi}: U \rightarrow X : \phi\bar{\psi} \in \mathcal{C}^\infty, \phi \in \mathcal{O}\}$
- $\bar{\mathcal{O}}(V; \mathbb{R}^m) = \{\bar{\phi}: V \rightarrow \mathbb{R}^m : \bar{\phi}\psi \in \mathcal{C}^\infty, \psi \in \mathcal{I}\}$



Composition

Proposition

Composition is well-defined for smooth objects where $(\bar{\mathcal{I}}, \bar{\mathcal{O}})$ satisfies **compatibility**.



Second Candidate

Definition (Second Attempt)

A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- ▶ X is a **topological space**
- ▶ $\mathcal{I}(U) \subseteq \text{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- ▶ $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- ▶ \mathcal{I} and \mathcal{O} are **compatible**,
- ▶ $\bar{\mathcal{I}}$ and $\bar{\mathcal{O}}$ are **also compatible**.

A **morphism** is a continuous map $f: X \rightarrow Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

The Whole Smooth: Too Many Smooth Spaces Spoil The Category

Lemma

$(X, \mathcal{I}, \mathcal{O})$ smooth space then $(X, \bar{\mathcal{I}}, \bar{\mathcal{O}})$ also smooth space **and moreover**

$$1_X: (X, \mathcal{I}, \mathcal{O}) \rightarrow (X, \bar{\mathcal{I}}, \bar{\mathcal{O}})$$

is an isomorphism.

Proof.

1. $\bar{\bar{\mathcal{I}}} = \bar{\mathcal{I}}$ and $\bar{\bar{\mathcal{O}}} = \bar{\mathcal{O}}$
2. $\phi \psi \in C^\infty$ for $\phi \in \bar{\mathcal{O}}$, $\psi \in \mathcal{I}$ and for $\phi \in \mathcal{O}$, $\psi \in \bar{\mathcal{I}}$ □

Third Candidate

Definition (Third Attempt)

A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- ▶ X is a **topological space**
- ▶ $\mathcal{I}(U) \subseteq \text{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- ▶ $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- ▶ \mathcal{I} and \mathcal{O} are **compatible**,
- ▶ \mathcal{I} and \mathcal{O} are **saturated**: $\mathcal{I} = \bar{\mathcal{I}}$, $\mathcal{O} = \bar{\mathcal{O}}$.

A **morphism** is a continuous map $f: X \rightarrow Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Comments

- ▶ **Non-smoothness** is detectable:
If $\phi: X \supseteq V \rightarrow \mathbb{R}^m$ is **not** in $\mathcal{O}(V; \mathbb{R}^m)$ then there is $\psi \in \mathcal{I}(U)$ such that $\phi\psi$ is **not** C^∞ .
- ▶ Analogy:

2nd Definition
versus
3rd Definition

smooth atlas
versus
maximal smooth atlas

Detox

Lemma

$(X, \mathcal{I}, \mathcal{O})$ is completely determined by

- ▶ $\mathcal{O}(V; \mathbb{R})$
- ▶ $\mathcal{I}(\mathbb{R})$

Proof.

1. $\phi: U \rightarrow \mathbb{R}^m$ is C^∞ if and only if each $p_i\phi: U \rightarrow \mathbb{R}$ is C^∞ , p_i projection
2. $\psi: U \rightarrow \mathbb{R}^m$ is C^∞ if and only if each $\psi\gamma: \mathbb{R} \rightarrow \mathbb{R}^m$ is C^∞ , $\gamma \in C^\infty(\mathbb{R}, U)$ [Boman's Theorem] \square

Fourth Candidate

Definition (Fourth Attempt)

A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- ▶ X is a **topological space**
- ▶ $\mathcal{I} \subseteq \text{Top}(\mathbb{R}, X)$,
- ▶ $\mathcal{O}(V) \subseteq \text{Top}(V, \mathbb{R})$, $V \subseteq X$ open.

such that

- ▶ \mathcal{I} and \mathcal{O} are **compatible**,
- ▶ \mathcal{I} and \mathcal{O} are **saturated**: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A **morphism** is a continuous map $f: X \rightarrow Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Nothing But the Smooth: Chicken and Egg

Question

Is “Smooth” built **on top** of “Continuous” or **alongside**?

- ▶ Topology **only** used for **local** functions.
- ▶ Can get topology from \mathcal{I} and \mathcal{O} .
- ▶ Topology from \mathcal{O} has **bump functions**.
- ▶ **Local** functions extend **globally**.

Fifth Candidate

Definition (Fifth Attempt)

A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- ▶ X is a **set**
- ▶ $\mathcal{I} \subseteq \text{Set}(\mathbb{R}, X)$,
- ▶ $\mathcal{O} \subseteq \text{Set}(X, \mathbb{R})$.

such that

- ▶ \mathcal{I} and \mathcal{O} are **compatible**,
- ▶ \mathcal{I} and \mathcal{O} are **saturated**: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A **morphism** is a map $f: X \rightarrow Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Fifth Candidate: Frölicher Space

Definition (Frölicher Space)

A **Frölicher space** is a triple $(X, \mathcal{C}, \mathcal{F})$ where:

- ▶ X is a **set**
- ▶ $\mathcal{C} \subseteq \text{Set}(\mathbb{R}, X)$,
- ▶ $\mathcal{F} \subseteq \text{Set}(X, \mathbb{R})$.

such that

- ▶ $\mathcal{C} = \{\psi: \mathbb{R} \rightarrow X : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \phi \in \mathcal{F}\}$
- ▶ $\mathcal{F} = \{\phi: X \rightarrow \mathbb{R} : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \psi \in \mathcal{C}\}$

A **morphism** is a map $f: X \rightarrow Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{C}, \phi \in \mathcal{F}$$

Smootheology

Theorem (Frölicher)

The category of Frölicher spaces is a

complete, co-complete, cartesian closed category.

Which was what we wanted!

Conclusion

By focussing on **morphisms** and looking for **simplicity** we found a natural path to **Frölicher spaces**.

Comparative

- ▶ Frölicher spaces are not the only candidate.
- ▶ Other versions due to
 - ▶ K. T. Chen (4 versions!)
 - ▶ J. M. Souriau (diffeological spaces)
 - ▶ J. W. Smith
 - ▶ R. Sikorski
 - ▶ More?
- ▶ Goal: extend category of smooth manifolds to contain more things.
- ▶ Motivation: particular examples and problems.

Countdown Categories

All involve **probing** using **test** objects.

Minor differences: underlying objects, test objects, direction.

Key difference:

- ▶ Closure condition on probing functions.

Closure: Things that **ought** to be the same **are** the same.

Examples

- ▶ Saturation: Frölicher, Smith.
- ▶ Sheaf: Chen, Souriau, Sikorski.
- ▶ Determined: Chen.

Examples: Souriau, Sikorski

Definition (Diffeological (Souriau) space)

Set X , diffeology $\mathcal{D}(U) \subseteq \text{Set}(U, X)$

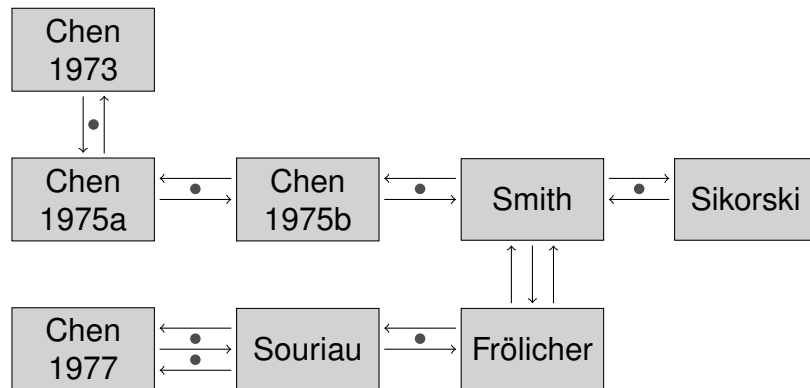
$\phi: U \rightarrow X$ in \mathcal{D} if it is locally in \mathcal{D} .

Definition (Sikorski space)

Top space X , functions $\mathcal{F} \subseteq \text{Top}(X, \mathbb{R})$

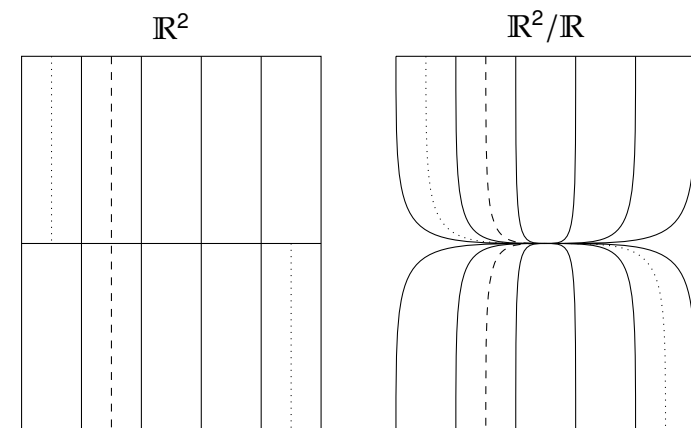
algebra, locality, $g(f_1, \dots, f_n) \in \mathcal{F}$ for $g \in C^\infty(\mathbb{R}^n, \mathbb{R})$

Schematic



• denotes adjunction

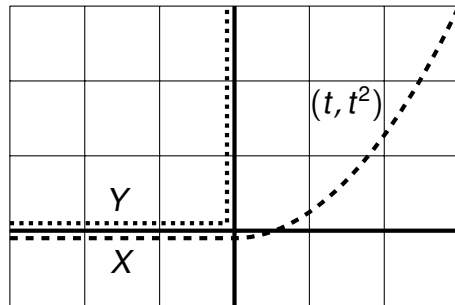
Distinctions: Souriau and Frölicher



----- Souriau
 Frölicher

Distinctions: Sikorski and Frölicher

$X, Y \subseteq \mathbb{R}^2$ as



Sikorski: $X \not\cong Y$, $t \mapsto |t|$ smooth on Y , **not** on X ,
Frölicher: $X \cong Y$, $t \mapsto |t|$ smooth on both.

Conclusion

- ▶ **Frölicher spaces** arise by taking seriously the notion of a **morphism of smooth manifolds**.
- ▶ The various categories fit into a neat setting but are **distinct**. So may have **distinct** behaviour.