Comparative Smootheology

Workshop on Smooth Structures in Logic, Category Theory, and Physics, Ottawa

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Contents

1. Smootheology

   Aim: Frölicher spaces are “obvious” generalisation of manifolds.

2. Comparative

   Aim: how the various categories of “smooth objects” fit together.

There now follows a party political broadcast
for the

Frölicher Party

Goal

Build a category extending smooth manifolds.
To contain:

- Orbifolds: quotients
- Stratifolds: push-outs
- Loop spaces: mapping objects
- $P_1(M)$: mapping objects AND quotients
- Embedding spaces: mapping objects AND subobjects
- And more …

C^5: complete, co-complete, cartesian closed category.
Guiding principle:

The Smooth,
the Whole Smooth,
and Nothing But the Smooth.

The Smooth

Building a Category

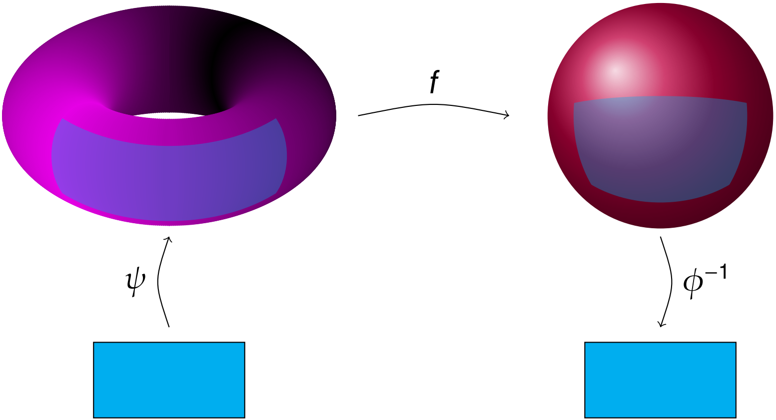
A category has

Objects

and

Morphisms

Morphism of Manifolds



$f\colon M \to N$ smooth if $\phi^{-1} f \psi$ is C^∞

Charts

- Charts control smooth structure
- Domains are “standard” spaces
- Use both charts and inverses
- Charts are homeomorphisms

Lose homeomorphism: too fragile.
Retain:

$$\mathcal{I}(U) = \{\psi \colon \mathbb{R}^m \supseteq U \to M\} \quad \text{input test functions}$$
$$\mathcal{O}(V; \mathbb{R}^m) = \{\phi \colon M \supseteq V \to \mathbb{R}^m\} \quad \text{output test functions}$$

First Candidate

Definition (First Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I}(U) \subseteq \mathrm{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \mathrm{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

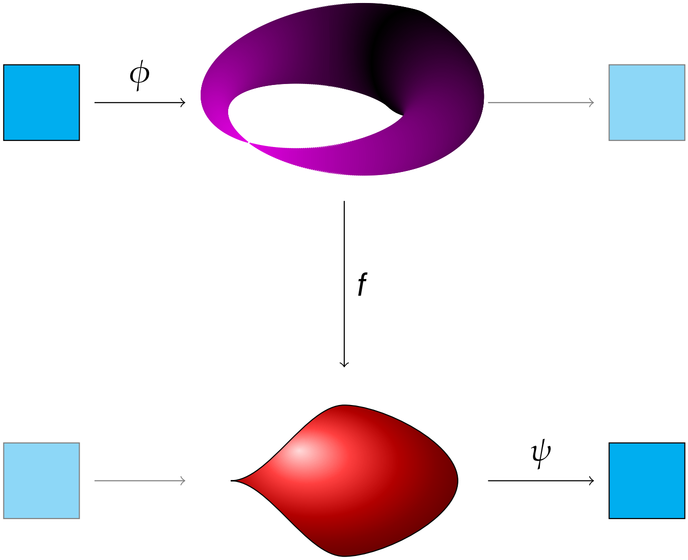
A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \text{ is } C^\infty \text{ for } \psi \in \mathcal{I}(U), \phi \in \mathcal{O}(V; \mathbb{R}^m)$$

Notation:

- smooth map = morphism
- $\psi \in \mathcal{I}$, $\phi \in \mathcal{O}$, $\theta \in C^\infty$

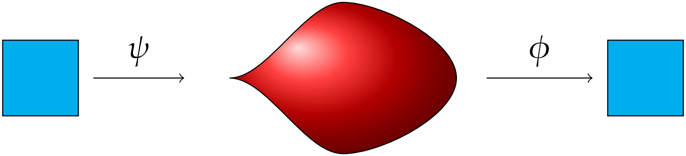
Morphisms



Categorical Construct?

1. Identities:
   $1_X : X \to X$ must be smooth.

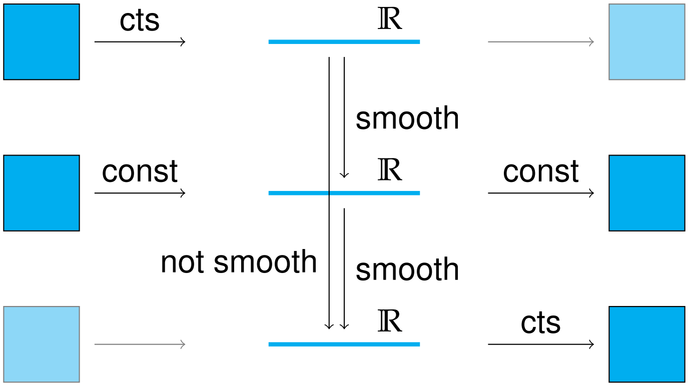
$$\psi \in \mathcal{I}, \phi \in \mathcal{O} \text{ then } \phi\psi = \phi 1_X \psi \in C^\infty$$



Compatibility Condition

Categorical Construct?

2. Composition:

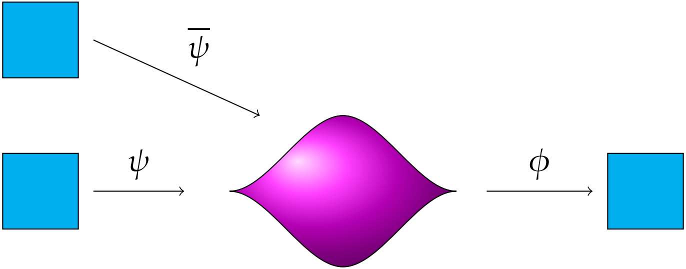


Composition

Definition

The completed inputs and outputs are

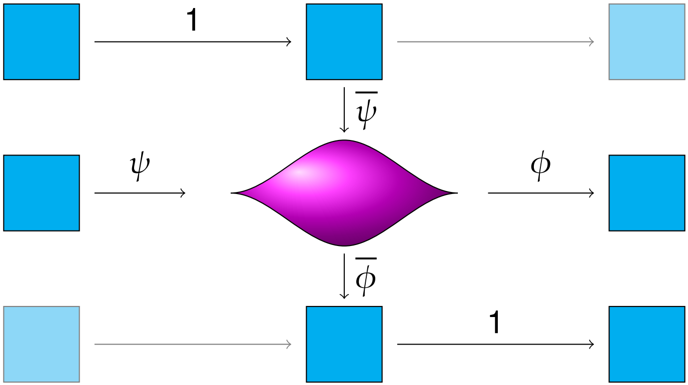
1. $\overline{\mathcal{I}}(U) = \{\overline{\psi}\colon U \to X : \phi\overline{\psi} \in C^\infty, \phi \in \mathcal{O}\}$
2. $\overline{\mathcal{O}}(V; \mathbb{R}^m) = \{\overline{\phi}\colon V \to \mathbb{R}^m : \overline{\phi}\psi \in C^\infty, \psi \in \mathcal{I}\}$



Composition

Proposition

Composition is well-defined for smooth objects where $(\overline{\mathcal{I}}, \overline{\mathcal{O}})$ satisfies compatibility.



Second Candidate

Definition (Second Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I}(U) \subseteq \mathrm{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \mathrm{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\overline{\mathcal{I}}$ and $\overline{\mathcal{O}}$ are also compatible.

A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

The Whole Smooth

Too Many Smooth Spaces Spoil The Category

Lemma

$(X, \mathcal{I}, \mathcal{O})$ smooth space then $(X, \overline{\mathcal{I}}, \overline{\mathcal{O}})$ also smooth space and moreover

$$1_X \colon (X, \mathcal{I}, \mathcal{O}) \to (X, \overline{\mathcal{I}}, \overline{\mathcal{O}})$$

is an isomorphism.

Proof.

1. $\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}$ and $\overline{\overline{\mathcal{O}}} = \overline{\mathcal{O}}$
2. $\phi\psi \in C^\infty$ for $\phi \in \overline{\mathcal{O}}$, $\psi \in \mathcal{I}$ and for $\phi \in \mathcal{O}$, $\psi \in \overline{\mathcal{I}}$ □

Third Candidate

Definition (Third Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I}(U) \subseteq \mathrm{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \mathrm{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Comments

- Non-smoothness is detectable:
  If $\phi\colon X \supseteq V \to \mathbb{R}^m$ is not in $\mathcal{O}(V; \mathbb{R}^m)$ then there is $\psi \in \mathcal{I}(U)$ such that $\phi\psi$ is not C^∞.
- Analogy:

2nd Definition	smooth atlas
versus	versus
3rd Definition	maximal smooth atlas

Detox

Lemma

$(X, \mathcal{I}, \mathcal{O})$ is completely determined by

- $\mathcal{O}(V; \mathbb{R})$
- $\mathcal{I}(\mathbb{R})$

Proof.

1. $\phi \colon U \to \mathbb{R}^m$ is C^∞ if and only if each $p_i\phi \colon U \to \mathbb{R}$ is C^∞, p_i projection
2. $\psi \colon U \to \mathbb{R}^m$ is C^∞ if and only if each $\psi\gamma \colon \mathbb{R} \to \mathbb{R}^m$ is C^∞, $\gamma \in C^\infty(\mathbb{R}, U)$ [Boman's Theorem] □

Fourth Candidate

Definition (Fourth Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I} \subseteq \mathrm{Top}(\mathbb{R}, X)$,
- $\mathcal{O}(V) \subseteq \mathrm{Top}(V, \mathbb{R})$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a continuous map $f\colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Nothing But The Smooth

Chicken and Egg

Question

Is “Smooth” built on top of “Continuous” or alongside?

- Topology only used for local functions.
- Can get topology from $\mathcal{I}$ and $\mathcal{O}$.
- Topology from $\mathcal{O}$ has bump functions.
- Local functions extend globally.

Fifth Candidate

Definition (Fifth Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a set
- $\mathcal{I} \subseteq \mathrm{Set}(\mathbb{R}, X)$,
- $\mathcal{O} \subseteq \mathrm{Set}(X, \mathbb{R})$.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Fifth Candidate: Frölicher Space

Definition (Frölicher Space)

A Frölicher space is a triple $(X, \mathcal{C}, \mathcal{F})$ where:

- X is a set
- $\mathcal{C} \subseteq \mathrm{Set}(\mathbb{R}, X)$,
- $\mathcal{F} \subseteq \mathrm{Set}(X, \mathbb{R})$.

such that

- $\mathcal{C} = \{\psi \colon \mathbb{R} \to X : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \phi \in \mathcal{F}\}$
- $\mathcal{F} = \{\phi \colon X \to \mathbb{R} : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \psi \in \mathcal{C}\}$

A morphism is a map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{C}, \phi \in \mathcal{F}$$

Smootheology

Theorem (Frölicher)

The category of Frölicher spaces is a

complete, co-complete, cartesian closed category.

Which was what we wanted!

Conclusion

By focussing on morphisms and looking for simplicity we found a natural path to Frölicher spaces.

Comparative

- Frölicher spaces are not the only candidate.
- Other versions due to
  - K. T. Chen (4 versions!)
  - J. M. Souriau (diffeological spaces)
  - J. W. Smith
  - R. Sikorski
  - More?
- Goal: extend category of smooth manifolds to contain more things.
- Motivation: particular examples and problems.

Countdown Categories

All involve probing using test objects.

Minor differences: underlying objects, test objects, direction.

Key difference:

- Closure condition on probing functions.

Closure: Things that ought to be the same are the same.

Examples

- Saturation: Frölicher, Smith.
- Sheaf: Chen, Souriau, Sikorski.
- Determined: Chen.

Examples: Souriau, Sikorski

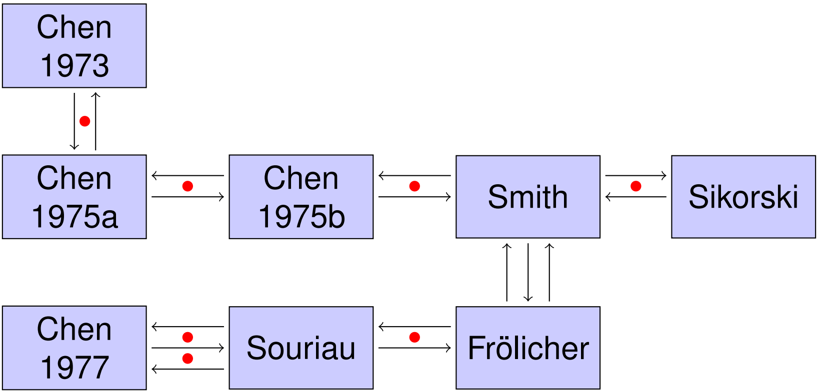
Definition (Diffeological (Souriau) space)

Set X, diffeology $\mathcal{D}(U) \subseteq \mathrm{Set}(U, X)$
$\phi\colon U \to X$ in $\mathcal{D}$ if it is locally in $\mathcal{D}$.

Definition (Sikorski space)

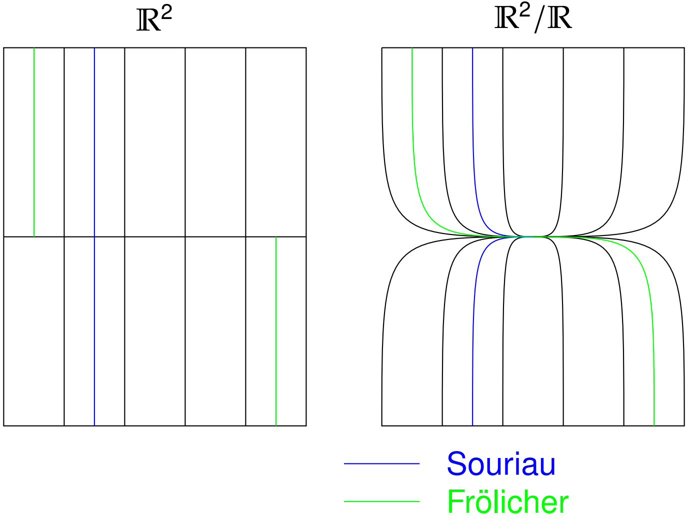
Top space X, functions $\mathcal{F} \subseteq \mathrm{Top}(X, \mathbb{R})$
algebra, locality, $g(f_1, \ldots, f_n) \in \mathcal{F}$ for $g \in C^\infty(\mathbb{R}^n, \mathbb{R})$

Schematic



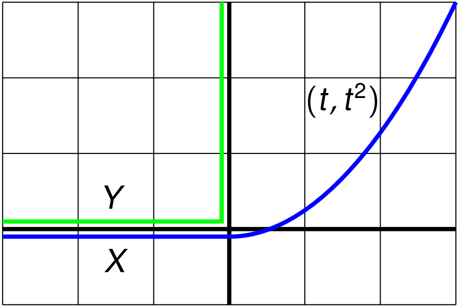
• denotes adjunction

Distinctions: Souriau and Frölicher



Distinctions: Sikorski and Frölicher

$X, Y \subseteq \mathbb{R}^2$ as



Sikorski: $X \not\cong Y$, $t \mapsto |t|$ smooth on Y, not on X,

Frölicher: $X \cong Y$, $t \mapsto |t|$ smooth on both.

Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
- The various categories fit into a neat setting but are distinct. So may have distinct behaviour.