Comparative Smootheology
Workshop on Loops, Strings and Moduli Spaces,
Chern Institute of Math, Tianjin, China

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Trondheim

3rd August 2009
1. Smootheology
   Aims:
   1.1 Motivate generalising manifolds.
   1.2 Frölicher spaces are “obvious” generalisation of manifolds.

2. Comparative
   Aim:
   2.1 Show how the various categories of “smooth objects” fit together.
   2.2 What is special about Frölicher spaces.
There now follows a party political broadcast for the Frölicher Party.
Manifolds are great, it’s a pity more things aren’t manifolds.
Manifolds

Manifolds are great, it’s a pity more things aren’t manifolds.
Manifolds

Manifolds are great, it’s a pity more things aren’t manifolds.
Manifolds

Manifolds are great, it’s a pity more things aren’t manifolds.

Goal: build a wider bridge.
<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is a manifold?</td>
</tr>
</tbody>
</table>
## Extending the Foundations

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is a manifold?</td>
<td>A <em>locally nice</em> space.</td>
</tr>
<tr>
<td>Question</td>
<td>Answer</td>
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<tr>
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<td>---------------------------------</td>
</tr>
<tr>
<td>What is a manifold?</td>
<td>Locally diffeomorphic to a model space.</td>
</tr>
</tbody>
</table>
## Question
What is a manifold?

## Answer
Locally diffeomorphic to a model space.

## Question
How do we extend “manifolds”? 
<table>
<thead>
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<tr>
<td>What is a manifold?</td>
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<tr>
<td>How do we extend “manifolds”?</td>
<td>Find more models.</td>
</tr>
</tbody>
</table>
Manifolds with . . .

Want to study cobordisms.
Manifolds with Boundary

Want to study cobordisms.
Manifolds with Boundary

Want to study cobordisms.

New model: half spaces.
Manifolds with . . .

Want to study cobordisms of cobordisms.
Manifolds with Corners

Want to study cobordisms of cobordisms.
Manifolds with Corners

Want to study cobordisms of cobordisms.

New model: quadrants.
Manifolds with . . .

Want to study smash products.
Manifolds with . . .

Want to study smash products.
Manifolds with . . .

Want to study smash products.
Manifolds with . . .

Want to study smash products.
Manifolds with Singularities

Want to study smash products.

New model: singular points.
Interlude

The models get more complicated faster than the spaces.

D. Hume
The models get more complicated faster than the spaces.

*The only thing capable of modelling the universe is*
Interlude

The models get more complicated faster than the spaces.

*The only thing capable of modelling the universe is the universe itself.*

D. Hume
Manifolds

Want to study loop spaces.
Want to study loop spaces.
Manifolds

Want to study loop spaces.
Want to study loop spaces.
Infinite Dimensional Manifolds

Want to study loop spaces.

New model: loops in $\mathbb{R}^n$. 
Manifolds

Want to study path spaces.
Manifolds

Want to study path spaces.
Manifolds

Want to study path spaces.
Manifolds

Want to study path spaces.
Not Manifolds at all

Want to study path spaces.

No good local models.
### Question

How do we extend “manifolds”?
<table>
<thead>
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<th>How do we extend “manifolds”?</th>
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<tbody>
<tr>
<td>Answer</td>
<td>What answer do we want?</td>
</tr>
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<td><strong>Answer</strong></td>
<td>What answer do we want?</td>
</tr>
<tr>
<td></td>
<td>Should have:</td>
</tr>
<tr>
<td></td>
<td>▶ Subobjects</td>
</tr>
<tr>
<td></td>
<td>▶ Quotients</td>
</tr>
<tr>
<td></td>
<td>▶ Mapping spaces</td>
</tr>
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</table>
Back to the Drawing Board

Question
How do we extend “manifolds”?

Answer
What answer do we want?
Should have:
▶ Subobjects
▶ Quotients
▶ Mapping spaces
All Categorical in nature.
The Search

Looking for a:

$C^5$: 
The Search

Looking for a:

\[ C^5: \]
\begin{align*}
&\text{complete,} \\
&\text{co-complete,} \\
&\text{cartesian closed} \\
&\text{category.}
\end{align*}
The Search

Looking for a:

$C^5$: complete, co-complete, cartesian closed category.

Guiding principle:

The Smooth,
The Search

Looking for a:

$C^5$: complete, co-complete, cartesian closed category.

Guiding principle:

The Smooth, the Whole Smooth,
The Search

Looking for a:

\[ C^5: \]
complete,
co-complete,
cartesian closed
category.

Guiding principle:

The Smooth,
the Whole Smooth,
and Nothing But the Smooth.
The Smooth
Building a Category

Question

What is a manifold?
<table>
<thead>
<tr>
<th>Question</th>
<th>Categorical Answer</th>
</tr>
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<tr>
<td>What is a manifold?</td>
<td>The structure needed to define smooth maps.</td>
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</table>
Building a Category

Question
What is a manifold?

Categorical Answer
The structure needed to define smooth maps.

A category has

Objects
and
Morphisms
Question
What is a manifold?

Categorical Answer
The structure needed to define smooth maps.

A category has

Objects

and

Morphisms
Morphism of Manifolds

\[ f : M \to N \text{ smooth if } \phi^{-1} f \psi \text{ is } C^\infty \]
The Role of the Charts

- Charts control smooth structure
The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
- Map is **smooth** if it looks smooth when we test it
The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
- Map is smooth if it looks smooth when we test it

The role of charts is to transport the question of smoothness to familiar spaces.
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- Charts control smooth structure
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The role of charts is to transport the question of smoothness to familiar spaces. Replace local models by test spaces.
The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
- Map is smooth if it looks smooth when we test it

The role of charts is to transport the question of smoothness to familiar spaces. Replace local models by test spaces.

Retain:

\[ I(U) = \{ \psi : \mathbb{R}^m \supseteq U \to M \} \]  
input test functions

\[ O(V; \mathbb{R}^m) = \{ \phi : M \supseteq V \to \mathbb{R}^m \} \]  
output test functions
The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
- Map is smooth if it looks smooth when we test it

The role of charts is to transport the question of smoothness to familiar spaces. Replace local models by test spaces. Retain:

\[
\mathcal{I}(U) = \{ \psi : \mathbb{R}^m \supseteq U \to M \} \quad \text{input test functions}
\]
\[
\mathcal{O}(V; \mathbb{R}^m) = \{ \phi : M \supseteq V \to \mathbb{R}^m \} \quad \text{output test functions}
\]

Look Ma! No homeomorphisms!
A smooth space is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I}(U) \subseteq \text{Top}(U, X), \ U \subseteq \mathbb{R}^m\) open,
- \(\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), \ V \subseteq X\) open.
Definition (First Attempt)

A **smooth space** is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I}(U) \subseteq \text{Top}(U, X), \ U \subseteq \mathbb{R}^m\) open,
- \(\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), \ V \subseteq X\) open.

A **morphism** is a continuous map \(f : X \to Y\) such that

\[ \phi f \psi \text{ is } C^\infty \text{ for } \psi \in \mathcal{I}(U), \phi \in \mathcal{O}(V; \mathbb{R}^m) \]
Definition (First Attempt)

A **smooth space** is a triple \((X, \mathcal{I}, \mathcal{O})\) where:
- \(X\) is a topological space
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A **morphism** is a continuous map \(f: X \rightarrow Y\) such that

\[
\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}
\]

**Notation:**
- smooth map = morphism
- \(\psi \in \mathcal{I}, \phi \in \mathcal{O}, \theta \in C^\infty\)
Morphisms

\[ \phi \quad \rightarrow \quad f \quad \rightarrow \quad \psi \]

\[
\begin{array}{c}
\text{Blue square} \\
\text{Red shape} \\
\text{Blue square}
\end{array}
\]

\[
\begin{array}{c}
\text{Light blue square} \\
\text{Red shape} \\
\text{Light blue square}
\end{array}
\]
1. Identities:
   \(1_X: X \rightarrow X\) must be smooth.

   \[\psi \in \mathcal{I}, \phi \in \mathcal{O} \text{ then } \phi \psi = \phi \cdot 1_X \psi \in \mathcal{C}^\infty\]
Categorical Construct?

1. Identities:
   \(1_X: X \rightarrow X\) must be smooth.

   \[\psi \in \mathcal{I}, \phi \in \mathcal{O}\] then \(\phi \psi = \phi 1_X \psi \in \mathcal{C}^\infty\)
1. Identities:
   \( 1_X : X \to X \) must be smooth.

   \[ \psi \in \mathcal{I}, \phi \in \mathcal{O} \quad \text{then} \quad \phi \psi = \phi 1_X \psi \in \mathcal{C}^\infty \]
Categorical Construct?

2. Composition:

- \text{cts} \rightarrow \mathbb{R} \rightarrow \text{const}
- \text{const} \rightarrow \mathbb{R} \rightarrow \text{const}
- \text{cts} \rightarrow \mathbb{R} \rightarrow \text{const}
Categorical Construct?

2. Composition:

\[ \text{cts} \quad \rightarrow \quad \mathbb{R} \quad \rightarrow \quad \text{smooth} \quad \rightarrow \quad \text{const} \quad \rightarrow \quad \text{cts} \]

\[ \text{const} \quad \rightarrow \quad \mathbb{R} \quad \rightarrow \quad \text{const} \quad \rightarrow \quad \text{cts} \]
2. Composition:

**Categorical Construct?**
Categorical Construct?

2. Composition:

- \( \text{cts} \rightarrow R \rightarrow \text{const} \rightarrow \text{smooth} \rightarrow \text{not smooth} \)
- \( \text{const} \rightarrow R \rightarrow \text{smooth} \rightarrow R \rightarrow \text{cts} \)
Composition

**Definition**

The completed inputs and outputs are

1. $\overline{I}(U) = \{\overline{\psi}: U \to X : \phi\overline{\psi} \in C^\infty, \phi \in O\}$
2. $\overline{O}(V; \mathbb{R}^m) = \{\overline{\phi}: V \to \mathbb{R}^m : \overline{\phi}\psi \in C^\infty, \psi \in I\}$
The completed inputs and outputs are

1. $\overline{I}(U) = \{\overline{\psi} : U \to X : \phi \overline{\psi} \in C^\infty, \phi \in \mathcal{O}\}$

2. $\overline{O}(V; \mathbb{R}^m) = \{\overline{\phi} : V \to \mathbb{R}^m : \overline{\phi} \psi \in C^\infty, \psi \in \mathcal{I}\}$
**Proposition**

Composition is well-defined for smooth objects where $(\mathcal{I}, \mathcal{O})$ satisfies *compatibility*. 
Composition is well-defined for smooth objects where $(\bar{I}, \bar{O})$ satisfies *compatibility*.
Composition

Proposition

Composition is well-defined for smooth objects where \((\overline{I}, \overline{O})\) satisfies compatibility.
First Candidate

**Definition (First Attempt)**

A **smooth space** is a triple \((X, I, O)\) where:

- **\(X\)** is a topological space
- **\(I(U) \subseteq \text{Top}(U, X), U \subseteq \mathbb{R}^m \text{ open},\)**
- **\(O(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), V \subseteq X \text{ open}.\)**

A **morphism** is a continuous map \(f: X \rightarrow Y\) such that

\[
\phi f \psi \in C^\infty \text{ for } \psi \in I, \phi \in O
\]
Definition (Second Attempt)

A smooth space is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I}(U) \subseteq \text{Top}(U, X), U \subseteq \mathbb{R}^m\) open,
- \(\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), V \subseteq X\) open.

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\overline{\mathcal{I}}\) and \(\overline{\mathcal{O}}\) are also compatible.

A morphism is a continuous map \(f: X \to Y\) such that

\[\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}\]
The Whole Smooth
**Lemma**

\[(X, \mathcal{I}, \mathcal{O}) \text{ smooth space then } (X, \overline{\mathcal{I}}, \overline{\mathcal{O}}) \text{ also smooth space and moreover} \]

\[1_X : (X, \mathcal{I}, \mathcal{O}) \to (X, \overline{\mathcal{I}}, \overline{\mathcal{O}}) \]

is an isomorphism.
Lemma


text

Proof.

1. $\overline{I} = \overline{I}$ and $\overline{O} = \overline{O}$
Lemma

\((X, \mathcal{I}, \mathcal{O})\) smooth space then \((X, \overline{\mathcal{I}}, \overline{\mathcal{O}})\) also smooth space
and moreover

\[
1_X: (X, \mathcal{I}, \mathcal{O}) \to (X, \overline{\mathcal{I}}, \overline{\mathcal{O}})
\]

is an isomorphism.

Proof.

1. \(\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}\) and \(\overline{\overline{\mathcal{O}}} = \overline{\mathcal{O}}\)

2. \(\phi \psi \in C^\infty\) for \(\phi \in \overline{\mathcal{O}}, \psi \in \overline{\mathcal{I}}\) and for \(\phi \in \mathcal{O}, \psi \in \overline{\mathcal{I}}\)
### Definition (Second Attempt)

A **smooth space** is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- **\(X\)** is a topological space
- **\(\mathcal{I}(U) \subseteq \text{Top}(U, X), U \subseteq \mathbb{R}^m \text{ open}\),**
- **\(\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), V \subseteq X \text{ open}\).**

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\overline{\mathcal{I}}\) and \(\overline{\mathcal{O}}\) are also compatible.

A **morphism** is a continuous map \(f : X \rightarrow Y\) such that

\[
\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}
\]
### Definition (Third Attempt)

A **smooth space** is a triple \((X, I, O)\) where:

- **\(X\) is a topological space**
- \(I(U) \subseteq \text{Top}(U, X), \ U \subseteq \mathbb{R}^m\) open,
- \(O(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), \ V \subseteq X\) open.

such that

- \(I\) and \(O\) are compatible,
- \(I\) and \(O\) are **saturated**: \(I = \overline{I}, \ O = \overline{O}\).

A **morphism** is a continuous map \(f: X \to Y\) such that

\[
\phi f \psi \in C^\infty \text{ for } \psi \in I, \phi \in O
\]
Analogy

2nd Definition versus 3rd Definition

smooth atlas versus maximal smooth atlas
Lemma

$(X, I, O)$ is completely determined by

- $O(V; \mathbb{R})$
- $I(\mathbb{R})$
**Lemma**

$(X, I, O)$ is completely determined by

- $O(V; \mathbb{R})$
- $I(\mathbb{R})$

**Proof.**

1. $\phi: U \to \mathbb{R}^m$ is $C^\infty$ if and only if each $p_i\phi: U \to \mathbb{R}$ is $C^\infty$, $p_i$ projection
Lemma

\( (X, \mathcal{I}, \mathcal{O}) \) is completely determined by

- \( \mathcal{O}(V; \mathbb{R}) \)
- \( \mathcal{I}(\mathbb{R}) \)

Proof.

1. \( \phi: U \to \mathbb{R}^m \) is \( C^\infty \) if and only if each \( p_i\phi: U \to \mathbb{R} \) is \( C^\infty \), \( p_i \) projection

2. \( \psi: U \to \mathbb{R}^m \) is \( C^\infty \) if and only if each \( \psi\gamma: \mathbb{R} \to \mathbb{R}^m \) is \( C^\infty \), \( \gamma \in C^\infty(\mathbb{R}, U) \) [Boman’s Theorem] 

\( \square \)
Definition (Third Attempt)

A **smooth space** is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I}(U) \subseteq \text{Top}(U, X), \ U \subseteq \mathbb{R}^m\) open,
- \(\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), \ V \subseteq X\) open.

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\mathcal{I}\) and \(\mathcal{O}\) are saturated: \(\mathcal{I} = \overline{\mathcal{I}}, \ \mathcal{O} = \overline{\mathcal{O}}\).

A **morphism** is a continuous map \(f : X \to Y\) such that

\[ \phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O} \]
A smooth space is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I} \subseteq \text{Top}(\mathbb{R}, X)\),
- \(\mathcal{O}(V) \subseteq \text{Top}(V, \mathbb{R})\), \(V \subseteq X\) open.

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\mathcal{I}\) and \(\mathcal{O}\) are saturated: \(\mathcal{I} = \overline{\mathcal{I}}, \mathcal{O} = \overline{\mathcal{O}}\).

A morphism is a continuous map \(f: X \to Y\) such that

\[
\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}
\]
Nothing But The Smooth
<table>
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<tr>
<td>Is “Smooth” built on top of “Continuous” or alongside?</td>
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Question

Is “Smooth” built on top of “Continuous” or alongside?

- Topology only used for local functions.
Egg and Chicken

**Question**

Is “Smooth” built **on top of** “Continuous” or **alongside**?

- Topology only used for local functions.
- Can get topology from $I$ and $O$. 
### Egg and Chicken

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- Topology only used for local functions.
- Can get topology from $\mathcal{I}$ and $\mathcal{O}$.
- Topology from $\mathcal{O}$ has **bump functions**.
Egg and Chicken

Question

Is “Smooth” built on top of “Continuous” or alongside?

- Topology only used for local functions.
- Can get topology from $\mathcal{I}$ and $\mathcal{O}$.
- Topology from $\mathcal{O}$ has bump functions.
- Local functions extend globally.
### Definition (Fourth Attempt)

A **smooth space** is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I} \subseteq \text{Top}(\mathbb{R}, X)\),
- \(\mathcal{O}(V) \subseteq \text{Top}(V, \mathbb{R}), \ V \subseteq X\) open.

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\mathcal{I}\) and \(\mathcal{O}\) are saturated: \(\mathcal{I} = \overline{\mathcal{I}}, \ \mathcal{O} = \overline{\mathcal{O}}\).

A **morphism** is a continuous map \(f : X \rightarrow Y\) such that

\[
\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}
\]
A smooth space is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a set
- \(\mathcal{I} \subseteq \text{Set}(\mathbb{R}, X)\),
- \(\mathcal{O} \subseteq \text{Set}(X, \mathbb{R})\).

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\mathcal{I}\) and \(\mathcal{O}\) are saturated: \(\mathcal{I} = \overline{\mathcal{I}}, \mathcal{O} = \overline{\mathcal{O}}\).

A morphism is a map \(f: X \rightarrow Y\) such that

\[ \phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O} \]
A Frölicher space is a triple \((X, C, F)\) where:

- \(X\) is a set
- \(C \subseteq \text{Set}(\mathbb{R}, X)\),
- \(F \subseteq \text{Set}(X, \mathbb{R})\).

such that

- \(C = \{\psi : \mathbb{R} \to X : \phi \psi \in C^\infty(\mathbb{R}, \mathbb{R}), \phi \in F\}\)
- \(F = \{\phi : X \to \mathbb{R} : \phi \psi \in C^\infty(\mathbb{R}, \mathbb{R}), \psi \in C\}\)

A morphism is a map \(f : X \to Y\) such that

\[\phi f \psi \in C^\infty \text{ for } \psi \in C, \phi \in F\]
Theorem (Frölicher)

The category of Frölicher spaces is a complete, co-complete, cartesian closed category.

Which was what we wanted!
Theorem (Frölicher)

The category of Frölicher spaces is a $C^5$. 

$W^5!$
Theorem (Frölicher)

The category of Frölicher spaces is a $C^5$.

$W^5!$

Conclusion (Part I)

By focussing on morphisms and looking for simplicity we found a natural path to Frölicher spaces.
Frölicher spaces are not the only candidate.
Comparative

- Frölicher spaces are not the only candidate.

- Other versions:
  - K. T. Chen: 4 versions!
  - J. M. Souriau: diffeological spaces
  - J. W. Smith
  - R. Sikorski
  - D. Spivak: derived smooth manifolds
  - J. Lurie: structured space
  - A. Kriegl and P. Michor
  - M. Kreck: stratifold
  - H. Hofer: polyfold
  - orbifold
  - differentiable stack

More?
Comparitive

Use

- local models
- test spaces
- both

In Comparitive Smootheology looked at: Test spaces + set-based Frölicher, Chen, Souriau, Smith, Sikorski.
Comparitive

Use

- local models
- test spaces
- both

In *Comparative Smootheology* looked at:

Test spaces + set-based

Frölicher, Chen, Souriau, Smith, Sikorski.
Figure: The relationships between the categories
### Characteristics of Frölicher Spaces

#### Cons

- Not locally cartesian closed
- Not easy to make non-set based
Characteristics of Frölicher Spaces

Pros

- Inclusion of manifolds is limit and colimit preserving. (into Hausdoff Frölicher spaces)
- “Smallest” extension
What Next?

- What of differential topology/geometry extends to Frölicher spaces?
- What extra structure is needed for those bits that don’t extend?

Example

Tangent “spaces” extend but are not vector spaces. Need extra structure to get addition.
Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
- The various categories fit into a neat setting but are **distinct**. So may have **distinct** behaviour.
Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
- The various categories fit into a neat setting but are distinct. So may have distinct behaviour.
- Examining what does and does not extend sheds light on the role of smoothness in differential topology.
Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
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- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
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- Examining what does and does not extend sheds light on the role of smoothness in differential topology.

Further Information

<table>
<thead>
<tr>
<th>My homepage</th>
<th>Andrew Stacey NTNU</th>
<th>Search</th>
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<tbody>
<tr>
<td>$n$—lab</td>
<td>nlab</td>
<td>Search</td>
</tr>
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